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Lecture Beyond Planar Graphs

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1 Overview

In the last lecture we learned about approximation schemes in planar graphs.

In this lecture we go beyond planar graphs and consider bounded-genus graphs, and more generally, minor-closed graph classes. In particular, we discuss the framework of *bidimensionality* that can be used to obtain approximation schemes for many problems in minor-closed graph classes. Afterwards, we consider *decomposition techniques* that can be applied to other types of problems.

The main *motivation* behind this line of work is that some networks are not planar but conform to some generalization of the concept of planarity, like being embeddable on a surface of bounded genus. Furthermore, by Kuratowski's famous theorem [Kur30], planarity can be characterized be *excluding* K_5 and $K_{3,3}$ as *minors*. Hence, it seems natural to study further classes with excluded minors as a generalization of planar graphs.

One of the *goals* of this research is to identify the largest classes of graphs for which we can still obtain good and efficient approximations, i.e. to find out how far beyond planarity we can go. Natural candidate classes are graphs of bounded genus, graphs excluding a fixed minor, powers thereof, but also further generalizations, like odd-minor-free graphs, graphs of bounded expansion and nowhere dense classes of graphs. Another objective is to derive *general approximation frameworks*, two of which we are going to discuss here: bidimensionality and contraction decomposition.

The main tools we are going to use come on one hand from *graph structure theory* and on the other hand from *algorithmic results*. On the structural side, we use the theory of Graphs on Surfaces [MT01], Graph Minor Theory [RS04], and in particular, the existence of decompositions into simpler parts [RS03]. On the algorithms side, some of the major tools are Lipton and Tarjan's Planar Separator Theorem [LT79], the seminal decomposition method due to Baker [Bak94], ideas from fixed-parameter algorithms [DF99], and algorithms for graphs of bounded treewidth [AP89, Bod88, Cou90].

One of the main results we are going to see is a PTAS for the traveling salesman problem (TSP) in weighted graphs that exclude a fixed minor that was recently obtained by Demaine, Hajiaghayi, and Kawarabayashi [DHK11].

2 Preliminaries

We usually denote graphs by letters G, H, and refer to their vertex/edge sets by $V(G)$ and $E(G)$, respectively. Unless otherwise mentioned, our graphs have n vertices and m edges. For a subset $U \subseteq V(G)$, we write $G[U]$ to denote the subgraph of G induced by U. The *r-neighborhood* of a vertex v, denoted by $N_r(v)$, is the set of vertices at distance at most r from v; we define $N(v) = N_1(v)$.

We let $d_G(u,v)$ denote the distance between vertices $u, v \in V(G)$.

2.1 Tree Decompositions and Treewidth

A *tree decomposition* of a graph G is a pair $(T, \mathcal{B})$, where T is a tree and $\mathcal{B} = \{B_u | u \in V(T)\}$ is a family of subsets of $V(G)$, called *bags*, such that (i) every vertex of G appears in some bag of $\mathcal{B}$; (ii) for every edge $e = \{u, v\}$ of G, there is a bag of $\mathcal{B}$ containing $\{u, v\}$; and (iii) for every vertex $v \in V(G)$ the set of bags containing v forms a connected subtree T_v of T. The *width* of a tree decomposition is the maximum size of any bag in $\mathcal{B}$ minus 1. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width over all possible tree decompositions of G. The *adhesion* of a tree decomposition is defined as $\max\{|B_u \cap B_t| \mid \{u, t\} \in E_T\}$. Many NP-hard optimization problems become fixed-parameter tractable when parameterized by the treewidth of the instance, by using dynamic programming on a given tree decomposition. The most well-known result in this area is Courcelle's theorem [Cou90] stating that any problem definable in monadic second-order logic is in FPT when parameterized by the treewidth and the length of the formula.

If T is a path, we call the resulting decomposition a *path decomposition.*

2.2 Minors, Models, Odd Minors

For an edge $e = uv$ in G, we define the operation G/e of *contracting* e as identifying u and v and removing all loops and duplicate edges. A graph H is a *minor* of G, written as $H \preceq G$, if it can be obtained from G by a series of vertex and edge deletions and contractions. We say G is an *H-minor-free graph* if it does not contain H as a minor. A class of graphs that is closed under building minors and does not contain all graphs is called a *proper minor-closed class of graphs.* A class of graphs is a proper minor-closed class if and only if it is H-minor-free for some fixed H. Examples of such classes include planar graphs, bounded-genus graphs, and linklessly embeddable graphs. An *apex-graph* is a planar graph augmented by an additional vertex that can have edges to any other vertex. A class of graphs is called *apex-minor-free* if it excludes a fixed apex-graph as a minor. It is a well-known fact that H-minor-free graphs have bounded average degree (depending only on $|H|$), i.e. they fulfill $m = \mathcal{O}_H(n)$; we use the notation $\mathcal{O}_H$ to denote that the constants hidden in the big-$\mathcal{O}$ depend on $|H|$.[1]

A *model* of H in G is a map that assigns to every vertex v of H, a connected subtree T_v of G such that the images of the vertices of H are all disjoint in G and there is an edge between them if there is an edge between the corresponding vertices in H. A graph H is a minor of G if and only if G contains a model of H. Now H is an *odd-minor* of G if additionally the vertices of the trees in the model of H in G can be 2-colored in such a way that (i) the edges of each tree T_v are bichormatic; and (ii) every edge e_G in G that connects two trees T_u and T_v and corresponds to an edge $e_H = uv$ of H is monochromatic. A graph is *odd-H-minor-free* if it excludes H as an odd minor. For example, bipartite graphs are odd-K_3-minor-free.

[1]this is necessary since in graph minor theory, the exact dependence is often not known.

2.3 Almost-embeddable graphs and the graph minor decomposition theorem

Let h be a constant and let S be a surface with $r \leq h$ boundary cycles $C_1, \dots, C_r$. A graph G is said to be *h-almost embeddable* in S if there exists a set $A \subseteq V(G)$ called *apices* with $|A| \leq h$ such that $G - A$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_r$ so that the following properties hold:

- G_0 has an embedding Π in S,
- $G_1, \dots, G_r$ are pairwise disjoint and called *vortices*,
- for $1 \leq i \leq r$, G_i has a path decomposition $(\{1, \dots, m_i\}, (B^i_j)_{1 \leq j \leq m_i})$ of width at most h,
- for $1 \leq i \leq r$ there are vertices $x^i_1, \dots, x^i_{m_i}$ such that $x^i_j \in B^i_j$ for $1 \leq j \leq m_i$ and $V(G_0) \cap V(G_i) = \{x^i_1, \dots, x^i_{m_i}\}$; $x^i_1, \dots, x^i_{m_i}$ are called *society vertices* of the vortex G_i,
- for $1 \leq i \leq r$ we have $\Pi(V(G_0)) \cap C_i = \{\Pi(x^i_1), \dots, \Pi(x^i_{m_i})\}$ and the points $\Pi(x^i_1), \dots, \Pi(x^i_{m_i})$ appear on C_i in this order (either clockwise or counter-clockwise).

Note that the vortices may contain internal vertices that do not appear in G_0.

For two graphs G_1 and G_2 whose intersection $E(G_1) \cap E(G_2)$ induces a h-clique, we define their *h-clique-sum* $G_1 \oplus G_2$ as the graph $G_1 \cup G_2$ with any number of edges in the clique $E(G_1) \cap E(G_2)$ deleted. Note that this operation is not well-defined and can have a number of possible outcomes.

The corner-stone structure theorem of Robertson and Seymour's Graph Minor Theory [RS03] is the following:

Theorem 1 ([RS03]). *Any H-minor-free graph can be written as the h-clique-sum of h-almost-embeddable graphs where h is a constant depending only on H.*

Another way to look at this theorem is to say that any H-minor-free graph admits a tree decomposition of adhesion at most h in which every bag is an h-almost-embeddable graph. This theorem has later been made algorithmic [DHK05, KW11].

3 Bidimensionality

In this section, we introduce bidimensionality for graph parameters [DFHT05b]. Roughly speaking, a graph parameter is bidimensional if it does not increase when performing certain operations, and it is large on specified grid-like graphs. More precisely, there are two types of bidimensionality, which we need to define: A graph parameter P is *$g(r)$-minor-bidimensional* (or just bidimensional) if it never increases under taking minors, and it is at least $g(r)$ on the $(r \times r)$-grid. The parameter P is *$g(r)$-contraction-bidimensional* if it never increases when contracting edges and it is at least $g(r)$ on *grid-like graphs* (see slides and [FGT09]).

To illustrate these definitions, let us look at some examples. A very easy graph parameter is the number of vertices of a graph, which is obviously r^2-minor-bidimensional. An easy example of a graph parameter that is contraction-decreasing but not minor-decreasing is the length of the longest cycle in a graph. This parameter is r^2-contraction-bidimensional. Likewise, the size of a dominating set (a subset of vertices such that each vertex is either in or adjacent to this subset) is contraction-bidimensional, with $g(t) = \theta(r^2)$, but not minor-decreasing. More examples of bidimensional parameters include the size of a vertex cover and the size of a feedback vertex set (see slides). These are both minor-bidimensional and contraction-bidimensional with $g(r) = \theta(r^2)$.

Theorem 2. *If a parameter k is bidimensional, then it satisfies a parameter-treewidth bound of the form* $\mathrm{tw}(G) = \mathcal{O}_H(\sqrt{k})$ *on any H-minor-free class of graphs. If it is contraction bidimensional, the same is true on all apex-minor-free classes of graphs.*

Proof Sketch. This follows immediately from another theorem of Demaine and Hajiaghayi [DH05b] that states that any H-minor-graph of treewidth w contains an $\Omega(w) \times \Omega(w)$-grid as a minor. □

3.1 Obtaining efficient approximation schemes (EPTAS)

Since Lipton and Tarjan's [LT79] separator theorem for planar graphs in 1979, several PTASes for optimization problems on planar graphs and their generalizations have been devised. They were either based on this or consequent separator theorems or on another seminal framework introduced by Baker [Bak94] in 1994 using layerwise decomposition. The bidimensionality theory captures many of these results and generalizes each of these methods in the following way:

The separator approach is based on finding small separators in the input graph, solving the problem on the resulting smaller graphs recursively, and merging the computed solutions. The size of the separator plays an important role in this process and has been usually bounded in terms of the size of the input graph. Using the parameter-treewidth bound, one can find small separators in terms of the *solution size* and this boosts the power of this approach by much. The idea is to find a tree decomposition with treewidth bounded in the size of the parameter and choose the most balanced cut that it provides. Since the treewidth is bounded by the size of the parameter, so is the size of the derived cut. Demaine and Hajiaghayi obtain the following result:

Theorem 3 ([DH05a])**.** *Consider a $\theta(r^2)$-minor-bidimensional problem that satisfies a certain separation property described below and that can be solved in time $h(\mathrm{tw}(G))n^{O(1)}$. Then the problem admits an EPTAS with running time $h(O(1/\varepsilon))n^{O(1)}$ on all H-minor-free graphs. The same results holds for $\theta(r^2)$-contraction-bidimensional problems on apex-minor-free graphs.*

The required separation property is somewhat technical and differs slightly for minor-bidimensional and contraction-bidimensional parameters but is roughly as follows:

- The solution on disconnected graphs is the union of solutions of each connected component.
- Given a solution to $G - C$, one can compute a solution to G at an additional cost of $\pm O(|C|)$.
- A solution S of G induces on a connected component X of $G - C$ a solution with size $|S \cap X| \pm O(|C|)$.

This results in PTASes in H-minor free graphs for vertex cover, face cover, minimum maximal matching and feedback vertex set, among others. On apex-minor-free graphs one obtains PTASes for various kinds of dominating set problems. Many of these were formerly known on planar graphs but, for example, a PTAS for feedback vertex set was not even known for planar graphs before.

3.2 Discussion

Very recently, Fomin et al. [FLRS11] revised and simplified the framework for obtaining EPTASes via bidimensionality.

Many contraction-bidimensional problems such as dominating set tend to still admit a PTAS beyond apex-minor-free graphs [DFHT05b, FGT09]. Hence, it is conjectured that they might have PTASes on all H-minor-free graphs. Another conjecture is that bidimensional problems admit PTASes even on *fixed powers* of H-minor-free graphs [DFHT05a].

Further directions to consider include:

- *Nontrivial weights*, such as finding k disjoint paths of minimum weight;
- *Directed graphs*; here a useful notion of directed treewidth is still lacking;
- *Subset type problems*. Using completely different techniques, PTASes have been obtained for subset-type problem such as Steiner tree and subset TSP in planar [BKM09] and bounded-genus [BDT09] graphs as well as recently for Steiner Forest [BHM10]. It is an important problem to determine wheather these PTASes can be generalized to H-minor-free graphs and also to see if there is some unifying framework behind them.

4 Decomposition Techniques

Another approach for obtaining approximation algorithms and schemes is to look for *simplifying decompositions* instead of small separator decompositions. For example, consider the following result:

Theorem 4 ([DHK10]). *Any odd-H-minor-free graph can be decomposed into two pieces, such that each one is of bounded treewidth.*

This theorem immediately results in 2-approximations for a large number of problems, e.g. chromatic number: simply find the optimal coloring of each of the two parts using a disjoint set of colors. Note that this problem is not approximable withing $n^{1-\varepsilon}$ in general graphs unless ZPP = NP.

4.1 Deletion decomposition view on Baker's approach

In order to obtain an approximation scheme, we need decompositions beyond just two parts. In 1994, Baker [Bak94] introduced a framework that essentially reduced the problem of finding an approximation scheme for certain types of problems to that being able to solve them efficiently on graphs of bounded treewidth. This approach was later generalized to apex-minor-free [Epp00] and H-minor-free graphs [Gro03]. Demaine et al. [DHK05] reinterpreted this idea in a slightly weaker but considerably simple fashion as follows:

Theorem 5 ([DHK05]). *For every graph H there is a constant c_H such that for any integer $p \geq 1$ and for every H-minor-free graph G, the vertices (edges) of G can be partitioned into p sets such that any $p-1$ of the sets induce a graph of treewidth at most $c_H p$. Furthermore, such a partition can be found in polynomial time.*

This gives rise to PTASes for H-minor-free graphs for several NP-complete problems, such as vertex cover, independent set, minimum color sum, maximum P-matching and max-cut. The basic idea is as follows: decompose the graph into $\mathcal{O}(1/\varepsilon)$ pieces such that deleting any one of them results in a graph of bounded treewidth. For each i, compute the optimum solution in the union of all pieces except piece i and return the minimum solution found. Since one of the pieces is guaranteed to contain only a small fraction of the optimum, the resulting solution will be nearly optimal. Tazari [Taz10] improved the running time of this algorithm, making the exponent of n independent of H.

4.2 Contraction Decomposition

As discussed earlier, many problems are closed under contractions but not under deletion. Hence, it would be desirable to have a similar theorem as above for the case of contraction instead of deletion. Indeed, this is possible, as was shown for planar graphs [Kle08], bounded-genus graphs [DHM07], apex-minor-free graphs [DHK09], and H-minor-free graphs [DHK11].

Theorem 6 ([DHK11])**.** *For every graph H there is a constant c_H such that for any integer $p \geq 1$ and for every H-minor-free graph G, the edges of G can be partitioned into $p+1$ sets such that* contracting *any one of the sets induce a graph of treewidth at most $c_H p$. Furthermore, such a partition can be found in polynomial time.*

Using this theorem and similar (but somewhat different) arguments as above, one can derive the following framework for obtaining PTASes on H-minor-free graphs:

Theorem 7. *Consider a minimization problem P on weighted graphs that is closed under contractions, solvable in polynomial time on graphs of bounded treewidth, and satisfying the following properties:*

1. *There is a polynomial-time algorithm that, given a weighted H-minor-free graph (G, w) and constant $\delta > 0$, computes an H-minor-free graph G' such that $\mathrm{OPT}(G') \geq \alpha w(G')$, for some constant $\alpha > 0$ (possibly depending on δ), and any c-approximate solution to G' can be converted into a $(1+\delta)c$-approximate solution to G in polynomial time. (G' is called a* (δ, α)-spanner *of G.)*
2. *There is a polynomial-time algorithm that, given a subsets S of edges of a weighted graph (G, w), and given an optimal solution for G/S, constructs a solution for G of value at most* $\mathrm{OPT}(G/S) + \beta w(S)$ *for some constant $\beta > 0$.*

For any fixed minor H, and for any fixed $0 < \varepsilon \leq 1$, there is a polynomial-time $(1+\varepsilon)$-approximation algorithm for problem P in H-minor-free graphs. Furthermore, if α grows as a function of n, then the running time becomes bounded by a polynomial times the cost of solving the problem on graphs of treewidth $\mathcal{O}(\alpha)$.

4.3 A PTAS for TSP in weighted H-minor-free graphs

The TSP is a testbed for many algorithmic ideas and indeed the first problem with a global connectivity requirement shown to admit a PTAS on unweighted planar graphs [GKP95]. Later PTASes were found on weighted planar graphs [AGK$^+$98, Kle08], weighted bounded-genus graphs [DHM07], unweighted apex-minor-free graphs [DHK09], and finally, weighted H-minor-free graphs [DHK11]. The latter result is obtained via an application of the generic theorem above and the following two observations:

1. existence of a polynomial time algorithm for TSP in bounded treewidth graphs with *singly exponential* dependence on the treewidth [DFT08];
2. existence of an $\mathcal{O}(\log n)$-size spanner for TSP in H-minor-free graphs [GS02].

This results in a PTAS with running time $n^{f_H(1/\varepsilon)}$ for TSP in H-minor-free graphs.

A further application is a PTAS for minimum-weight c-edge-connected submultigraph in H-minor-free graphs (in this variant, an edge may be used several times to increase connectivity but the algorithm pays for each use).

4.4 Rough sketch of proof idea of contraction decomposition theorem

Recall from the preliminaries that any H-minor-free graph can be written as h-clique-sums of h-almost-embeddable graphs. First, observe that the h-almost-embeddable graphs are rather easy to handle: we already know how to deal with bounded-genus graphs [DHM07] and apices and vortices can basically be ignored (they only contribute a constant factor to the treewidth). The really hard part is to deal with the clique-sums and the fact that when building a clique-sum, we may chose to delete some edges at will. That is, some edges of an almost-embeddable summand turn out to be *virtual*, i.e. not present in the actual H-minor-free graph that we are considering. Now, if we find a contraction decompositon for a summand and are required to contract a virtual edge, we are in trouble – that edge does not exist and hence cannot be contracted.

Demaine et al. [DHK11] overcome this problem by showing that except for some special cases, the virtual edges can in a sense be replaced by certain edge-disjoint shortest paths in the other summand; therefore, if we are required to contract such an edge, we can instead contract one of those paths. The proof of this is very technical and delves deep into the structure theory of H-minor-free graphs.

One of the main ideas is involved is to reprove the contraction decomposition theorem of bounded-genus graphs using *radial colorings*. The radial graph can be thought of as (something like) the union of the primal and dual graphs. Now we color each edge at radial distance r from some root as $r \mod k$. Now, it can be shown that any k consecutive layers have bounded treewidth, provided the first k do. The next step is to show that if we start with a bounded number of shortest paths (the ones chosen to replace the virtual edges), then the first k layers will indeed have bounded treewidth. This is shown by arguing that if the treewidth of those first k layers were large, then we would have a large grid; but grids can be used to shortcut shortest paths and this leads to a contradiction.

5 Fixed-Parameter Algorithms

Another major area where the techniques mentioned above are massively applied is that of *parameterized complexity* [DF99, FG06]. As this is not the focus of this presentation, we just very briefly overview some results.

A parameterized problem is said to be *fixed-parameter tractable* (FPT) if for any instance of size n with parameter k it can be solved in time $f(k)n^{\mathcal{O}(1)}$, for some computable function f solely dependent on k. The *standard parameterization* of a problem is the size of the solution, e.g. the size of a minimum vertex cover, or the size of a maximum independent set. But the parameter can actually be any property of the instance, such as the treewidth, the genus, or the size of an excluded minor. Whereas many problems have been shown to be fixed-parameter tractable, a large number of others (such as k-clique) have been shown to be very unlikely to admit such efficient algorithms. This is based on the assumption that W[1] $\neq$ FPT, which is the analog of P $\neq$ NP in parameterized complexity theory.

5.1 Bidimensionality and FPT

The bidimensionality theory provides a framework for capturing many FPT results on H-minor-free graphs by means of Theorem 2. As we mentioned earlier, many of the considered problems can be solved on graphs of bounded treewidth in time $2^{O(\mathrm{tw}(G))}n^{O(1)}$. This implies that all these problems admit *subexponential* fixed parameter algorithms with running time $2^{O(\sqrt{k})}n^{O(1)}$ on the mentioned graph classes, where k is the bidimensional parameter, typically the solution size. Examples of these problems include vertex cover, minimum maximal matching, dominating set, and unweighted longest path.

As an example, consider the k-vertex cover problem. To determine if an H-minor-free graph contains a vertex cover of size at most k, we first approximate its treewidth. If the treewidth is large, then there is a large grid, and hence the answer is no. Otherwise the treewidth is small, i.e. $\mathcal{O}_H(\sqrt{k})$, and we can obtain the exact answer in subexponential time.

It is important to note that for contraction-bidimensional parameters, these results are limited to apex-minor-free graphs. Still, this does not imply that problems that are contraction-closed but not minor-closed do not admit FPT algorithms beyond this class: for example, the k-dominating set problem has been shown to admit a $2^{O(\sqrt{k})}n^{O(1)}$-time FPT algorithm on H-minor-free graphs, map graphs and in fact, all fixed powers of H-minor-free graphs [DFHT05b, DFHT05a].

5.2 Contraction decomposition and the minimum k-way cut problem

Let us consider the k-cut problem where our goal is to remove the fewest number of edges so as to break the graph into at least k connected components. The parameter is k (and not the solution size).

Using the contraction decomposition theorem, this problem is easily seen to be FPT in H-minor-free graphs. Recall that H-minor-free graphs have constant average degree c_H. This implies that OPT $\leq c_H k$. Now, if we apply contraction decomposition to partition the edges of the graph into $c_H k + 1$ parts, then there is an optimal solution that avoids one of the parts. Hence, for each part

i, we can compute the optimal solution on the graph in which part i is contracted in polynomial time (because it is of bounded treewidth) and return the minimum, resulting in an FPT algorithm.

Kawarabayashi and Thorup [KT11] very recently showed how to use graph minor theory to generalize this result to *general graphs* for the case when the solution is of bounded size. However, they use completely different techniques for this aim.

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