COS 341, November 6, 2000

Handout Number 5

Solving Recurrence with Generating Functions

The first problem is to solve the recurrence relation system $a_0 = 1$, and $a_n = a_{n-1} + n$ for $n \geq 1$.

Let $A(x) = \sum_{n\geq 0} a_n x^n$. Multiply both side of the recurrence by x_n and sum over $n \geq 1$. This gives

$$\sum_{n\geq 1} a_n x^n = x \sum_{n\geq 1} a_{n-1} x^{n-1} + \sum_{n\geq 1} n x^n.$$

Note that

$$\begin{aligned}\sum_{n\geq 1} n x^n &= \sum_{n\geq 0} n x^n \\ &= x \frac{d}{dx} \left(\sum_{n\geq 0} x^n\right) \\ &= x \frac{d}{dx} \frac{1}{1-x} \\ &= x \frac{1}{(1-x)^2}.\end{aligned}$$

Thus, in term of $A(x)$, we obtain

$$A(x) - 1 = xA(x) + \frac{x}{(1-x)^2}.$$

Rearranging terms, we get

$$(1-x)A(x) = 1 + \frac{x}{(1-x)^2}.$$

Hence,

$$A(x) = \frac{1}{1-x} + x\dot{(}1-x)^{-3}.$$

We can now get a_n by expanding $A(x)$ as a series

$$A(x) = \sum_{n\geq 0} x^n + x \sum_{n\geq 0} \binom{-3}{n} (-1)^n x^n.$$

This gives, for all $n \geq 0$,

$$a_n = 1 + \binom{-3}{n-1} (-1)^{n-1}.$$

Now

$$\begin{aligned}\binom{-3}{k} &= \frac{(-3)(-4)\cdots(-3-k+1)}{k!} \\ &= \frac{(-1)^k(k+2)(k+1)}{2} \\ &= (-1)^k\binom{k+2}{2}.\end{aligned}$$

Thus,

$$a_n = 1 + \binom{n+1}{2}.$$

This is the same answer as we obtained earlier by different means.

The next problem for solution is the *Rabbit Island* problem. Before studying it, let us note the following identity, valid for any distinct numbers b and c:

$$\frac{1}{(1-bx)(1-cx)} = \frac{1}{b-c}\left(\frac{b}{1-bx} - \frac{c}{1-cx}\right). \tag{1}$$

It can be directly verified by taking common denominators of the terms on the right-hand-side, and simplyfing the expression. A more systematic way to do this is to solve the system of equations for variables λ, μ,

$$\lambda + \mu = 1, \quad \lambda b + \mu c = 0.$$

The solution satisfyies the equation

$$1 = \lambda(1-bx) + \mu(1-cx),$$

and gives

$$\begin{aligned}\frac{1}{(1-bx)(1-cx)} &= \frac{\lambda(1-bx)+\mu(1-cx)}{(1-bx)(1-cx)} \\ &= \frac{\lambda}{1-cx} + \frac{\mu}{1-bx}.\end{aligned}$$

This is a special case of the *partial fraction decomposition.* You might find it challenging to extend the discussion to show that, if b, c, d are distinct,

$$\frac{1}{(1-bx)(1-cx)(1-dx)} = \frac{\lambda}{1-dx} + \frac{\mu}{1-cx} + \frac{\gamma}{1-bx},$$

with sosme appropriate choice of λ, μ, γ.

In the Rabbit Island problem, we need to solve the recurrence $a_0 = 0, a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Let $A(x) = \sum_{n\geq 0} a_n x^n$. As in the previous problem, let us multiply the recurrence by x^n and sum over $n \geq 2$. This gives

$$\sum_{n\geq 2} a_n x^n = x\sum_{n\geq 2} a_{n-1}x^{n-1} + x^2\sum_{n\geq 2} a_{n-2}x^{n-2}.$$

In terms of $A(x)$, we have $A(x) - a_0 - a_1 x = x(A(x) - a_0) + x^2 A(x)$. This leads to

$$A(x) = \frac{x}{1 - x - x^2}. \tag{2}$$

It remains to expand $A(x)$ into a power series, so that we can identify a_n.

Now note that

$$\begin{aligned}
1 - x - x^2 &= 1 - x + \frac{x^2}{4} - \frac{5x^2}{4} \\
&= \left(1 - \frac{x}{2}\right)^2 - \left(\frac{\sqrt{5}x}{2}\right)^2 \\
&= \left(1 - \frac{x}{2} - \frac{\sqrt{5}x}{2}\right)\left(1 - \frac{x}{2} + \frac{\sqrt{5}x}{2}\right) \\
&= (1 - bx)(1 - cx),
\end{aligned}$$

where $b = (1 + \sqrt{5})/2$ and $c = (1 - \sqrt{5})/2$. Using (1) and (2), we can expand $A(x)$ as

$$\begin{aligned}
A(x) &= \frac{x}{(1 - bx)(1 - cx)} \\
&= \frac{bx}{b - c}\frac{1}{1 - bx} - \frac{cx}{b - c}\frac{1}{1 - cx} \\
&= \frac{bx}{b - c}\sum_{n \geq 0}(bx)^n - \frac{cx}{b - c}\sum_{n \geq 0}(cx)^n \\
&= \frac{1}{\sqrt{5}}\sum_{n \geq 0}(b^{n+1} - c^{n+1})x^{n+1} \\
&= \frac{1}{\sqrt{5}}\sum_{n \geq 1}(b^n - c^n)x^n.
\end{aligned}$$

Thus, for all $n \geq 1$, we have

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right).$$

For $n = 1$, this formula gives $a_1 = 1$, as was to be expected. Incidentally, the above formula also gives the correct value $a_0 = 0$.

The numbers a_n are called *Fibonacci numbers*, and often denoted by F_n. Note that $b = 1.6\cdots$ and $c = -0.6\cdots$. Thus, c^n is numerically a very small number, while b^n is large. For reasonably large n, say $n > 10$, F_n can be obtained by evaluating $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$, and rounding it to the closest integer.

The third problem we tackle is the recurrence $a_0 = 0$, $a_1 = 1$, and $a_n = \sum_{1 \leq i \leq n-1} a_i a_{n-i}$ for $n \geq 2$. The quantity a_n is the number of ways to parenthesize an expression $y_1 + y_2 + \cdots + y_n$.

Let $A(x) = \sum_{n\geq 0} a_n x^n = \sum_{n\geq 1} a_n x^n$. The recurrence relation gives

$$\begin{aligned}
\sum_{n\geq 2} a_n x^n &= \sum_{n\geq 2} \sum_{1\leq i\leq n-1} a_i x^i a_{n-i} x^{n-i} \\
&= (\sum_{i\geq 1} a_i x^i)(\sum_{j\geq 1} a_j x^j) \\
&= (A(x))^2.
\end{aligned}$$

This means $A(x) - x = (A(x))^2$, and hence $A(x)^2 - A(x) + x = 0$. Solving the quadratic equation for $A(x)$, we obtain two possible solutions: $A(x) = (1+\sqrt{1-4x})/2$ and $A(x) = (1-\sqrt{1-4x})/2$. The former solution can be discarded, since it would give $a_0 = A(0) = 1$, which contradicts our assumption $a_0 = 0$. Thus,

$$\begin{aligned}
A(x) &= \frac{1-\sqrt{1-4x}}{2} \\
&= \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2} \\
&= \frac{1}{2} - \frac{1}{2}\sum_{n\geq 0}\binom{1/2}{n}(-4x)^n.
\end{aligned}$$

We infer from it $a_0 = 0$, and for $n \geq 1$,

$$a_n = -\frac{1}{2}\binom{1/2}{n}(-4)^n.$$

Note that, for $n \geq 2$

$$\begin{aligned}
\binom{1/2}{n} &= \frac{1}{n!}\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-3\right)\cdots\left(\frac{1}{2}-(n-1)\right) \\
&= \frac{1}{n!}\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right) \\
&= \frac{1}{2^n}\frac{1}{n!}(-1)^{n-1}1\cdot 3\cdots(2n-3) \\
&= \frac{1}{2^n}\frac{1}{n!}(-1)^{n-1}\frac{1\cdot 2\cdot 3\cdots(2n-2)}{2\cdot 4\cdots(2n-2)} \\
&= \frac{1}{2^n}\frac{1}{n!}(-1)^{n-1}\frac{(2n-2)!}{2^{n-1}(n-1)!} \\
&= (-1)^{n-1}\frac{2}{4^n}\frac{1}{n}\binom{2n-2}{n-1}.
\end{aligned}$$

This leads to

$$a_n = -\frac{1}{2}\binom{1/2}{n}(-4)^n$$

$$
\begin{aligned}
&= -\frac{1}{2}(-4)^n \cdot (-1)^{n-1} \frac{2}{4^n} \frac{1}{n} \binom{2n-2}{n-1} \\
&= \frac{1}{n} \binom{2n-2}{n-1},
\end{aligned}
$$

for $n \geq 2$. The above formula also holds for $n = 1$ since both sides are equal to 1. (Recall $\binom{0}{0} = \frac{0!}{0!0!} = 1$.) The numbers a_n are often called the *Catalan numbers*.