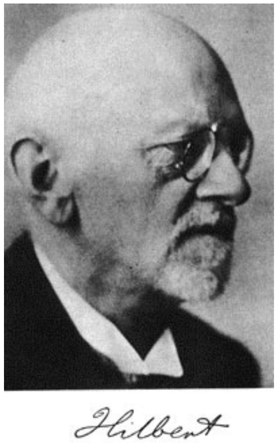
Lecture T3: Computability



" Every mathematical problem can be solved. We are convinced of that. After all, one of the things that attracts us most when we apply ourselves to a mathematical problem is precisely that within us we always hear the call: here is the problem, search for the solution, you can find it by pure thought, for in mathematics there is no *ignorabimus*. "

A Puzzle ("Post's Correspondence Problem")

Given a set of cards:

- **N card types (can use as many of each type as possible).**
- **Each card has a top string and bottom string.**

Example 1:

BAB	A	AB	BA
A	ABA	B	B
0	1	2	3

N = 4

Puzzle:

- **Is it possible to arrange cards so that top and bottom strings are the same?**

A Puzzle ("Post's Correspondence Problem")

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Example 1:

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0	1	2	3

N = 4

Puzzle:

- **Is it possible to arrange cards so that top and bottom strings are the same?**

Solution 1.

A	BA	BAB	AB	A
ABA	B	A	B	ABA
1	3	0	2	1

A Puzzle ("Post's Correspondence Problem")

Given a set of cards:

- **N card types (can use as many of each type as possible).**
- **Each card has a top string and bottom string.**

Example 2:

A	ABA	B	A
BAB	B	A	B
0	1	2	3

N = 4

Puzzle:

- **Is it possible to arrange cards so that top and bottom strings are the same?**

A Puzzle ("Post's Correspondence Problem")

Given a set of cards:
- N card types (can use as many of each type as possible).
- Each card has a top string and bottom string.

Example 2:

A	ABA	B	A
BAB	B	A	B
0	1	2	3

N = 4

Puzzle:
- Is it possible to arrange cards so that top and bottom strings are the same?

Solution 2.

PCP Puzzle Contest

S[	X	X	11A	1	[A	]	[	B1	B]	[1A]E
S[11111X][	1X	A	A1	1	[B	]	[	1B	A]	E
0	1	2	3	4	5	6	7	8	9	10

Contest:
- Additional restriction: string must start with 'S'.
- Be the first to solve this puzzle!
  - extra credit for first correct solution
- Check solution by putting STRING ONLY (blanks and line breaks OK) in a file `solution.txt`, then type
  `pcp126 < solution.txt`

Hopeless challenge for the bored:
- Write a program that reads a set of Post cards, and determines whether or not there is a solution.

Overview

Formal language.
- Rigorously express computational problems.
- Ex: L = { 2, 3, 5, 7, 11, 13, 17, . . . }

Abstract machines recognize languages.
- Ex. Is 977 prime? Is 977 in L?
- Essence of computers.

This lecture:
- What is an "algorithm"?
- Is it possible, in principle, to write a program to solve any problem (recognize any language)?

Background

Abstract models of computation help us learn:
- Nature of machines needed to solve problems.
- Relationship between problems and machines.
- Intrinsic difficulty of problems.

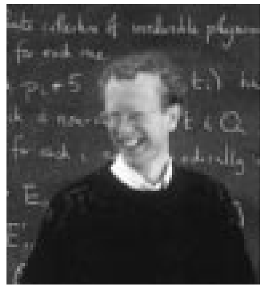
As we make machines more powerful, we can recognize more languages.
- Are there languages that no machine can recognize?
- Are there limits on the power of machines that we can imagine?

Pioneering work in the 1930's. (Princeton = center of universe)
- Turing, Church, von Neumann, Gödel. (inspiration from Hilbert)
- Automata, languages, computability, complexity, logic, rigorous definition of "algorithm."

Undecidable Problems

Hilbert's 10th Problem.

- "*Devise a process according to which it can be determined by a finite number of operations whether a given multivariate polynomial has an integral root.*"
- Example 1: $f(x,y,z) = 6x^3yz^2 + 3xy^2 - x^3 - 10$
- Example 2: $f(x,y) = x^2 + y^2 - 3$
- Example 3: $f(x,y,z) = x^n + y^n - z^n$



Andrew Wiles, 1995

Undecidable Problems

Hilbert's 10th Problem.

- "*Devise a process according to which it can be determined by a finite number of operations whether a given multivariate polynomial has an integral root.*"
- Problem resolved in very surprising way. (Matijasevič, 1970)
- How can we assert such a mind-boggling statement?

Undecidable Problems

Hilbert's 10th Problem.
Post's Correspondence Problem.
Halting Problem.

- Write a C program that reads in another program and its inputs, and decides whether or not it goes into an infinite loop.
  - infinite loop often signifies a bug
- Program 1.
  - 8 6 4 2 4 2 4 2 4 2 4 2 4 2 4
  - 9 7 5 3 1

odd.c

```
. . .

while (x > 1) {
  if (x > 2)
     x = x – 2;
  else
     x = x + 2;
}
```


Undecidable Problems

Hilbert's 10th Problem.
Post's Correspondence Problem.
Halting Problem.

- Write a C program that reads in another program and its inputs, and decides whether or not it goes into an infinite loop.
  - infinite loop often signifies a bug
- Program 2.
  - 8 4 2 1
  - 7 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1

hailstone.c

```
. . .
while (x > 1) {
  if (x % 2 == 0)
     x = x / 2;
  else
     x = 3*x + 1;
}
```

Undecidable Problems

Hilbert's 10th Problem.
Post's Correspondence Problem.
Halting Problem.
Program Equivalence.
Optimal Data Compression.
Virus Identification.

Impossible to write C program to solve any of these problem!

TM : As Powerful As TOY Machine

Turing machines are strictly more powerful than FSA, PDA, LBA because of infinite tape memory.

- Power = ability to recognize languages.

Turing machines are at least as powerful as a TOY machine:

- Encode state of memory, PC, etc. onto Turing tape.
- Develop TM states for each instruction.
- Can do because all instructions:
  - examine current state
  - make well-define changes depending on current state

Works for all real machines.

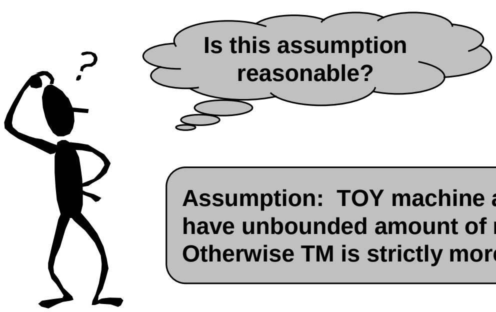
- Can simulate at machine level, gate level, ...

TM : Equal Power as TOY and C

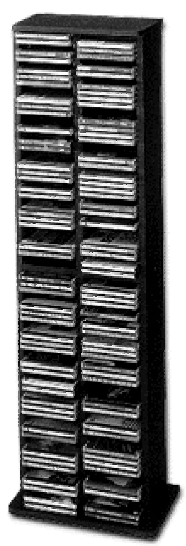
Turing machines are equivalent in power to C programs.

- C program ⇒ TOY program (Lecture A2)
- TOY program ⇒ TM (previous slide)
- TM ⇒ C program (TM simulator, Lecture T2)

Works for all real programming languages.



Assumption: TOY machine and C program have unbounded amount of memory. Otherwise TM is strictly more powerful.



Church-Turing Thesis

Church-Turing thesis (1936):

Q. Which problems can a Turing machine solve?
A. Any problem that any real computer can solve.

"Thesis" and not a mathematical theorem.

Implications:

- Provides rigorous definition for algorithm.
- Universality among computational models.
  - if a problem can be solved by TM, then it can be solved on EVERY general-purpose computer.
  - if a problem can't be solved by TM, then it can't be solve on ANY physical computer

Evidence Supporting Church-Turing Thesis

Imagine TM with more power.

- Composition of TM's, multiple heads, more tapes, 2D tapes.
- Nondeterminism.

Different ways to define "computable."

- TM, circuits, grammar, λ-calculus, μ-recursive functions.
- Conway's game of life.

Conventional computers.

- ENIAC, TOY, Pentium III, . . .

New speculative models of computation.

- DNA computers, quantum computers, soliton computers.

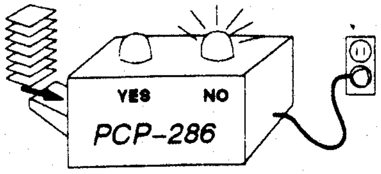
A More Powerful Computer

Post machine (PCP-286).

- Input: set of Post cards.
- Output.
  - YES light if PCP is solvable for these cards
  - NO light if PCP has no solution

PCP is strictly more powerful than:

- Turing machine.
- TOY machine.
- C programming language.
- iMac.
- Any conceivable super-computer.



Why doesn't it violate Church-Turing thesis?

TM: A General Purpose Machine

Each TM solves one particular problem.

- Ex: is the integer x prime?
- Analogy: computer algorithm.
- Similar to ancient special-purpose computers (Analytic Engine) prior to von Neumann stored-program computers.

Goal: "general purpose machine" that can solve many problems.

- Simulate the operations of any special-purpose TM.
- Analogy: computer than can execute any algorithm.
- How?

Representation of a Turing Machine

Special-purpose TM consists of 3 ingredients.

- TM program.
- Initial tape contents.
- Current TM state.

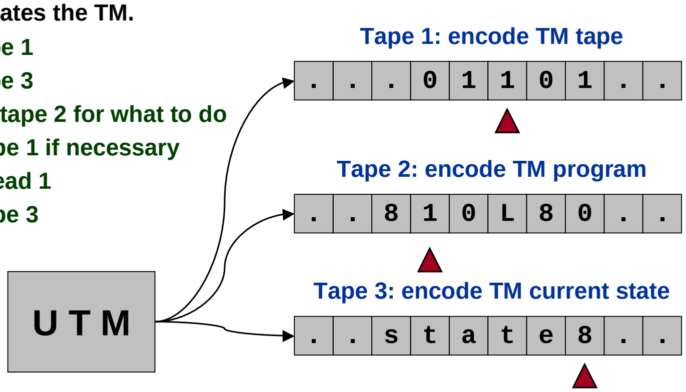
Universal Turing Machine

Universal Turing Machine (UTM),

- A specific TM that simulates operations of any TM.

How to create.

- Encode 3 ingredients of TM using 3 tapes.
- UTM simulates the TM.
  - read tape 1
  - read tape 3
  - consult tape 2 for what to do
  - write tape 1 if necessary
  - move head 1
  - write tape 3



Universal Turing Machine

Universal Turing Machine (UTM),

- A specific TM that simulates operations of any TM.

How to create.

- Encode 3 ingredients of TM using 3 tapes.
- UTM simulates the TM.

- Like the fetch-increment-execute cycle of TOY.

- Can reduce 3-tape TM to single tape one.

Implications of Universal Turing Machine

Existence of UTM has profound implications.

- "Invention" of general-purpose computer.
  - stimulated development of stored-program computers (von Neumann machines)
- "Invention" of software.
- Universal framework for studying limitations of computing devices.
- Can simulate any machine (including itself)!

Halting Problem

Halting problem.

- Devise a TM that reads in another TM (encoded in binary) and its initial tape, and determines whether or not that TM would ever reach a yes or no state.
- Write a C program that reads in another program and its inputs, and determines whether or not it goes into an infinite loop.

Halting problem is unsolvable.

- No TM can solve this problem.
- Not possible to write a C program either.

We prove that the halting problem is not solvable.

- Intuition of proof: self-reference.

Warmup: Grelling's Paradox

Grelling's paradox:

- Divide all adjectives into two categories:
  - autological: self-descriptive
  - heterological: not self-descriptive

autological adjectives	heterological adjectives
pentasyllabic	bisyllabic
awkwardnessful	palindromic
recherché	edible
. . .	. . .

- How do we categorize heterological?

Warmup: Grelling's Paradox

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heterological	

- How do we categorize heterological?
  - suppose it's autological

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~~heterological~~	heterological

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  - suppose it's heterological

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autological adjectives	heterological adjectives
pentasyllabic	bisyllabic
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recherché	edible
~~heterological~~	~~heterological~~

- How do we categorize heterological?
  - not possible
  - we can't have words with these meanings!
    (or we can't partition adjectives into these two groups)

Halting Problem Proof

Assume the existence of Halt(f,x) that takes as input: any function f and its input x, and outputs yes if f(x) halts, and no otherwise.

- Proof by contradiction.
- Note: Halt(f, x) always returns yes or no. (infinite loop not possible)

Halt(f, x)

```
#define YES 1
#define NO  0

int Halt(char f[], char x[]) {
   if ( ??? )
      return YES;
   else
      return NO;
}
```

function f and its input x encoded as strings

Halting Problem Proof

Assume the existence of Halt(f,x) that takes as input: any function f and its input x, and outputs yes if f(x) halts, and no otherwise.

- Construct program Strange(f) as follows:
  - calls Halt(f, f)
  - halts if Halt(f, f) outputs no
  - goes into infinite loop if Halt(f, f) outputs yes
- In other words:
  - if f(f) does not halt then Strange(f) halts
  - if f(f) halts then Strange(f) does not halt

f is a string so legal to use for either input

Strange(f)

```
void Strange(char f[]) {
   if (Halt(f, f) == NO)
      return;
   else
      while(1)
         ;     // infinite loop
}
```


Halting Problem Proof

Assume the existence of Halt(f,x) that takes as input: any function f and its input x, and outputs yes if f(x) halts, and no otherwise.

- Construct program Strange(f) as follows:
  - calls Halt(f, f)
  - halts if Halt(f, f) outputs no
  - goes into infinite loop if Halt(f, f) outputs yes
- In other words:
  - if f(f) does not halt then Strange(f) halts
  - if f(f) halts then Strange(f) does not halt
- Call Strange with ITSELF as input.
  - if Strange(Strange) does not halt then Strange(Strange) halts
  - if Strange(Strange) halts then Strange(Strange) does not halt
- Either way, a contradiction. Hence Halt(f,x) cannot exist.

Consequences

Halting problem is "not artificial."

- Undecidable problem reduced to simplest form to simplify proof.
- Closely related to practical problems.
  - Hilbert's 10th problem, Post's correspondence problem, program equivalence, optimal data compression

How to show new problem X is undecidable?

- Use fact that Halting problem is undecidable.
- Design algorithm to solve Halting problem, using (alleged) algorithm for X as a subroutine.
- See Reduction in Lecture T5.

Implications

Practical:

- Work with limitations.
- Recognize and avoid unsolvable problems.
- Learn from structure.
  - same theory tells us about efficiency of algorithms (see T4)

Philosophical (caveat: ask a philosopher):

- We "assume" that any step-by-step reasoning will solve any technical or scientific problem.
- "Not quite" says the halting problem.
- Anything that is like (could be) a computer has the same flaw:

Summary

What is an algorithm?

- Informally, step-by-step procedure for solving a problem.
- Formally, Turing machine.

Turing's key ideas:

- Computing is same as manipulating symbols.
  - can encode numbers as strings
- Existence of general-purpose computer (UTM).
  - programmable machine

What is a general-purpose computer (UTM)?

- Can be "programmed" to implement any algorithm.
- iMac, Dell, Sun UltraSparc, TOY (assuming unlimited memory).

Is it possible, in principle, to write a program to solve any problem?

- No.