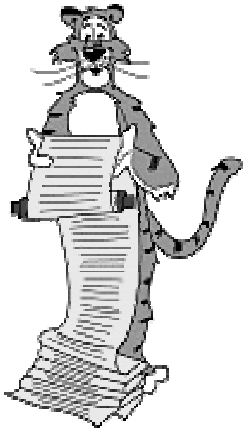
Lecture A4: Boolean Circuits



George Boole
(1815 – 1864)

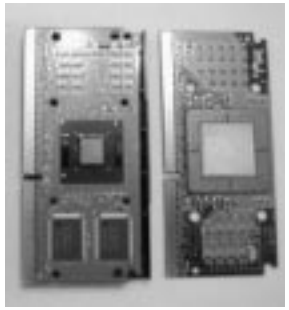
Claude Shannon
(1916 – 2001)

Architecture

Lecture A1 – A3.
- TOY machine.

Lecture A4 – A5.
- Digital circuits.

Lecture A6.
- Putting it all together.



Digital Circuits

What is a digital system?

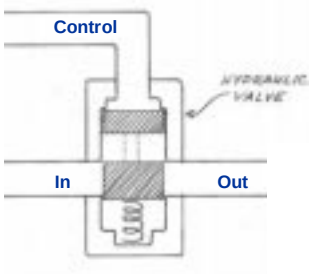
Why digital systems?

Basic abstraction.
- On, off.
- Switch that can turn something on or off.

Digital circuits and you.
- Computer microprocessors.
- Antilock brakes.
- VCR.
- Cell phone.

Abstract Switch

Abstract switch.
- Electrical (transistor).
- Electro-mechanical (relay).
- Hydraulic (water valve).

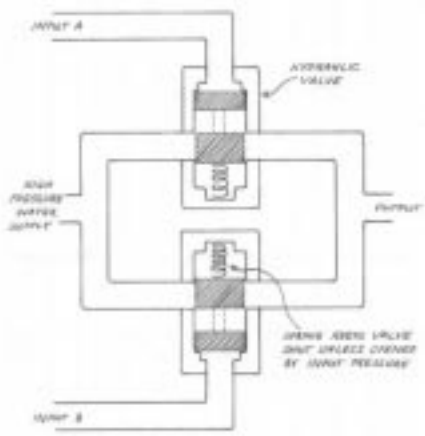


Hydraulic valve.
- 3 connections: input, output, control.
- Pressure on control pushes on a piston that turns on water flow from input to output.
  - valve is always entirely open or closed (digital)
  - amplification, restoring logic design
- Control pipe affects output pipe, but output does not affect control.
  - establishes forward flow of information over time

Computer

Hydraulic computer. (The Pattern on the Stone)

- Build computer by connecting hydraulic valves and pipe.



Hydraulic OR Gate

Computer

Electrical computer.

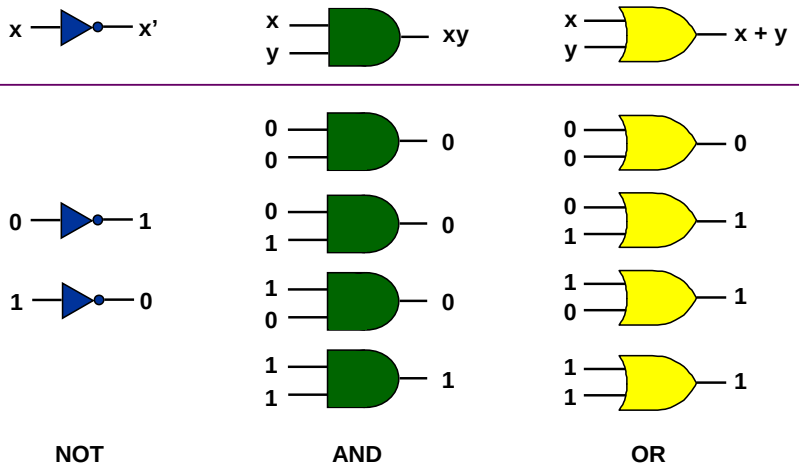
- Same fundamental abstraction.

	Hydraulic Computer	Electrical Computer
Connector	Pipe	Wire
Signal	Water pressure	Voltage
Switch	Hydraulic valve	Metal-oxide transistor

Logic Gates

Logical gates.

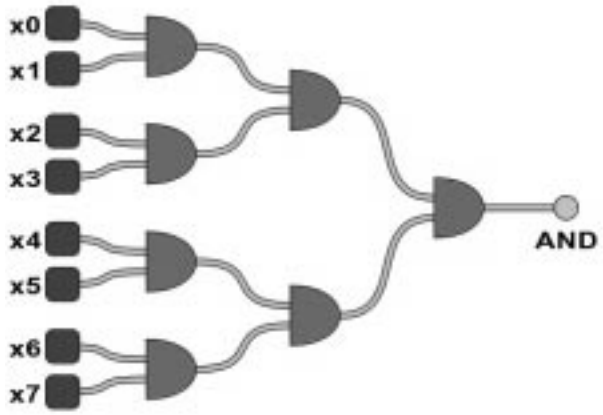
- Fundamental building blocks.



Multiway AND Gates

AND(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7).

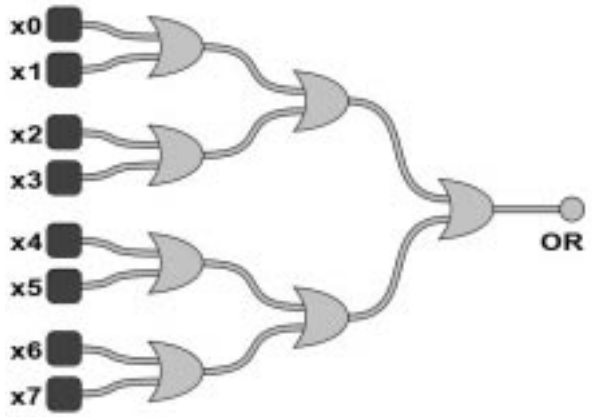
- 1 if all inputs are 1.
- 0 otherwise.



Multiway OR Gates

OR(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7).

- 1 if at least one input is 1.
- 0 otherwise.



Boolean Algebra

History.

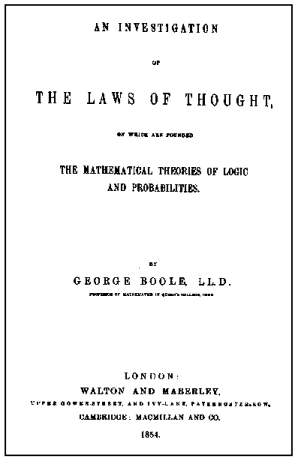
- Developed by Boole to solve mathematical logic problems (1847).
- Shannon first applied to digital circuits (1937).

Basics.

- Boolean variable: value is 0 or 1.
- Boolean function: function whose inputs and outputs are 0, 1.

Relationship to circuits.

- Boolean variables: signals.
- Boolean functions: circuits.

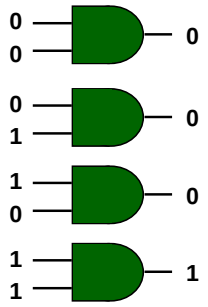

AN INVESTIGATION
OF
THE LAWS OF THOUGHT,
ON WHICH ARE FOUNDED
THE MATHEMATICAL THEORIES OF LOGIC
AND PROBABILITIES.
BY
GEORGE BOOLE, LL.D.
LONDON:
WALTON AND MABERLY,
CAMBRIDGE: MACMILLAN AND CO.
1854.

Truth Table

Truth table.

- Systematic method to describe Boolean function.
- One row for each possible input combination.
- N inputs $\Rightarrow$ 2^N rows.

AND Truth Table		
x	y	AND
0	0	0
0	1	0
1	0	0
1	1	1



AND

Truth Table

Truth table.

- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one

Truth Table for All Boolean Functions of 2 Variables									
x	y	ZERO	AND		X		Y	XOR	OR
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth Table for All Boolean Functions of 2 Variables									
x	y	NOR	EQ	Y'		X'		NAND	ONE
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth Table

Truth table.

- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one
- 256 Boolean functions of 3 variables.
  - every 8-bit value represents one
- 2^2^N Boolean functions of N variables!

Some Functions of 3 Variables							
x	y	z	AND	OR	MAJ	ODD	MUX
0	0	0	0	0	0	0	0
0	0	1	0	1	0	1	0
0	1	0	0	1	0	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	1	1
1	0	1	0	1	1	0	0
1	1	0	0	1	1	0	1
1	1	1	1	1	1	1	1

Universality of AND, OR, NOT

Any Boolean function can be expressed using AND, OR, NOT.

- "Universal."
- XOR(x,y) = xy' + x'y

Expressing XOR Using AND, OR, NOT							
x	y	x'	y'	x'y	xy'	x'y + xy'	XOR
0	0	1	1	0	0	0	0
0	1	1	0	1	0	1	1
1	0	0	1	0	1	1	1
1	1	0	0	0	0	0	0

Exercise: {AND, NOT}, {OR, NOT}, {NAND}, {AND, XOR} are universal.

Sum-of-Products

Any Boolean function can be expressed using AND, OR, NOT.

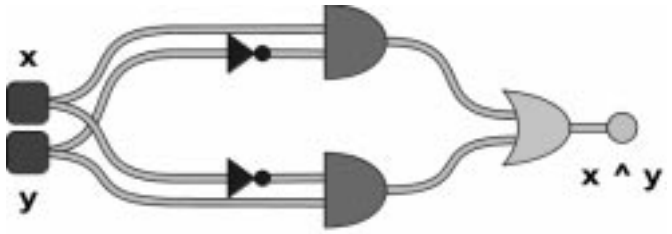
- Sum-of-products is systematic procedure.
  - form AND term for each 1 in Boolean function
  - OR terms together

Expressing MAJ Using Sum-of-Products								
x	y	z	MAJ	x'yz	xy'z	xyz'	xyz	x'yz + xy'z + xyz' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	0	1
1	1	0	1	0	0	1	0	1
1	1	1	1	0	0	0	1	1

Translate Boolean Formula to Boolean Circuit

Use sum-of-products form.

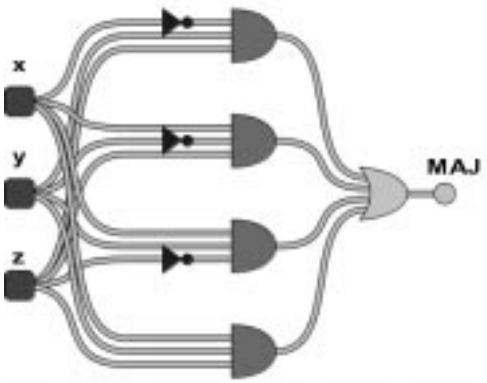
- XOR(x, y) = xy' + x'y.



Translate Boolean Formula to Boolean Circuit

Use sum-of-products form.

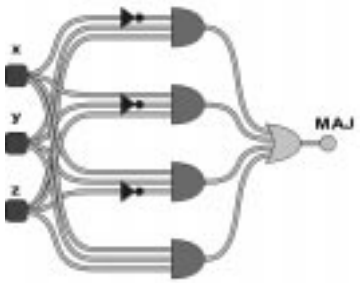
- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz.



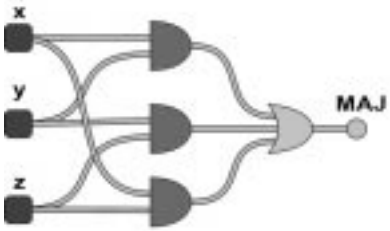
Simplification Using Boolean Algebra

Many possible circuits for each Boolean function.

- Sum-of-products not optimal.
- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz = xy + yz + xz.



size = 8, depth = 4



size = 4, depth = 3

Recipe for Making Combinational Circuit

Ingredients.

- AND gates.
- OR gates.
- NOT gates.
- Wire.

White Chocolate Mousse Cake, Balducci's

Instructions.

- Step 1: represent input and output signals with Boolean variables.
- Step 2: construct truth table to carry out computation.
- Step 3: derive (simplified) Boolean expression using sum-of products.
- Step 4: transform Boolean expression into circuit.

ODD Parity Circuit

ODD(x, y, z).

- 1 if odd number of inputs are 1.
- 0 otherwise.

Expressing ODD Using Sum-of-Products								
x	y	z	ODD	x'y'z	x'yz'	xy'z'	xyz	x'y'z + x'yz' + xy'z' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0	1
0	1	0	1	0	1	0	0	1
0	1	1	0	0	0	0	0	0
1	0	0	1	0	0	1	0	1
1	0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	1	1	0	0	0	1	1

ODD Parity Circuit

ODD(x, y, z).

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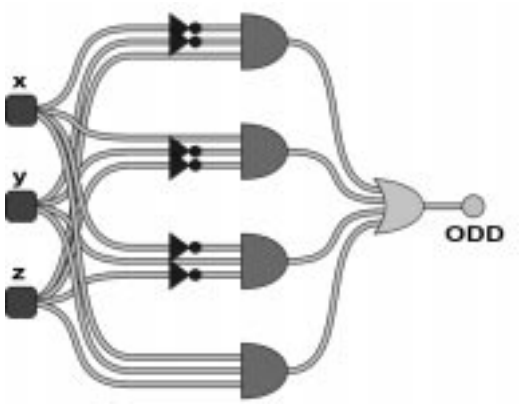


Let's Make an Adder Circuit

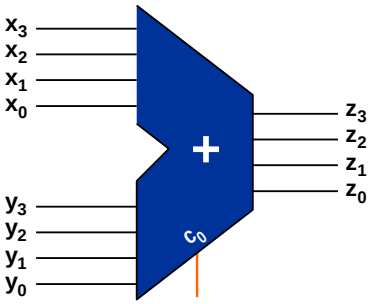
Goal: x + y = z.

- We build 4-bit adder: 8 inputs, 4 outputs. (Same idea scales to 128-bit adder.)
- Key computer component.

	1	1	1	0
	2	4	8	7
+	3	5	7	9
	6	1	6	6

Step 1.

- Represent input and output in binary.



	1	1	0	0
	0	0	1	0
+	0	1	1	1
	1	0	0	1

	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Let's Make an Adder Circuit

Goal: x + y = z.

Step 2.

- Build truth table.
- Why is this a bad idea?

					c_0
	x_3	x_2	x_1	x_0	
+	y_3	y_2	y_1	y_0	
	z_3	z_2	z_1	z_0	

4-Bit Adder Truth Table

c_0	x_3	x_2	x_1	x_0	y_3	y_2	y_1	y_0	z_3	z_2	z_1	z_0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	1	1	0	0	1	1
0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	0	1	0	1	0	1	0	1
.	.	.	.	.	.	.	.	.	.	.	.	.
1	1	1	1	1	1	1	1	1	1	1	1	1

$2^{8+1} = 512$ rows!

Let's Make an Adder Circuit

Goal: x + y = z.

Step 3.

- Derive (simplified) Boolean expression.

					c_3	c_2	c_1	$c_0 = 0$
	x_3	x_2	x_1	x_0				
+	y_3	y_2	y_1	y_0				
	z_3	z_2	z_1	z_0				

Carry Bit

x_i	y_i	c_i	c_{i+1}	MAJ
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Summand Bit

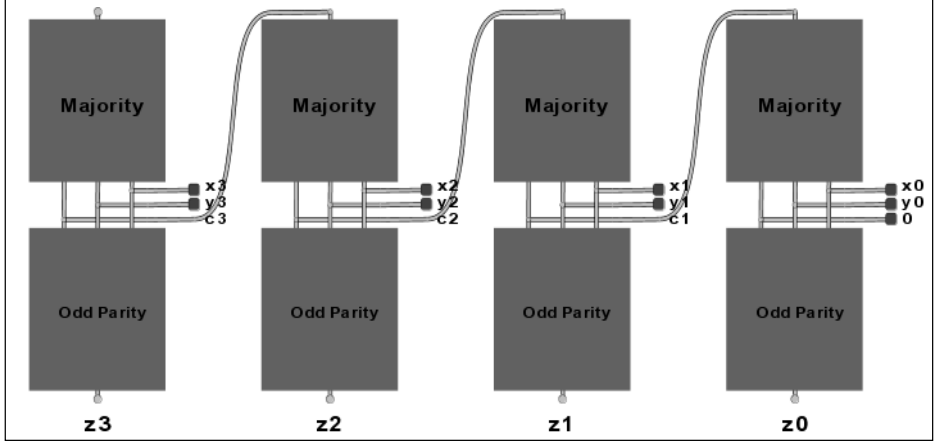
x_i	y_i	c_i	z_i	ODD
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

Let's Make an Adder Circuit

Goal: x + y = z.

Step 4.

- **Transform Boolean expression into circuit.**

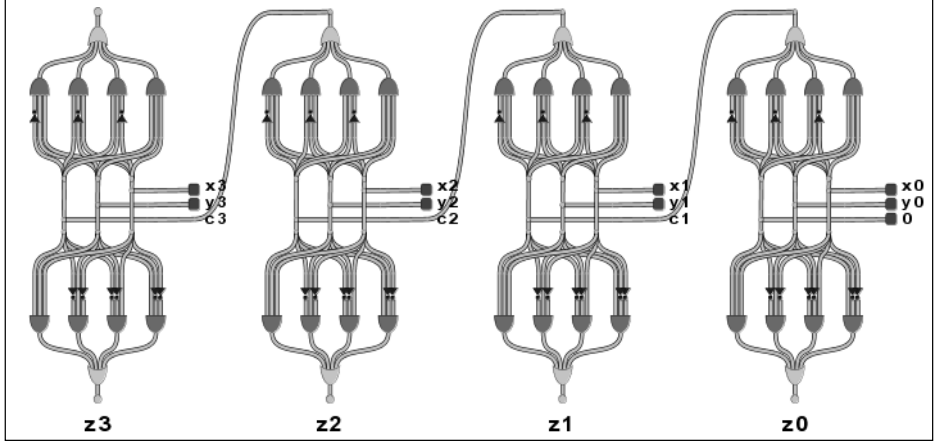


Let's Make an Adder Circuit

Goal: x + y = z.

Step 4.

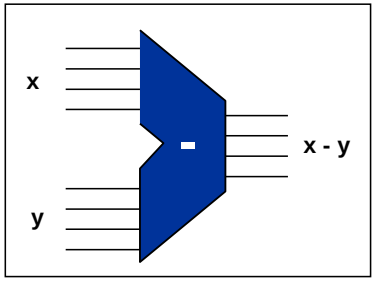
- **Transform Boolean expression into circuit.**



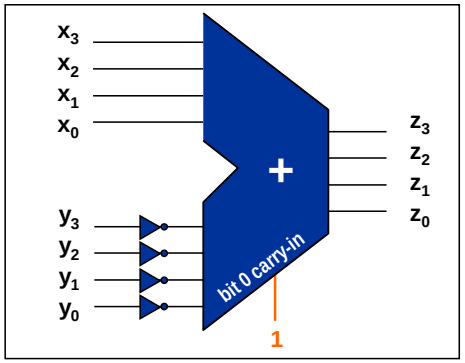
Subtractor

Subtractor circuit: z = x - y.

- **One approach: design like adder circuit.**
- **Better idea: reuse adder circuit.**
  - **2's complement: to negate an integer, flip bits, then add 1**



4-Bit Subtractor Interface

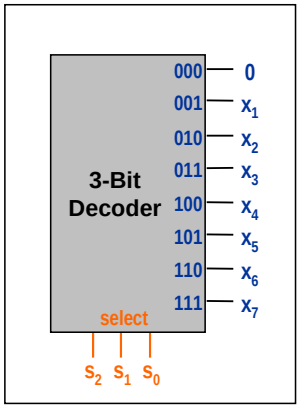


4-Bit Subtractor Implementation

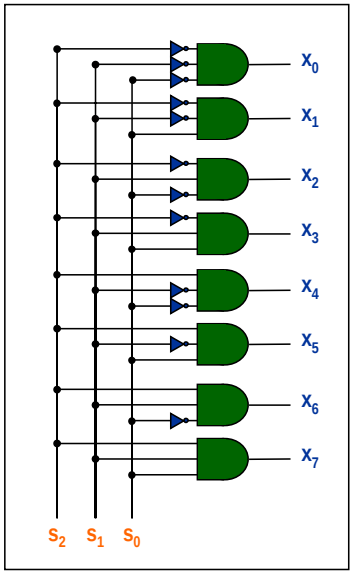
N-Bit Decoder

N-bit decoder.

- **N address inputs, 2^N data outputs.**
- **Addressed output bit is 1; all others are 0.**



3-Bit Decoder Interface

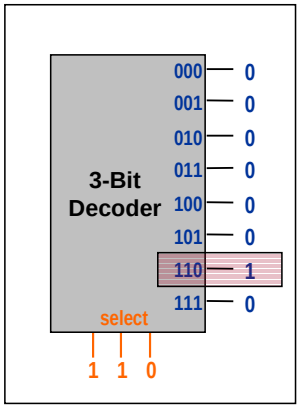


3-Bit Decoder Implementation

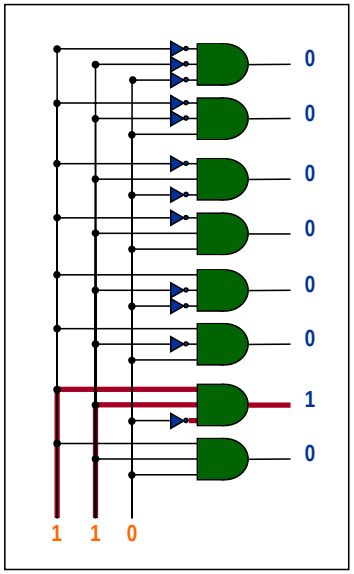
N-Bit Decoder

N-bit decoder.

- N address inputs, 2^N data outputs.
- Addressed output bit is 1; all others are 0.



3-Bit Decoder Interface

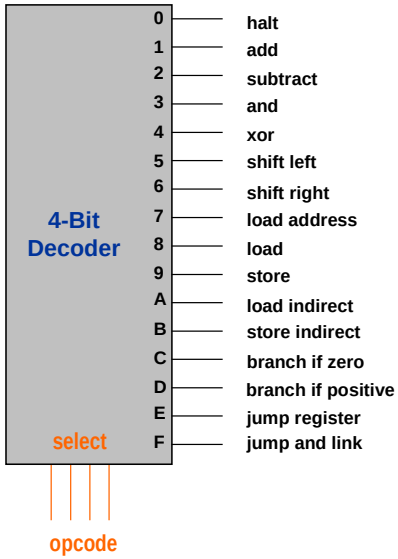


3-Bit Decoder Implementation

N-Bit Decoder

Application.

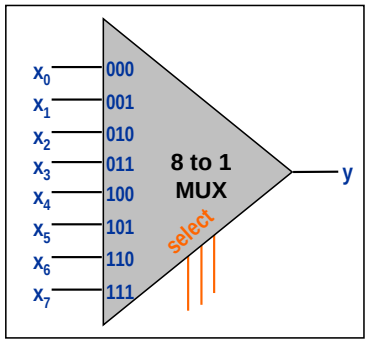
- Convert from binary to "unary."
- Decode opcode into instruction type.



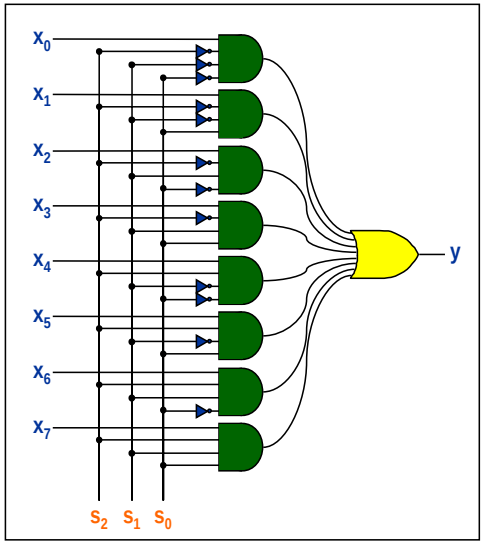
8-to-1 Multiplexer

2^N-to-1 multiplexer.

- N select inputs, 2^N data inputs, 1 output.
- Copies "selected" data input bit to output.



8-to-1 Mux Interface

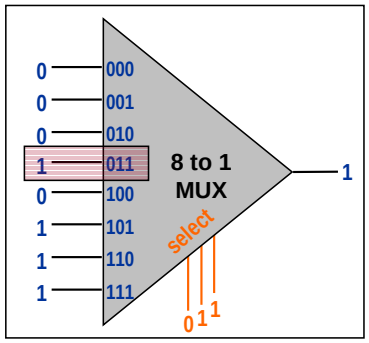


8-to-1 Mux Implementation

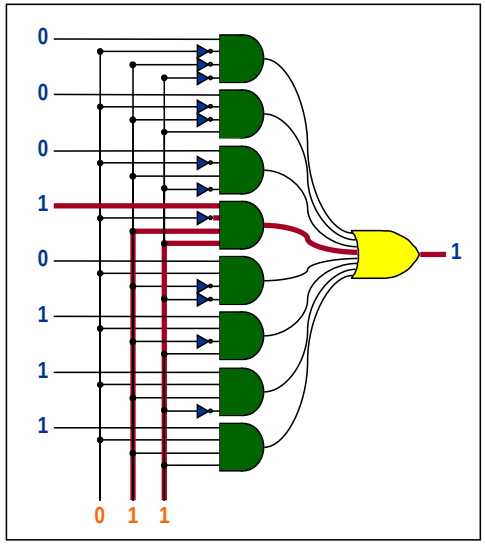
8-to-1 Multiplexer

2^N-to-1 multiplexer.

- N select inputs, 2^N data inputs, 1 output.
- Copies "selected" data input bit to output.



8-to-1 Mux Interface



8-to-1 Mux Implementation

Multiplexer: Application

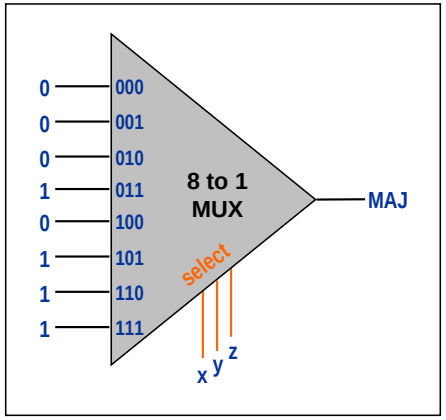
Use 2^N-to-1 mux to implement any Boolean function of N variables.

- N select inputs: Boolean variables.
- 2^N data inputs: truth table.

MAJ(x, y, z).

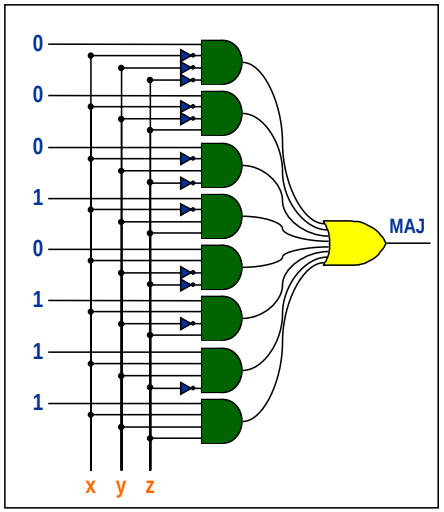
- 1 if at least two inputs are 1.
- 0 otherwise.

x	y	z	MAJ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1



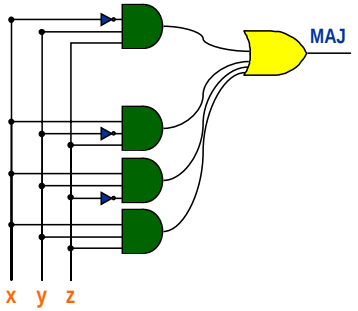
Majority Implementation

Sum-Of-Products Revisited



Majority Implementation with Mux

Simplify circuit by exploiting that data inputs are fixed.

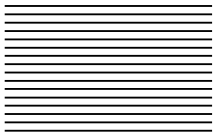


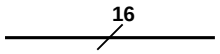
Sum of Products

Bus

16-bit bus.

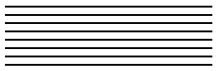
- Bundle of 16 wires.
- Memory transfer, register transfer.

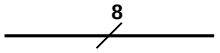




8-bit bus.

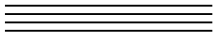
- Bundle of 8 wires.
- Memory address.

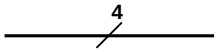




4-bit bus.

- Bundle of 4 wires.
- Register address.

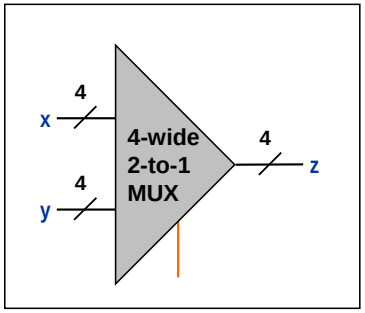




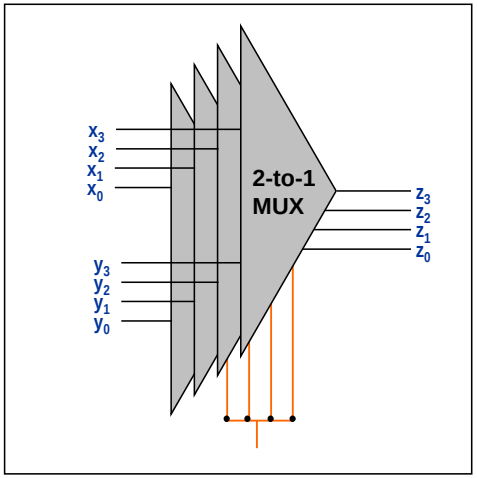
4-Wide 2-to-1 Multiplexer

Goal: select from one of two 4-bit buses.

- Implement by layering 4 2-to-1 multiplexers.



Interface

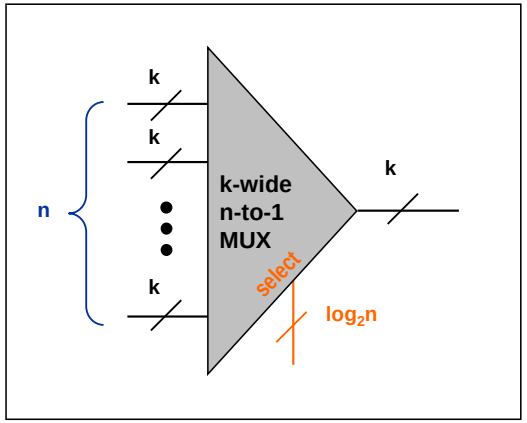


Implementation

k-Wide n-to-1 Multiplexer

Goal: select from one of n k-bit buses.

- Implement by layering k n-bit muxes.



Interface

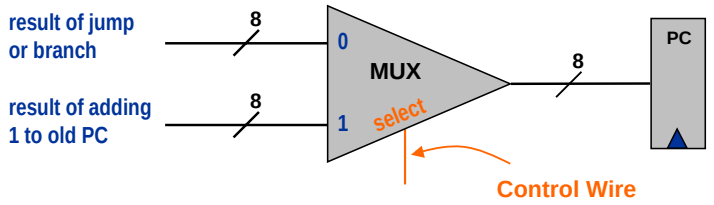
Multiplexer: Application

Program counter

- Result of jump or branch instruction.
- Adding 1 to old program counter.

Use 8-wide 2-to-1 mux to route appropriate 8-bit address to PC.

- Unspecified detail: how to set control wire?

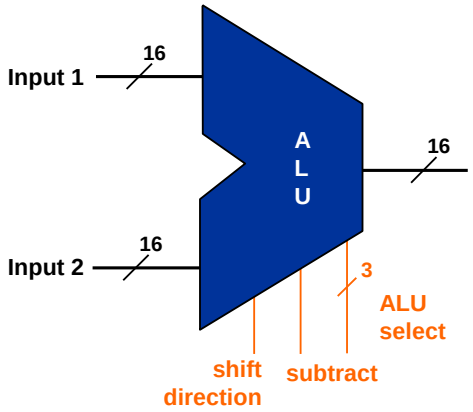


Arithmetic Logic Unit: Interface

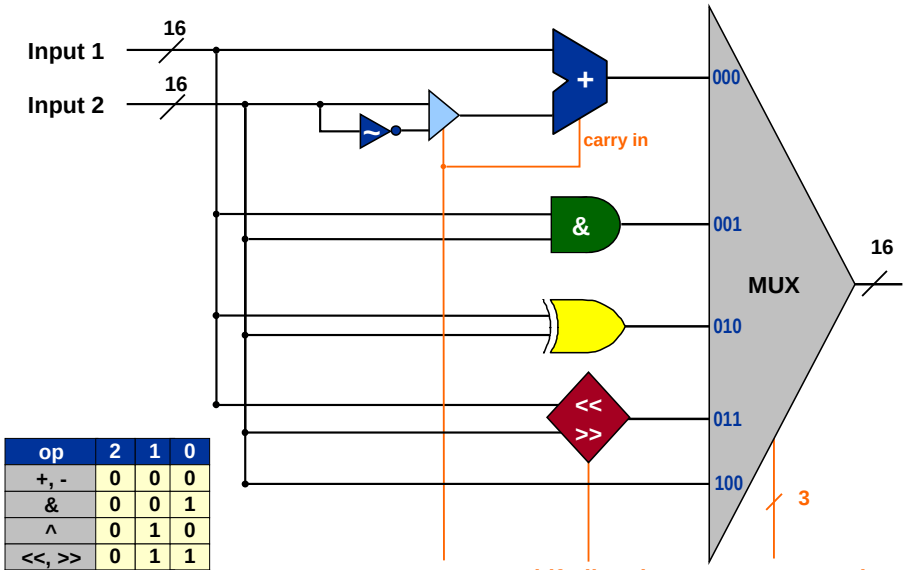
ALU Interface.

- Add, subtract, bitwise and, bitwise xor, shift left, shift right, copy.
- Associate 3-bit integer with 5 primary ALU operations.
  - ALU performs operations in parallel
  - control wires select which result ALU outputs

op	2	1	0
+, -	0	0	0
&	0	0	1
^	0	1	0
<<, >>	0	1	1
input 2	1	0	0



Arithmetic Logic Unit: Implementation

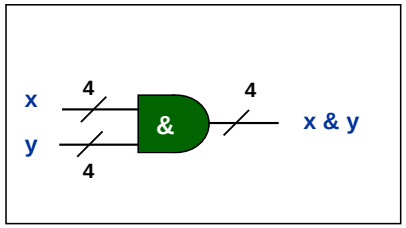


op	2	1	0
+, -	0	0	0
&	0	0	1
^	0	1	0
<<, >>	0	1	1
input 2	1	0	0

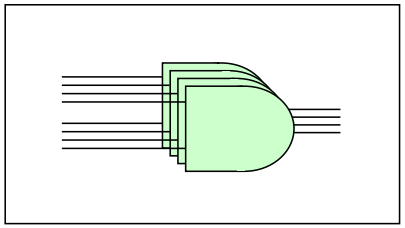
Bitwise AND, XOR, NOT

Bitwise logical operations.

- Inputs x and y: n-bits each.
- Output z: n-bits.
- Apply logical operation to each corresponding pair of bits.



Bitwise And Interface



Bitwise And Implementation

Abstraction and Encapsulation

Lessons for ADT apply to hardware!

- Interface describes behavior of circuit.
- Implementation gives details of how to build it.