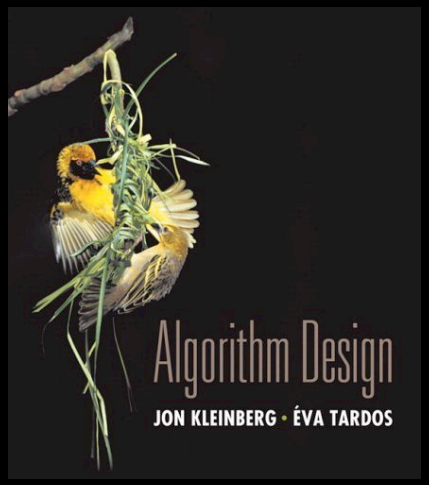



# Chapter 3

## Graphs

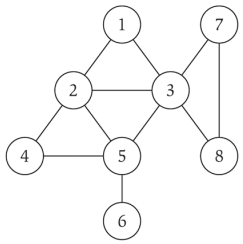


# 3.1 Basic Definitions and Applications

## Undirected Graphs

Undirected graph. $G = (V, E)$

- V = nodes.
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: n = |V|, m = |E|.




V = { 1, 2, 3, 4, 5, 6, 7, 8 }
E = { 1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6 }
n = 8
m = 11



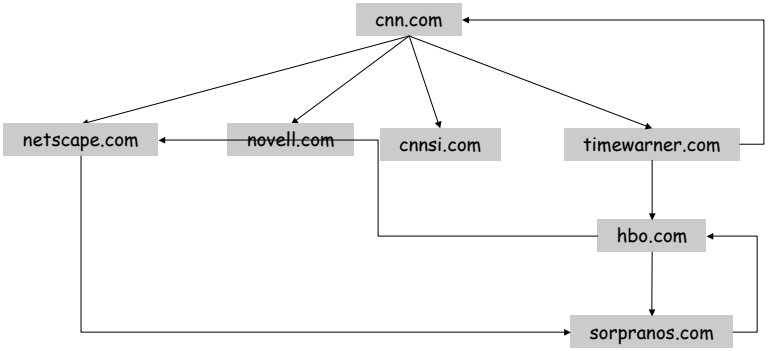
## Some Graph Applications

| *Graph* | *Nodes* | *Edges* |
|---|---|---|
| transportation | street intersections | highways |
| communication | computers | fiber optic cables |
| World Wide Web | web pages | hyperlinks |
| social | people | relationships |
| food web | species | predator-prey |
| software systems | functions | function calls |
| scheduling | tasks | precedence constraints |
| circuits | gates | wires |

## World Wide Web

Web graph.

- Node: web page.
- Edge: hyperlink from one page to another.

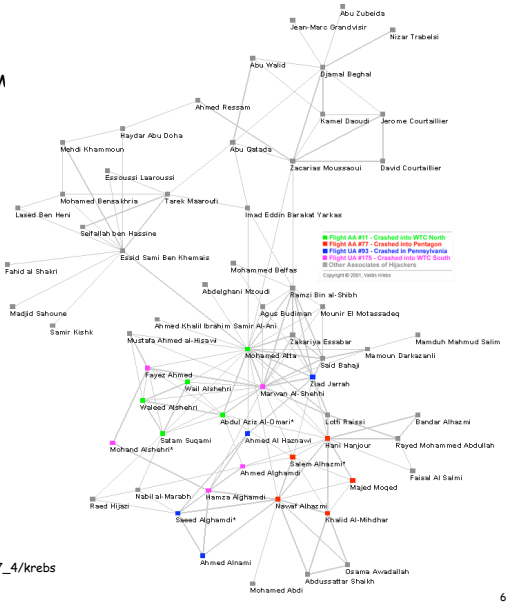





## 9-11 Terrorist Network

Social network graph.

- Node: people.
- Edge: relationship between tw



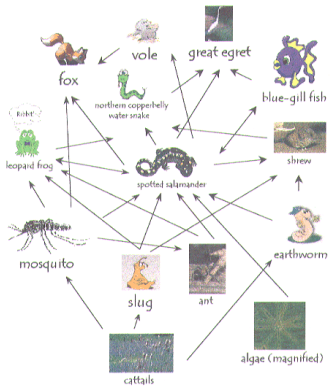
Reference: Valdis Krebs, http://www.firstmonday.org/issues/issue7_4/krebs



## Ecological Food Web

Food web graph.

- Node = species.
- Edge = from prey to predator.



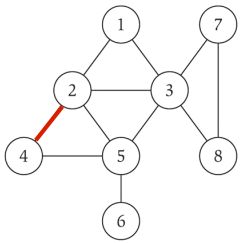

Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff



## Graph Representation: Adjacency Matrix

Adjacency matrix. n-by-n matrix with $A_{uv}$ = 1 if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to $n^2$.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



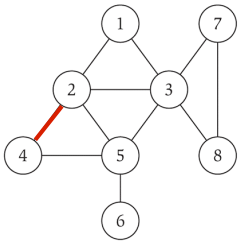

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 3 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 |
| 5 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 7 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 8 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |

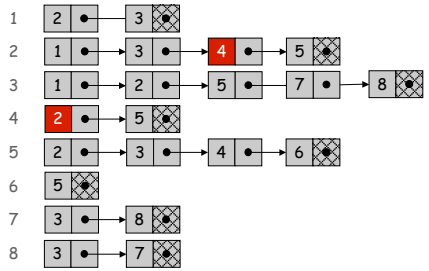
## Graph Representation: Adjacency List

Adjacency list. Node indexed array of lists.

- Two representations of each edge.
- Space proportional to m + n.
- Checking if (u, v) is an edge takes O(deg(u)) time.
- Identifying all edges takes $\Theta(m + n)$ time.

degree = number of neighbors of u



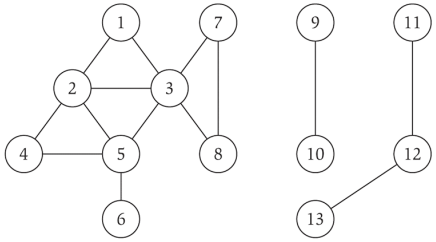



## Paths and Connectivity

Def. A path in an undirected graph G = (V, E) is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair $v_i, v_{i+1}$ is joined by an edge in E.

Def. A path is simple if all nodes are distinct.

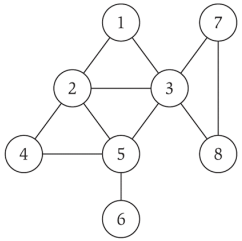
Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.






## Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first k-1 nodes are all distinct.




cycle C = 1-2-4-5-3-1

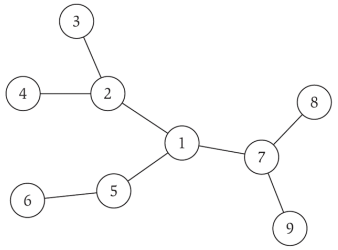


## Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.

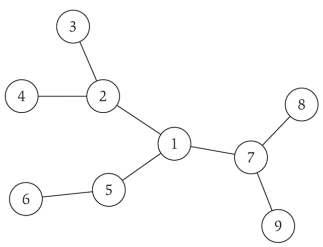
- G is connected.
- G does not contain a cycle.
- G has n-1 edges.



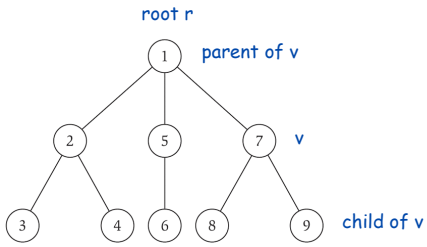
## Rooted Trees

Rooted tree. Given a tree T, choose a root node r and orient each edge away from r.

Importance. Models hierarchical structure.




a tree

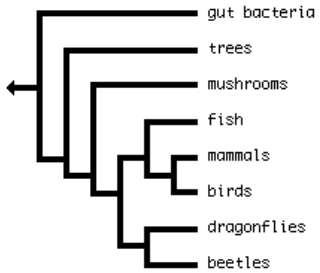



the same tree, rooted at 1



## Phylogeny Trees

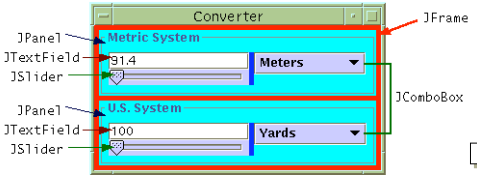
Phylogeny trees. Describe evolutionary history of species.

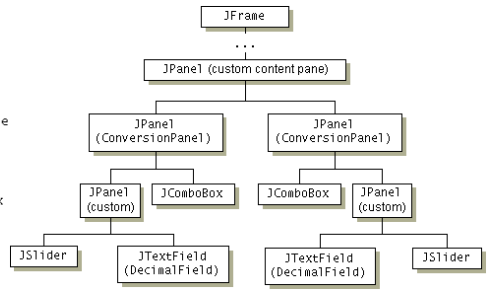





## GUI Containment Hierarchy

GUI containment hierarchy. Describe organization of GUI widgets.




Reference: http://java.sun.com/docs/books/tutorial/uiswing/overview/anatomy.html



# 3.2 Graph Traversal

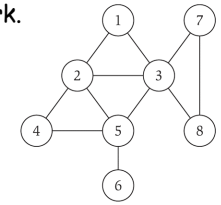
## Connectivity

s-t connectivity problem. Given two node s and t, is there a path between s and t?

s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

Applications.

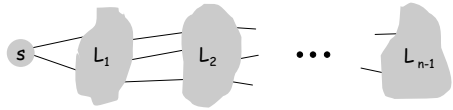
- Friendster.
- Maze traversal.
- Kevin Bacon number.
- Fewest number of hops in a communication network.






## Breadth First Search

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.




BFS algorithm.

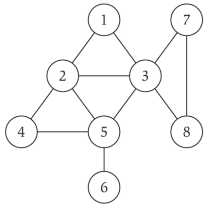
- $L_0$ = { s }.
- $L_1$ = all neighbors of $L_0$.
- $L_2$ = all nodes that do not belong to $L_0$ or $L_1$, and that have an edge to a node in $L_1$.
- $L_{i+1}$ = all nodes that do not belong to an earlier layer, and that have an edge to a node in $L_i$.

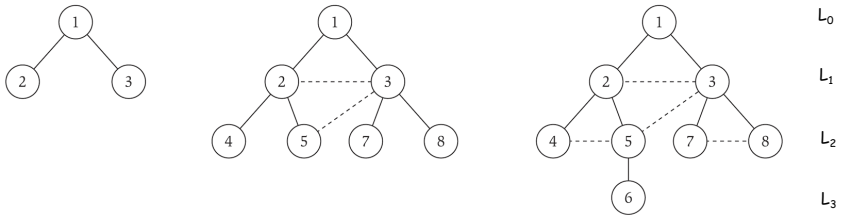
Theorem. For each i, $L_i$ consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.



## Breadth First Search

Property. Let T be a BFS tree of G = (V, E), and let (x, y) be an edge of G. Then the level of x and y differ by at most 1.






## Breadth First Search: Analysis

Theorem. The above implementation of BFS runs in O(m + n) time if the graph is given by its adjacency representation.

Pf.

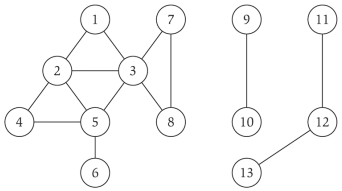
- Easy to prove $O(n^2)$ running time:
  - at most n lists L[i]
  - each node occurs on at most one list; for loop runs ≤ n times
  - when we consider node u, there are ≤ n incident edges (u, v), and we spend O(1) processing each edge

- Actually runs in O(m + n) time:
  - when we consider node u, there are deg(u) incident edges (u, v)
  - total time processing edges is $\Sigma_{u \in V} \deg(u) = 2m$ ▪

each edge (u, v) is counted exactly twice in sum: once in deg(u) and once in deg(v)

## Connected Component

Connected component. Find all nodes reachable from s.



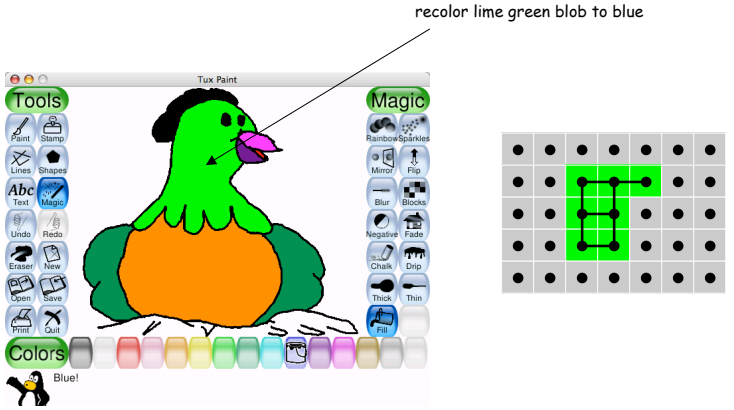

Connected component containing node 1 = { 1, 2, 3, 4, 5, 6, 7, 8 }.



## Flood Fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

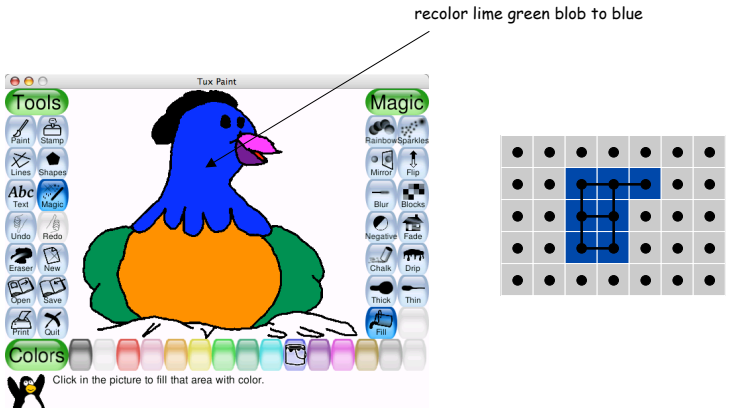





## Flood Fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.





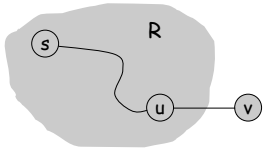

## Connected Component

Connected component. Find all nodes reachable from s.

---

$R$ will consist of nodes to which $s$ has a path
Initially $R = \{s\}$
While there is an edge $(u, v)$ where $u \in R$ and $v \notin R$
  Add $v$ to $R$
Endwhile

---




it's safe to add v

Theorem. Upon termination, R is the connected component containing s.

- BFS = explore in order of distance from s.
- DFS = explore in a different way.

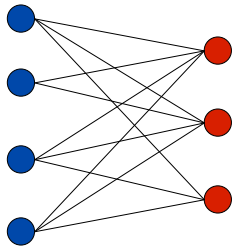
# 3.4 Testing Bipartiteness

## Bipartite Graphs

Def. An undirected graph G = (V, E) is bipartite if the nodes can be colored red or blue such that every edge has one red and one blue end.

Applications.

- Stable marriage: men = red, women = blue.
- Scheduling: machines = red, jobs = blue.



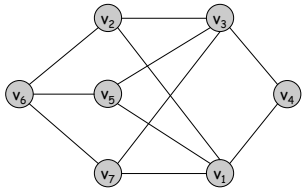
a bipartite graph



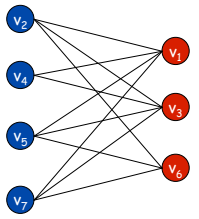
## Testing Bipartiteness

Testing bipartiteness. Given a graph G, is it bipartite?

- Many graph problems become:
  - easier if the underlying graph is bipartite (matching)
  - tractable if the underlying graph is bipartite (independent set)
- Before attempting to design an algorithm, we need to understand structure of bipartite graphs.




a bipartite graph G



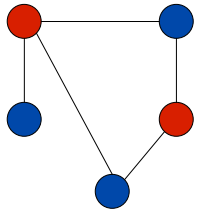

another drawing of G



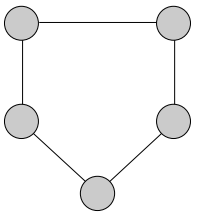
## An Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

Pf. Not possible to 2-color the odd cycle, let alone G.



bipartite
(2-colorable)



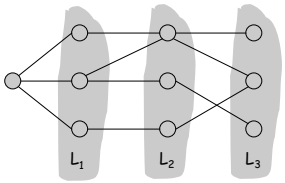
not bipartite
(not 2-colorable)

## Bipartite Graphs

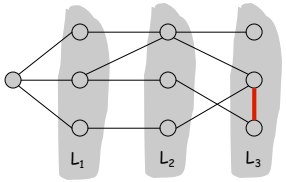
Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).




Case (i)




Case (ii)



## Bipartite Graphs

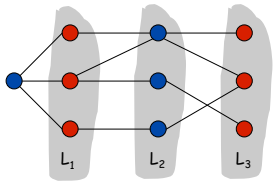
Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

Pf. (i)

- Suppose no edge joins two nodes in adjacent layers.
- By previous lemma, this implies all edges join nodes on same level.
- Bipartition: red = nodes on odd levels, blue = nodes on even levels.




Case (i)



## Bipartite Graphs

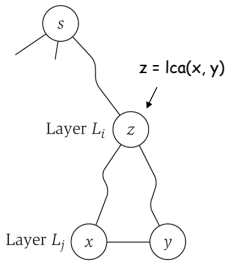
Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

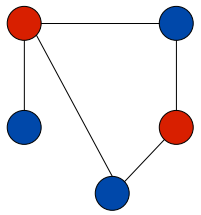
- Suppose $(x, y)$ is an edge with $x$, $y$ in same level $L_j$.
- Let $z$ = lca$(x, y)$ = lowest common ancestor.
- Let $L_i$ be level containing $z$.
- Consider cycle that takes edge from $x$ to $y$, then path from $y$ to $z$, then path from $z$ to $x$.
- Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j\text{-}i)}_{\text{path from } y \text{ to } z} + \underbrace{(j\text{-}i)}_{\text{path from } z \text{ to } x}$, which is odd. ▪



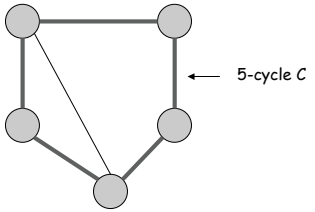



## Obstruction to Bipartiteness

Corollary. A graph $G$ is bipartite iff it contain no odd length cycle.



bipartite
(2-colorable)

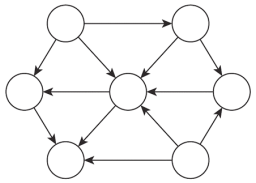



not bipartite
(not 2-colorable)

# 3.5 Connectivity in Directed Graphs



Directed graph. $G = (V, E)$
- Edge (u, v) goes from node u to node v.



Ex. Web graph - hyperlink points from one web page to another.
- Directedness of graph is crucial.
- Modern web search engines exploit hyperlink structure to rank web pages by importance.





Directed reachability. Given a node s, find all nodes reachable from s.

Directed s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

Graph search. BFS extends naturally to directed graphs.

Web crawler. Start from web page s. Find all web pages linked from s, either directly or indirectly.





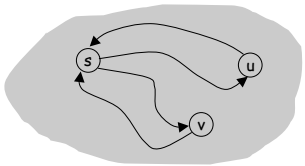
Def. Node u and v are mutually reachable if there is a path from u to v and also a path from v to u.

Def. A graph is strongly connected if every pair of nodes is mutually reachable.

Lemma. Let s be any node. G is strongly connected iff every node is reachable from s, and s is reachable from every node.

Pf. $\Rightarrow$ Follows from definition.
Pf. $\Leftarrow$ Path from u to v: concatenate u-s path with s-v path.
Path from v to u: concatenate v-s path with s-u path. ▪

ok if paths overlap

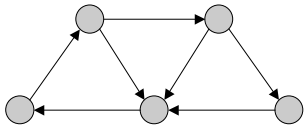


## Strong Connectivity: Algorithm

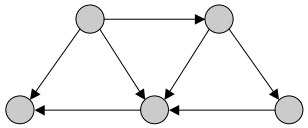
Theorem. Can determine if G is strongly connected in O(m + n) time.

Pf.

- Pick any node s.
- Run BFS from s in G.
- Run BFS from s in $G^{rev}$. (reverse orientation of every edge in G)
- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from previous lemma. ▪



strongly connected



not strongly connected



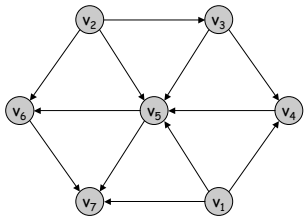
# 3.6 DAGs and Topological Ordering

## Directed Acyclic Graphs

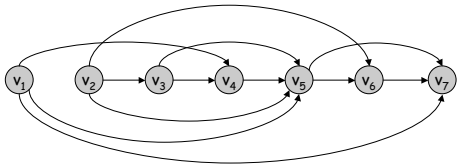
Def. An DAG is a directed graph that contains no directed cycles.

Ex. Precedence constraints: edge $(v_i, v_j)$ means $v_i$ must precede $v_j$.

Def. A topological order of a directed graph G = (V, E) is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge $(v_i, v_j)$ we have i < j.




a DAG




a topological ordering



## Precedence Constraints

Precedence constraints. Edge $(v_i, v_j)$ means task $v_i$ must occur before $v_j$.

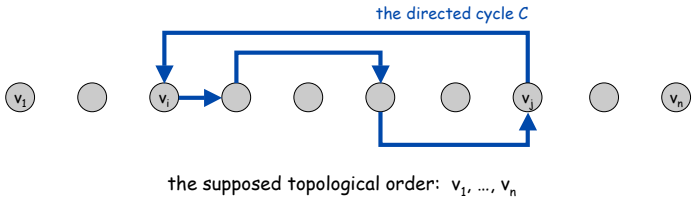
Applications.

- Course prerequisite graph: course $v_i$ must be taken before $v_j$.
- Compilation: module $v_i$ must be compiled before $v_j$. Pipeline of computing jobs: output of job $v_i$ needed to determine input of job $v_j$.

Lemma. If G has a topological order, then G is a DAG.

Pf. (by contradiction)

- Suppose that G has a topological order $v_1, \ldots, v_n$ and that G also has a directed cycle C. Let's see what happens.
- Let $v_i$ be the lowest-indexed node in C, and let $v_j$ be the node just before $v_i$; thus $(v_j, v_i)$ is an edge.
- By our choice of i, we have i < j.
- On the other hand, since $(v_j, v_i)$ is an edge and $v_1, \ldots, v_n$ is a topological order, we must have j < i, a contradiction. ▪




the supposed topological order: $v_1, \ldots, v_n$





Lemma. If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

Q. If so, how do we compute one?

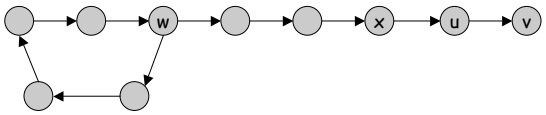




Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

- Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.
- Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.
- Then, since u has at least one incoming edge (x, u), we can walk backward to x.
- Repeat until we visit a node, say w, twice.
- Let C denote the sequence of nodes encountered between successive visits to w. C is a cycle. ▪





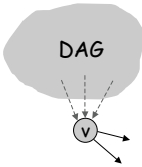



Lemma. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

- Base case: true if n = 1.
- Given DAG on n > 1 nodes, find a node v with no incoming edges.
- G - { v } is a DAG, since deleting v cannot create cycles.
- By inductive hypothesis, G - { v } has a topological ordering.
- Place v first in topological ordering; then append nodes of G - { v }
- in topological order. This is valid since v has no incoming edges. ▪

```
To compute a topological ordering of G:
Find a node v with no incoming edges and order it first
Delete v from G
Recursively compute a topological ordering of G−{v}
  and append this order after v
```



## Topological Sorting Algorithm: Running Time

Theorem. Algorithm finds a topological order in $O(m + n)$ time.

Pf.

- Maintain the following information:
  - `count[w]` = remaining number of incoming edges
  - S = set of remaining nodes with no incoming edges
- Initialization: $O(m + n)$ via single scan through graph.
- Update: to delete v
  - remove v from S
  - decrement `count[w]` for all edges from v to w, and add w to S if c `count[w]` hits 0
  - this is $O(1)$ per edge ▪