COS 341, December 13, 2000

Handout Number 9

The following are last year's final exam problems and their solutions. Keep in mind that, in contrast to the current year, the exam last year was a take-home exam.

Last Year's Final Exam Problems

In the following, i, j, k, n take on integer values only.

Problem 1 [20 points] Let $b_0, b_1, b_2, \cdots$ be the sequence defined by the following recurrence relation:

$$\begin{aligned} b_0 &= 1, \\ b_1 &= 2, \\ b_n &= b_{n-1} + \sum_{1 \le k \le n-1} b_k b_{n-1-k} \quad \text{for } n \ge 2. \end{aligned}$$

Let $B(x) = \sum_{k \ge 0} b_k x^k$. Derive a closed-form formula for $B(x)$.

Problem 2 [20 points] For any integer $n > 0$, let $G_n = (V, E)$ be a graph on $2n$ vertices, where $V = \{1, 2, 3, \cdots, 2n\}$, and $E = \{\{i, j\} \mid 1 \le i < j \le 2n, j \ne n + i\}$. Answer the following questions, each with a concise but rigorous justification.

(a) For what values of n are G_n Eulerian?

(b) For what values of n do G_n contain a Hamiltonian circuit?

(c) What is $\omega(G_n)$, the size of the largest clique in G_n?

(d) What is $\chi(G_n)$, the chromatic number of G_n?

Remarks In other words, G_n is obtained from the complete graph on $2n$ vertices by deleting n edges (no two of which have any endpoints in common). Thus, G_n has exactly $\binom{2n}{2} - n$ edges.

Problem 3 [20 points] For any integer $n > 0$, let $H_n = (V, E)$ be a graph on $4n + 1$ vertices, where $V = \{1, 2, 3, \cdots, 4n, 4n + 1\}$, and

$$E = \{\{i, i+1\} \mid 1 \le i \le 4n-1\} \cup \{\{1, 4n\}, \{4n+1, n\}, \{4n+1, 2n\}, \{4n+1, 3n\}, \{4n+1, 4n\}\}.$$

Thus, H_n has exactly $4n + 4$ edges. Let s_n be the number of spanning trees for H_n. Determine s_n as a closed-form expression of n.

Solutions

Problem 1

$$\begin{aligned}\sum_{n\geq 2} b_n x^n &= \sum_{n\geq 2} b_{n-1}x^n + \sum_{n\geq 2} x^n \sum_{1\leq k\leq n-1} b_k b_{n-1-k} \\ &= x\sum_{n\geq 2} b_{n-1}x^{n-1} + x\sum_{m\geq 1} x^m \sum_{1\leq k\leq m} b_k b_{m-k}.\end{aligned}$$

This implies

$$B(x) - b_0 - b_1 x = x(B(x) - b_0) + x(b_1 x + b_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots),$$

ie,

$$B(x) - 1 - 2x = x(B(x) - 1) + x(B(x) - 1)B(x).$$

This leads to $xB(x)^2 - B(x) + (1+x) = 0$, and hence using $B(0) = b_0 = 1$ we have

$$B(x) = \frac{1 - \sqrt{1 - 4x(1+x)}}{2x}.$$

Problem 2

(a) For $n = 1$, G_n consists of two isolated vertices and is thus by definition Eulerian. For $n > 1$, G_n is Eulerian since (A) it is connected (vertex 1 is connected to vertex $n+1$ through $1 - 2 - (n+1)$, and vertex 1 has an edge to each of the remaining vertices) and (B) every vertex has even degree (in fact $2n-2$).

(b) For $n = 1$, G_n consists of two isolated vertices and has no Hamiltonian circuit. For $n > 1$, G_n has the following Hamiltonian circuit $1, 2, 3, \cdots, n-1, n, n+1, n+2, \cdots, 2n, 1$.

(c) The answer is $\omega(G_n) = n$. Note that $\omega(G_n) \geq n$ since $\{1, 2, \cdots, n\}$ is a clique; $\omega(G_n) < n+1$ since any clique can contain at most one of the vertices $i, n+i$ for each $1 \leq i \leq n$.

(d) The answer is $\chi(G_n) = n$. Note that $\chi(G_n) \geq n$ since $\{1, 2, \cdots, n\}$ is a clique and thus each vertex in it has to be painted with a different color; $\chi(G_n) \leq n$ since we can just paint both vertices $i, n+i$ with color i, for each $1 \leq i \leq n$.

Problem 3 Let $E_0 = \{\{4n+1, n\}, \{4n+1, 2n\}, \{4n+1, 3n\}, \{4n+1, 4n\}\}$, and

$$\begin{aligned} E_1 &= \{\{4n, 1\}, \{1, 2\}, \{2, 3\}, \cdots, \{n-1, n\}\}, \\ E_2 &= \{\{n, n+1\}, \{n+1, n+2\}, \{n+2, n+3\}, \cdots, \{2n-1, 2n\}\}, \\ E_3 &= \{\{2n, 2n+1\}, \{2n+1, 2n+2\}, \{2n+2, 2n+3\}, \cdots, \{3n-1, 3n\}\}, \\ E_4 &= \{\{3n, 3n+1\}, \{3n+1, 3n+2\}, \{3n+2, 3n+3\}, \cdots, \{4n-1, 4n\}\}. \end{aligned}$$

Then $E = \cup_{0 \le i \le 4} E_i$.

A spanning tree of H_n has $4n$ edges, and can be specified by the 4 edges missing from E. For $\alpha \in \{0,1,2,3,4\}$, let $s_{n,\alpha}$ be the number of spanning trees of H_n for which α of the missing edges are from E_0. Then

$$s_n = \sum_{0 \le \alpha \le 4} s_{n,\alpha}.$$

Clearly, $s_{n,4} = 0$ since at least one edge from E_0 is needed to keep vertex $4n+1$ from being isolated.

To calculate $s_{n,3}$, we count first how many spanning trees there are that contain $\{4n+1, n\}$ but no other edge from E_0. A spanning tree is now specified by the one missing edge from $\cup_{1 \le i \le 4} E_i$, so that number is $| \cup_{1 \le i \le 4} E_i| = 4n$. We can prove the same result if we count the number of spanning trees that contain any one specific edge but no other edges in E_0. Thus,

$$s_{n,3} = 4 \cdot 4n = 16n.$$

To calculate $s_{n,2}$, let a_n be the number of spanning trees containing $\{4n+1, n\}, \{4n+1, 2n\}$ but no other edges in E_0; let b_n be the number of spanning trees containing $\{4n+1, n\}, \{4n+1, 3n\}$ but no other edges in E_0. Clearly,

$$s_{n,2} = 4a_n + 2b_n.$$

We compute a_n. A spanning tree of this type is specified by a missing edge chosen from E_2, and a missing edge from $E_1 \cup E_3 \cup E_4$. Thus,

$$a_n = |E_2| \cdot |E_1 \cup E_3 \cup E_4| = 3n^2.$$

Similarly,

$$b_n = |E_2 \cup E_3| \cdot |E_1 \cup E_4| = 4n^2.$$

This leads to

$$s_{n,2} = 4 \cdot 3n^2 + 2 \cdot 4n^2 = 20n^2.$$

To calculate $s_{n,1}$, let c_n be the number of spanning trees containing $\{4n+1, n\}, \{4n+1, 2n\}, \{4n+1, 3n\}$ but no other edges in E_0. Then $s_{n,1} = 4c_n$. To compute c_n, note that such a spanning tree is specified by a missing edge from each of the sets $E_2, E_3, E_4 \cup E_1$. Thus, $c_n = |E_2| \cdot |E_3| \cdot |E_4 \cup E_1| = 2n^3$. Hence,

$$s_{n,1} = 4 \cdot 2n^3 = 8n^3.$$

To calculate $s_{n,0}$, note that such a spanning tree is specified by a missing edge from each of the sets E_1, E_2, E_3, E_4. Thus,

$$s_{n,0} = |E_1| \cdot |E_2| \cdot |E_3| \cdot |E_4| = n^4.$$

Putting everything together, we have

$$s_n = \sum_{0 \leq \alpha \leq 4} s_{n,\alpha} = n^4 + 8n^3 + 20n^2 + 16n.$$