COS 341, October 9, 2000
Handout Number 3

Math Casino

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Let $N = 10^4$. Roulettes in the Math Casino have the following rule. For each run, a random integer n, $1 \le n \le N$ is uniformly generated. If $\lfloor n^{1/4} \rfloor$ divides n, you win \$5; otherwise, the \$1 bet you place is forfeited. Should you play the game? One way to make a rational decision is to calculate the probability p of winning a game, and play the game if and only if $p \ge 1/6$. How can we compute p?

By definition $p = m/N$, where m is the number of integers $1 \le n \le 10^4$ that satisfy "$\lfloor n^{1/4} \rfloor$ divides n". Let A_j be the set of integers $1 \le n \le N$ satisfying $\lfloor n^{1/4} \rfloor = j$ (i.e., $j^4 \le n \le (j+1)^4 - 1$) and "j dividing n". Clearly,

$$m = \sum_{1 \le j \le 10} |A_j|.$$

By inspection, we have $A_{10} = \{N\}$ and hence $|A_{10}| = 1$. Let $1 \le j \le 9$. Starting at $n = j^4$, every j-th integer in the range $j^4 \le n \le (j+1)^4 - 1$ satisfies j dividing n. Hence,

$$\begin{aligned}
|A_j| &= \lceil \frac{(j+1)^4 - 1 - j^4 + 1}{j} \rceil \\
&= \lceil 4j^2 + 6j + 4 + \frac{1}{j} \rceil \\
&= 4j^2 + 6j + 4 + 1 \\
&= 4j^2 + 6j + 5.
\end{aligned}$$

This shows

$$\begin{aligned}
m &= 1 + \sum_{1 \le j \le 9} (4j^2 + 6j + 5) \\
&= 1 + 4 \sum_{1 \le j \le 9} j^2 + 6 \sum_{1 \le j \le 9} j + 9 \cdot 5 \\
&= 46 + 4 \sum_{1 \le j \le 9} j^2 + 6 \frac{1}{2} 9 \cdot 10 \\
&= 316 + 4 \sum_{1 \le j \le 9} j^2. \qquad (1)
\end{aligned}$$

It remains to evaluate $a \equiv \sum_{1 \le j \le 9} j^2$. Using the equality $\sum_{2 \le k \le s} \binom{k}{2} = \binom{s+1}{3}$, we have

$$\begin{aligned} a &= \sum_{1 \le j \le 9} j^2 \\ &= \sum_{1 \le j \le 9} (2\binom{j}{2} + j) \\ &= 2 \sum_{1 \le j \le 9} \binom{j}{2} + \frac{1}{2} 9 \cdot 10 \\ &= 2\binom{9+1}{3} + 45 \\ &= 285. \end{aligned}$$

From (1) we then have

$$m = 316 + 4 \cdot 285 = 1456.$$

Hence $p = m/N = 0.1456 < 1/6$, and you shouldn't play this game. This solves the Math Casino Problem.