COS 341, November 27, 2000
Handout Number 8

Prüfer's Algorithm

Theorem 1 (Cayley Formula) The number of trees on n vertices is equal to n^{n-2}.

This theorem has many different proofs. We present a proof due to Prüfer by producing a one-to-one correspondence between the set $\mathcal{T}_n$ of all n-vertex trees and the set $\mathcal{L}_n$ of all $(n-2)-tuples$ of integers in $\{1, 2, \cdots, n\}$. Prüfer gave the following algorithm which takes a tree $T = (\{1, 2, \cdots, n\}, E)$ as input, and outputs a sequence $r_1, r_2, \cdots, r_{n-2}$ where r_i are integers between 1 and n. Let $n > 2$. Prüfer's algorithm keeps two variables k and F, where k is an integer and F is a tree whose vertex set is a subset of $\{1, 2, \cdots, n\}$.

Set $k \leftarrow 1$; $F \leftarrow T$;
Repeat the following loop while F has more than 2 vertices:
    Let i be the smallest leaf in F (ie, vertex of degree 1),
        and let $\{i, j\}$ be the unique edge in F incident to i;
    set $r_k \leftarrow j$ and $k \leftarrow k + 1$;
    delete the vertex i and the edge $\{i, j\}$ from F;

As an example, consider the tree $T = (V, E)$ where $V = \{1, 2, \cdots, 9\}$ and with edges $\{1, 9\}, \{2, 3\}, \{3, 6\}, \{3, 9\}, \{4, 8\}, \{5, 8\}, \{7, 8\}, \{8, 9\}$. Applying the algorithm, after the first loop we get $r_1 = 9$, since 1 is a leaf and $\{1, 9\} \in F$ ($F = T$ at that time), similarly after the second loop we get $r_2 = 3$ etc. Eventually, the sequence produced is $9, 3, 8, 8, 3, 9, 8$.

To show that Prüfer's algorithm produces a one-to-one correspondence between $\mathcal{T}_n$ and $\mathcal{L}_n$, it suffices to show that for any $(r_1, r_2, \cdots, r_{n-2}) \in \mathcal{L}_n$, there exists an input tree $T \in \mathcal{T}_n$ for which the algorithm outputs $(r_1, r_2, \cdots, r_{n-2})$. This proof will be left as a problem in Homework Set 9.