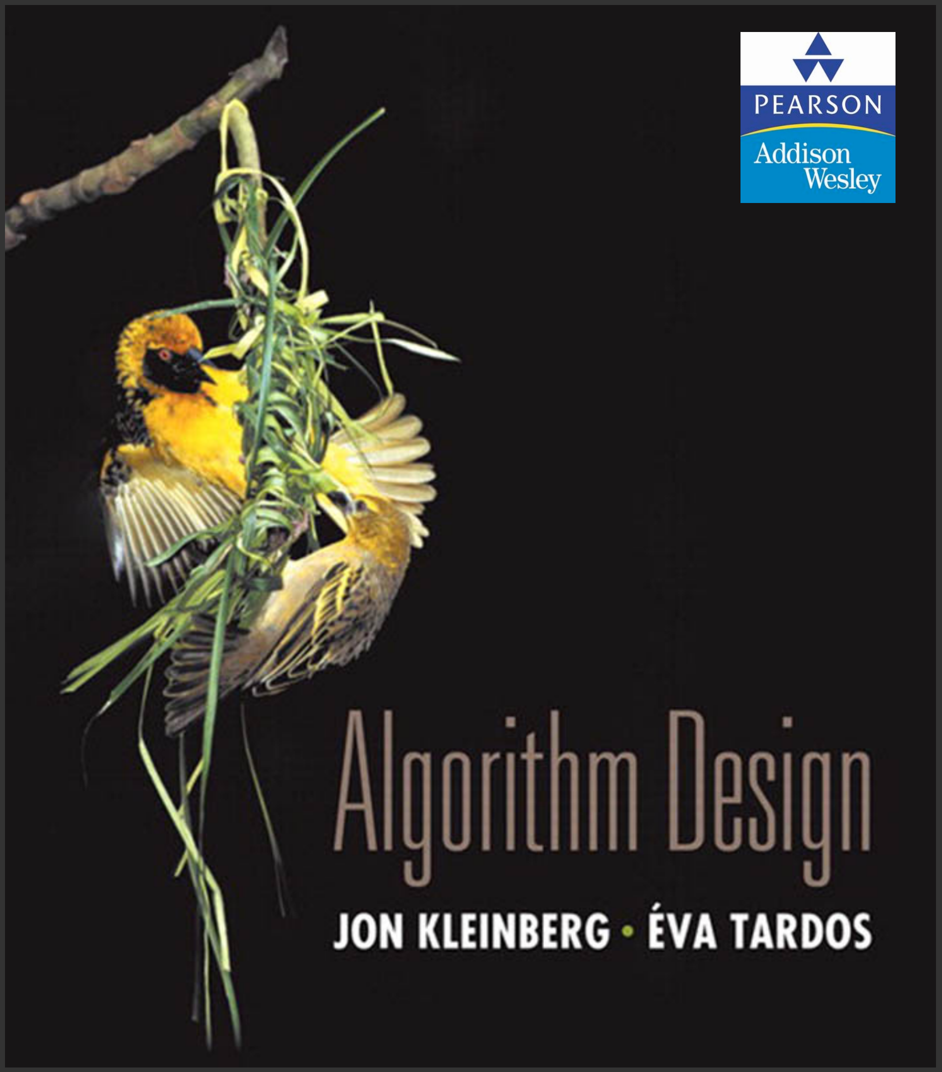


11. APPROXIMATION ALGORITHMS

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 2/26/20 11:14 AM

Coping with NP-completeness

Q. Suppose I need to solve an **NP**-hard optimization problem.
What should I do?

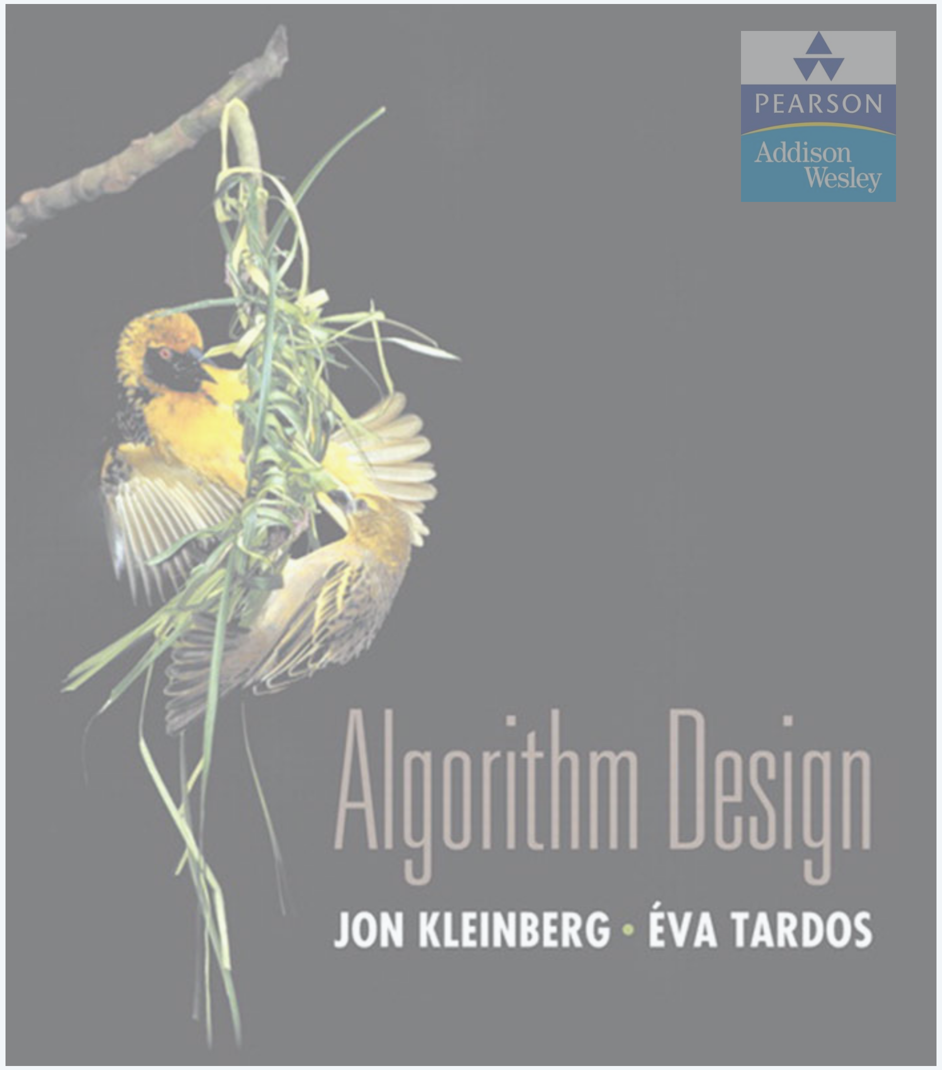
A. Sacrifice one of three desired features.

i. Runs in polynomial time.
ii. Solves arbitrary instances of the problem.
iii. Finds optimal solution to problem.

ρ-approximation algorithm.

- Runs in polynomial time.
- Solves arbitrary instances of the problem
- Finds solution that is within ratio ρ of optimum.

Challenge. Need to prove a solution's value is close to optimum, without even knowing what is optimum value.



SECTION 11.1

11. APPROXIMATION ALGORITHMS

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

Load balancing

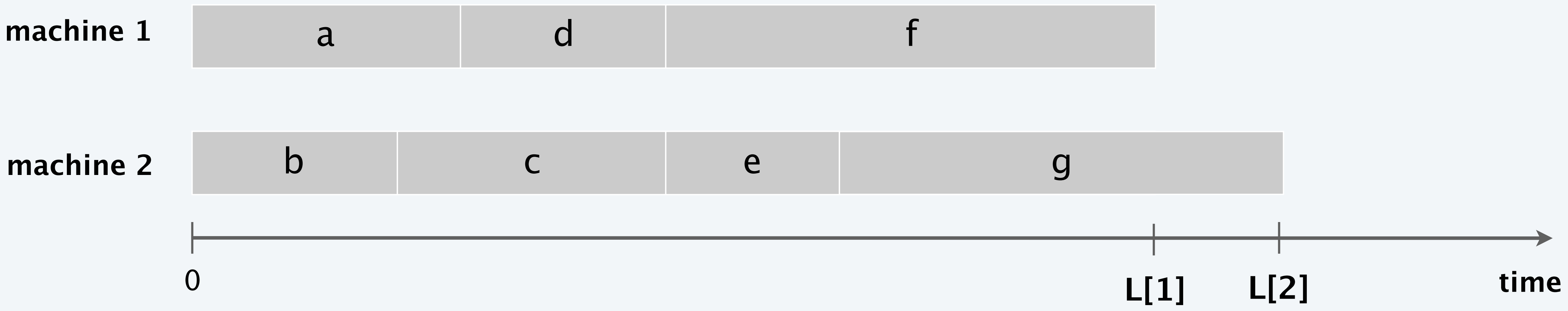
Input. m identical machines; $n \geq m$ jobs, job j has processing time t_j.

- Job j must run contiguously on one machine.
- A machine can process at most one job at a time.

Def. Let $S[i]$ be the subset of jobs assigned to machine i.
The load of machine i is $L[i] = \Sigma_{j \in S[i]}\ t_j$.

Def. The makespan is the maximum load on any machine $L = \max_i L[i]$.

Load balancing. Assign each job to a machine to minimize makespan.

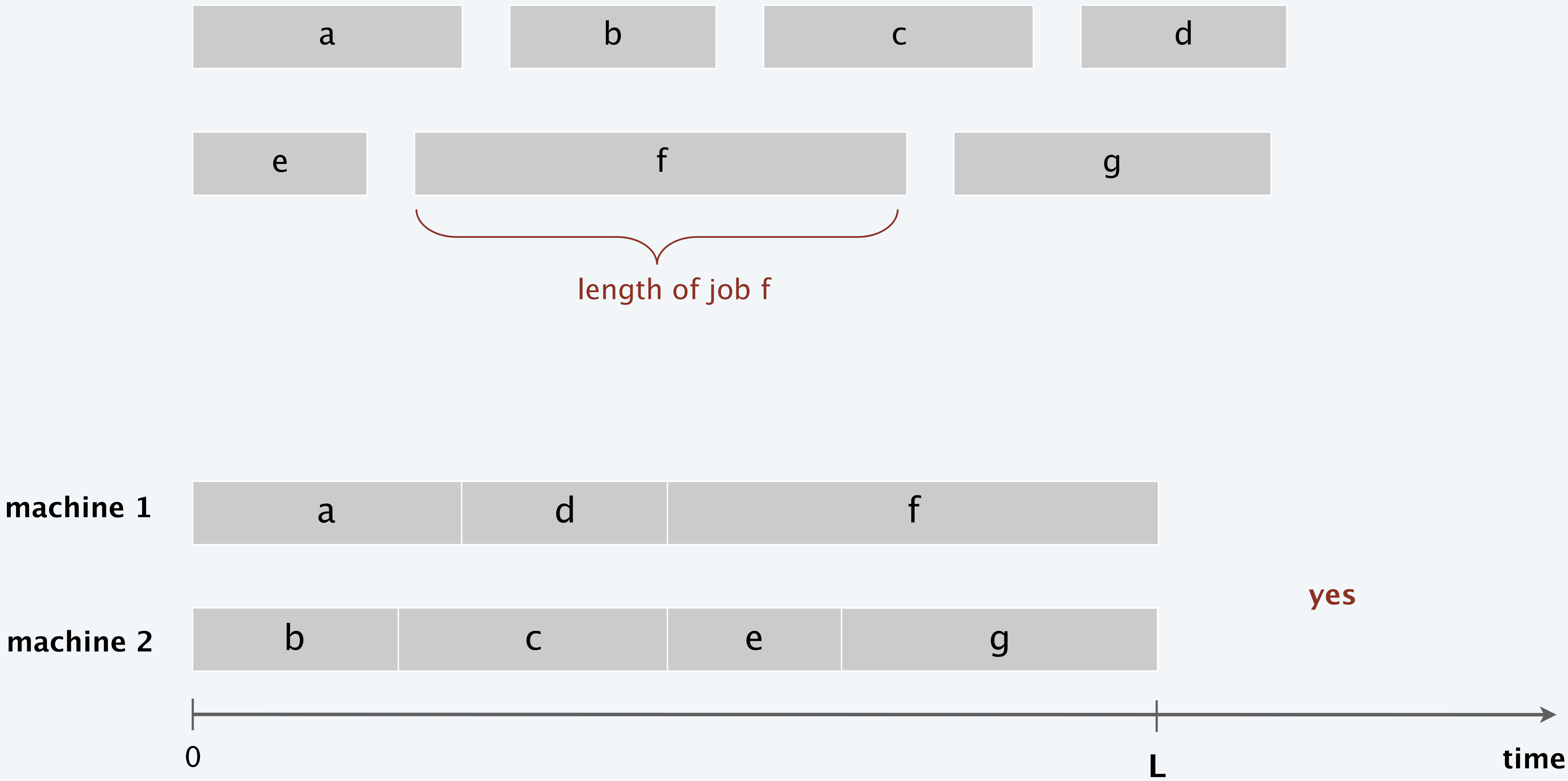


Load balancing on 2 machines is NP-hard

Claim. Load balancing is hard even if $m = 2$ machines.

Pf. PARTITION $\le_P$ LOAD-BALANCE.

NP-complete by Exercise 8.26



Load balancing: list scheduling

List-scheduling algorithm.

- Consider n jobs in some fixed order.
- Assign job j to machine i whose load is smallest so far.

```
LIST-SCHEDULING $(m, n, t_1, t_2, \ldots, t_n)$

FOR $i = 1$ TO $m$
    $L[i] \leftarrow 0.$            ⟵ load on machine $i$
    $S[i] \leftarrow \varnothing.$            ⟵ jobs assigned to machine $i$

FOR $j = 1$ TO $n$
    $i \leftarrow \text{argmin}_k\ L[k].$        ⟵ machine $i$ has smallest load
    $S[i] \leftarrow S[i] \cup \{j\}.$        ⟵ assign job $j$ to machine $i$
    $L[i] \leftarrow L[i] + t_j.$        ⟵ update load of machine $i$

RETURN $S[1], S[2], \ldots, S[m].$
```

Implementation. $O(n \log m)$ using a priority queue for loads $L[k]$.

Load balancing: list scheduling analysis

Theorem. [Graham 1966] Greedy algorithm is a 2-approximation.

- First worst-case analysis of an approximation algorithm.
- Need to compare resulting solution with optimal makespan L^*.

Lemma 1. For all k : the optimal makespan $L^* \geq t_k$.

Pf. Some machine must process the most time-consuming job. ▪

Lemma 2. The optimal makespan $L^* \geq \frac{1}{m} \sum_k t_k$.

Pf.

- The total processing time is $\Sigma_k \, t_k$.
- One of m machines must do at least a $1 / m$ fraction of total work. ▪

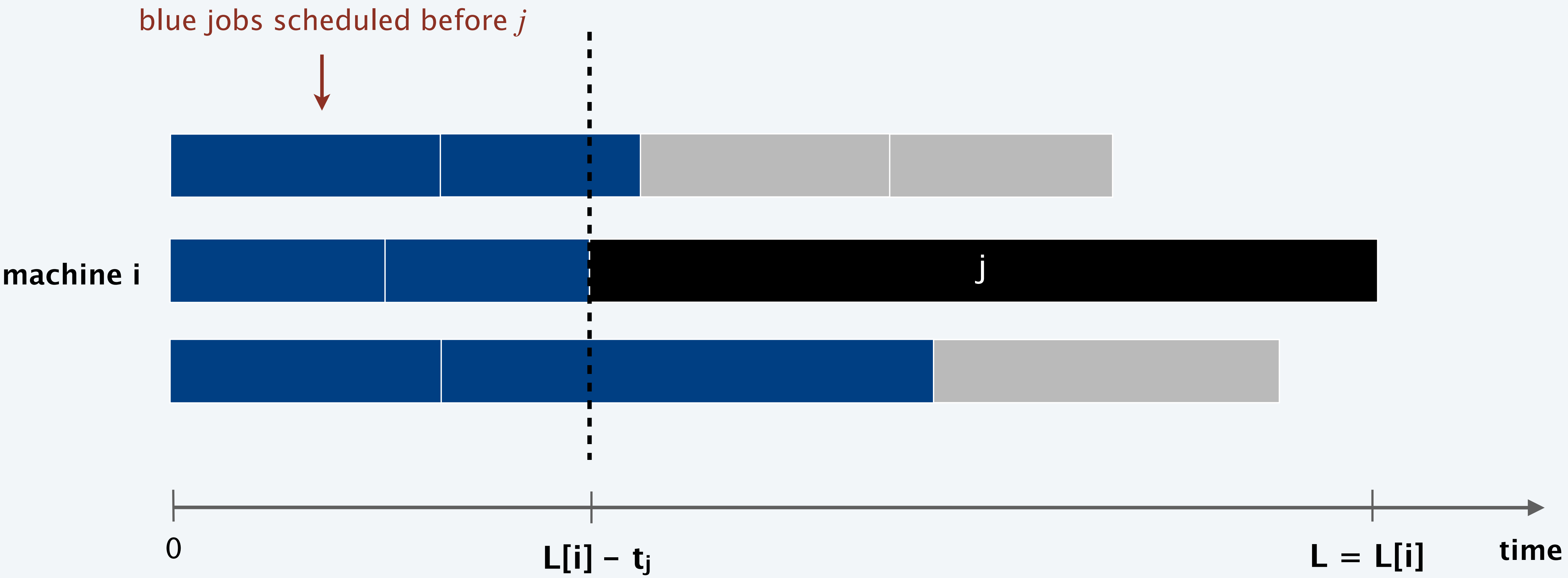
Load balancing: list scheduling analysis

Theorem. Greedy algorithm is a 2-approximation.

Pf. Consider load $L[i]$ of bottleneck machine i. ← machine that ends up with highest load

- Let j be last job scheduled on machine i.
- When job j assigned to machine i, i had smallest load.

  Its load before assignment is $L[i] - t_j$; hence $L[i] - t_j \le L[k]$ for all $1 \le k \le m$.



Load balancing: list scheduling analysis

Theorem. Greedy algorithm is a 2-approximation.

Pf. Consider load $L[i]$ of bottleneck machine i. ← machine that ends up with highest load

- Let j be last job scheduled on machine i.
- When job j assigned to machine i, i had smallest load.
  Its load before assignment is $L[i] - t_j$; hence $L[i] - t_j \le L[k]$ for all $1 \le k \le m$.
- Sum inequalities over all k and divide by m:

$$\begin{aligned} L[i] - t_j &\le \frac{1}{m} \sum_k L[k] \\ &= \frac{1}{m} \sum_k t_k \\ \text{Lemma 2} \longrightarrow \quad &\le L^*. \end{aligned}$$

- Now, $L = L[i] = \underbrace{(L[i] - t_j)}_{\le L^*} + \underbrace{t_j}_{\le L^*} \le 2L^*$ ▪

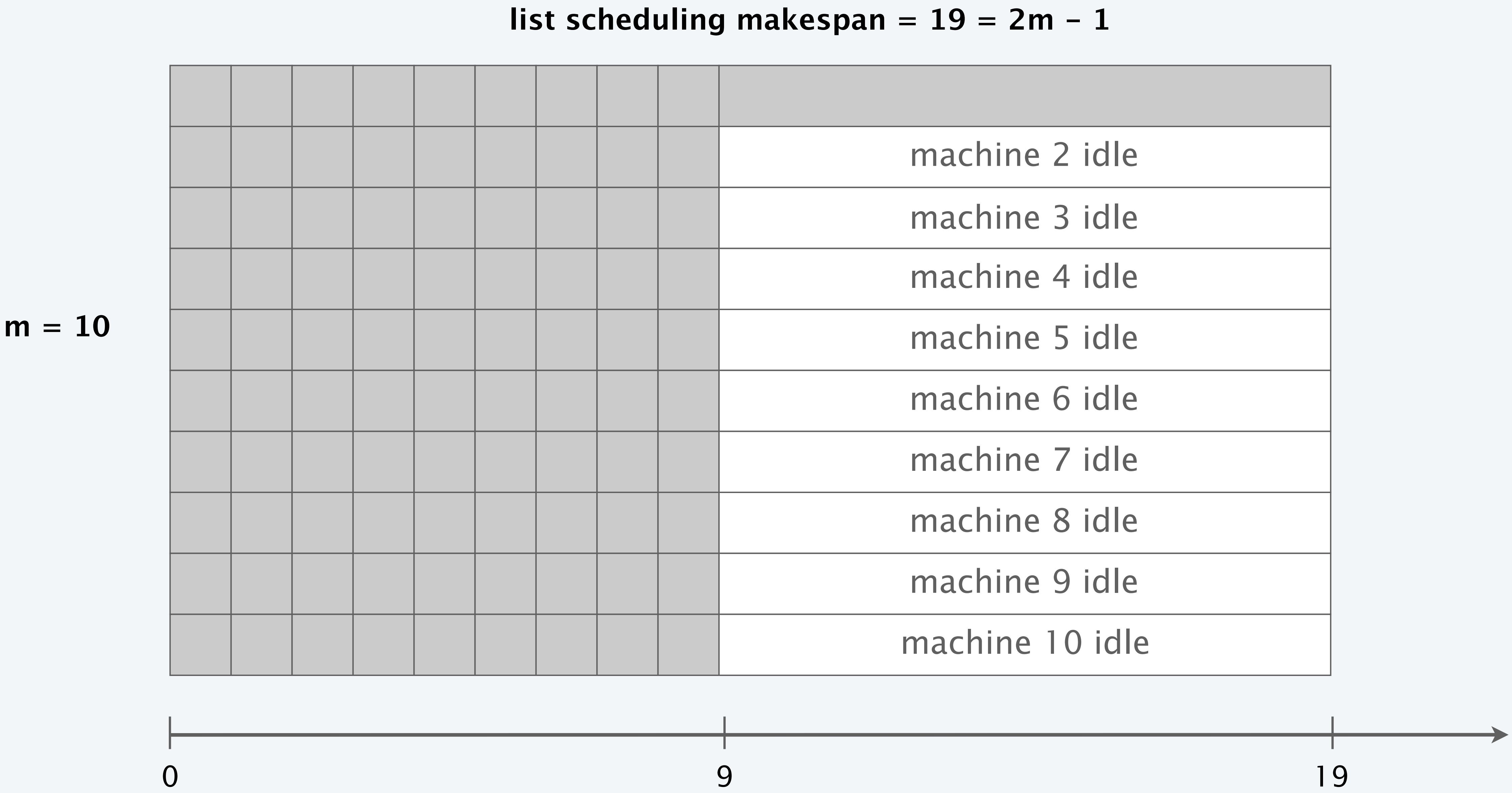
$\uparrow$ above inequality　　$\uparrow$ Lemma 1

Load balancing: list scheduling analysis

Q. Is our analysis tight?

A. Essentially yes.

Ex: m machines, first $m\,(m-1)$ jobs have length 1, last job has length m.



Load balancing: list scheduling analysis

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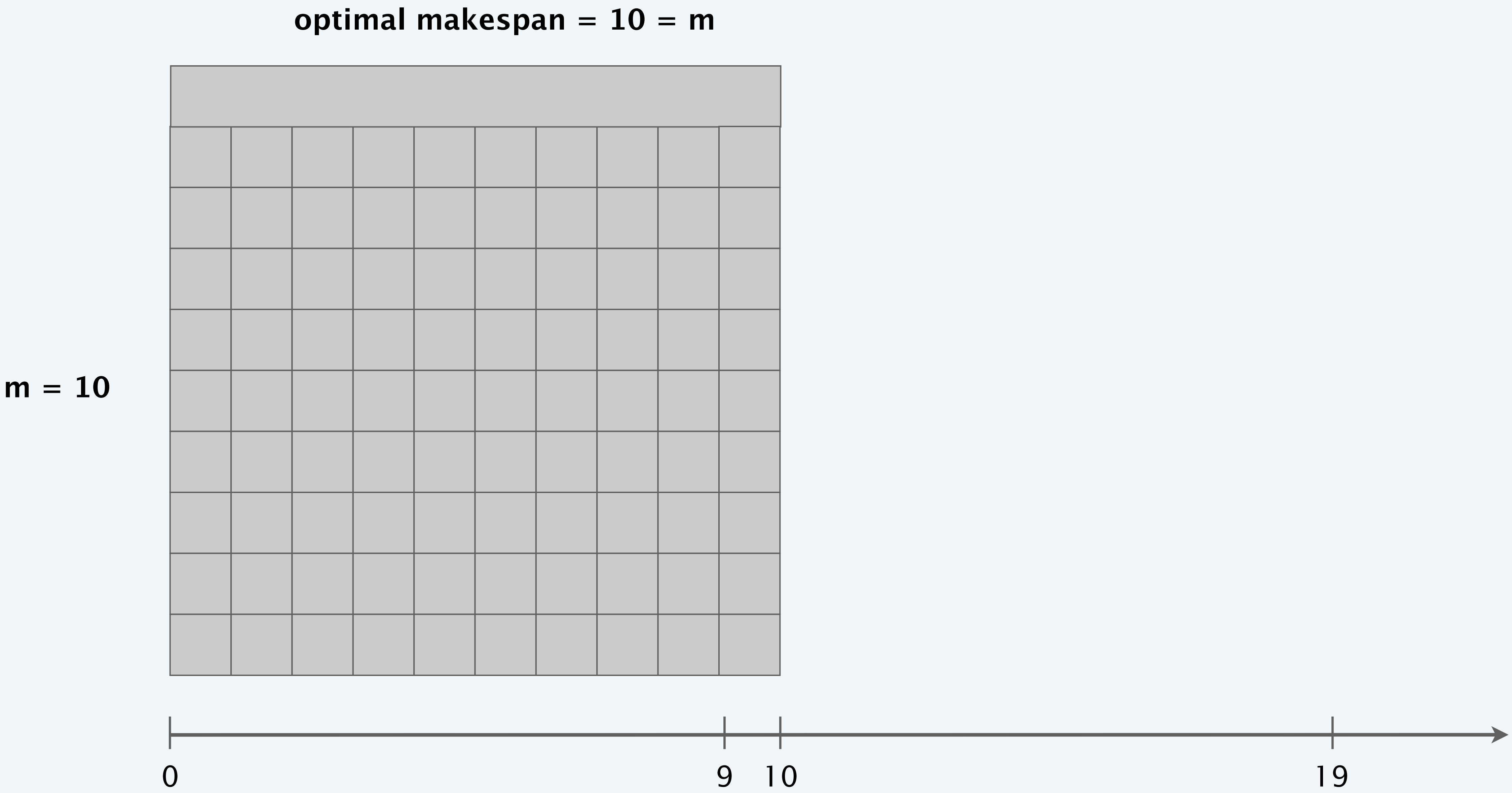


Load balancing: LPT rule

Longest processing time (LPT). Sort n jobs in decreasing order of processing times; then run list scheduling algorithm.

```
LPT-LIST-SCHEDULING $(m, n, t_1, t_2, \ldots, t_n)$

SORT jobs and renumber so that $t_1 \geq t_2 \geq \ldots \geq t_n$.

FOR $i = 1$ TO $m$
    $L[i] \leftarrow 0$.            ← load on machine $i$
    $S[i] \leftarrow \varnothing$.  ← jobs assigned to machine $i$

FOR $j = 1$ TO $n$
    $i \leftarrow \text{argmin}_k\, L[k]$.      ← machine $i$ has smallest load
    $S[i] \leftarrow S[i] \cup \{j\}$.          ← assign job $j$ to machine $i$
    $L[i] \leftarrow L[i] + t_j$.               ← update load of machine $i$

RETURN $S[1], S[2], \ldots, S[m]$.
```

Load balancing: LPT rule

Observation. If bottleneck machine i has only 1 job, then optimal.

Pf. Any solution must schedule that job. ▪

Lemma 3. If there are more than m jobs, $L^* \geq 2\,t_{m+1}$.

Pf.

- Consider processing times of first $m+1$ jobs $t_1 \geq t_2 \geq \ldots \geq t_{m+1}$.
- Each takes at least t_{m+1} time.
- There are $m+1$ jobs and m machines, so by pigeonhole principle, at least one machine gets two jobs. ▪

Theorem. LPT rule is a 3/2-approximation algorithm.

Pf. [similar to proof for list scheduling]

- Consider load $L[i]$ of bottleneck machine i.
- Let j be last job scheduled on machine i. ←

assuming machine i has at least 2 jobs, we have $j \geq m+1$

$$L = L[i] = \underbrace{(L[i] - t_j)}_{\leq L^*} + \underbrace{t_j}_{\leq \frac{1}{2} L^*} \leq \frac{3}{2} L^* \;\blacksquare$$

as before → $\leq L^*$

$\leq \frac{1}{2} L^*$ ← Lemma 3 (since $t_{m+1} \geq t_j$)

Load balancing: LPT rule

Q. Is our 3/2 analysis tight?

A. No.

Theorem. [Graham 1969] LPT rule is a 4/3-approximation.

Pf. More sophisticated analysis of same algorithm.

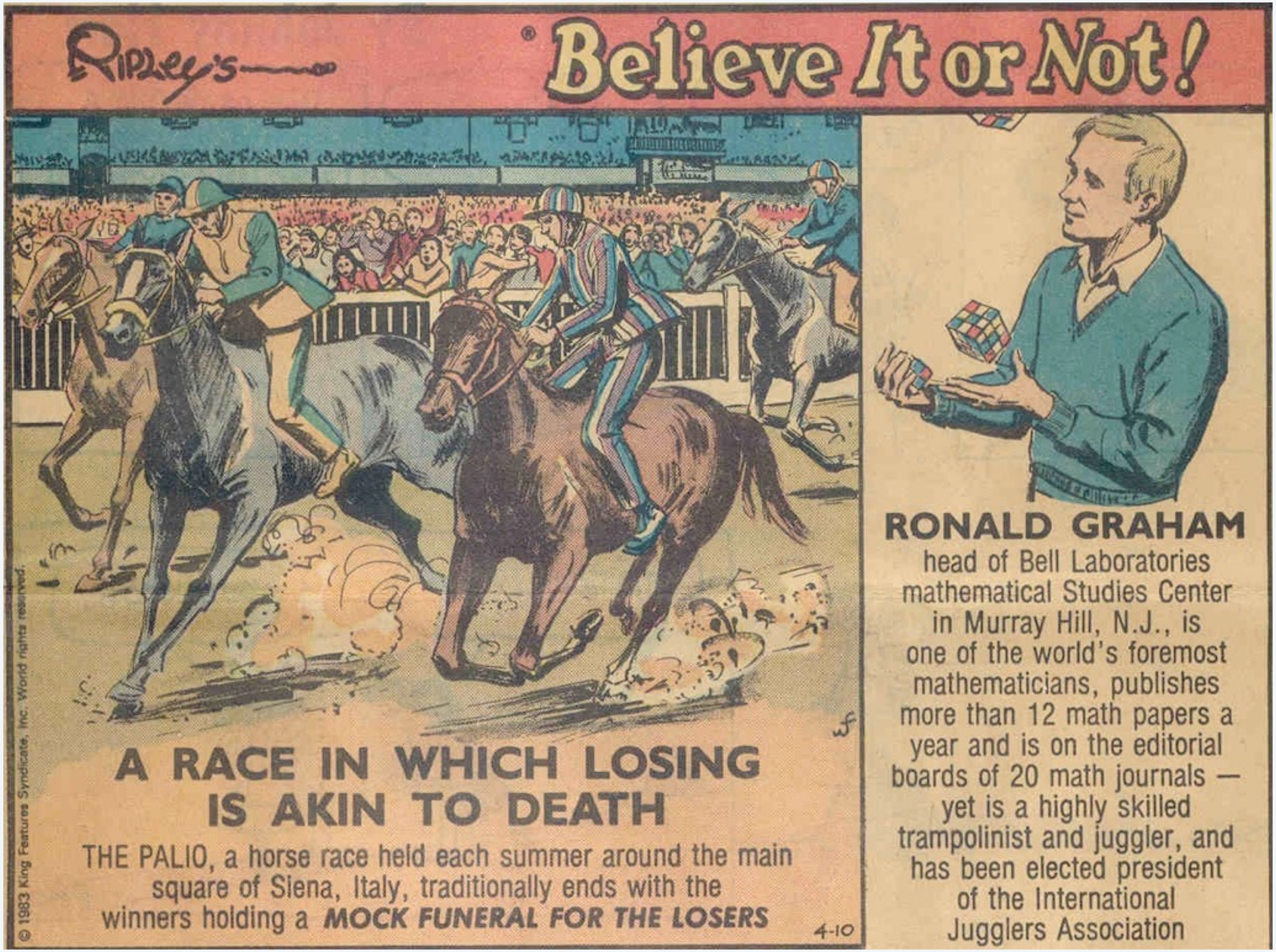
Q. Is Graham's 4/3 analysis tight?

A. Essentially yes.

Ex.

- m machines
- $n = 2m + 1$ jobs
- 2 jobs of length $m, m + 1, \ldots, 2m - 1$ and one more job of length m.
- Then, $L / L^* = (4m - 1) / (3m)$

Believe it or not



Ripley's® Believe It or Not!

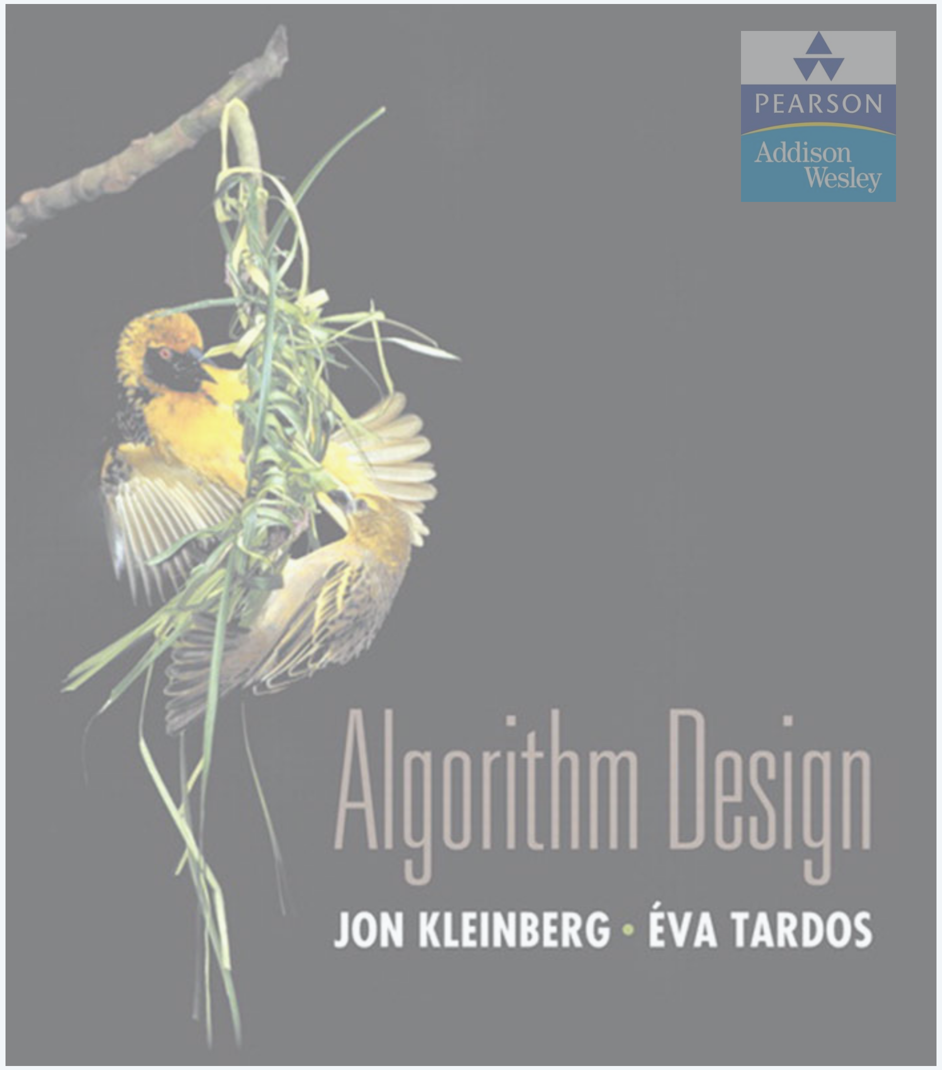
A RACE IN WHICH LOSING IS AKIN TO DEATH

THE PALIO, a horse race held each summer around the main square of Siena, Italy, traditionally ends with the winners holding a ***MOCK FUNERAL FOR THE LOSERS***

4-10

RONALD GRAHAM

head of Bell Laboratories mathematical Studies Center in Murray Hill, N.J., is one of the world's foremost mathematicians, publishes more than 12 math papers a year and is on the editorial boards of 20 math journals — yet is a highly skilled trampolinist and juggler, and has been elected president of the International Jugglers Association



SECTION 11.2

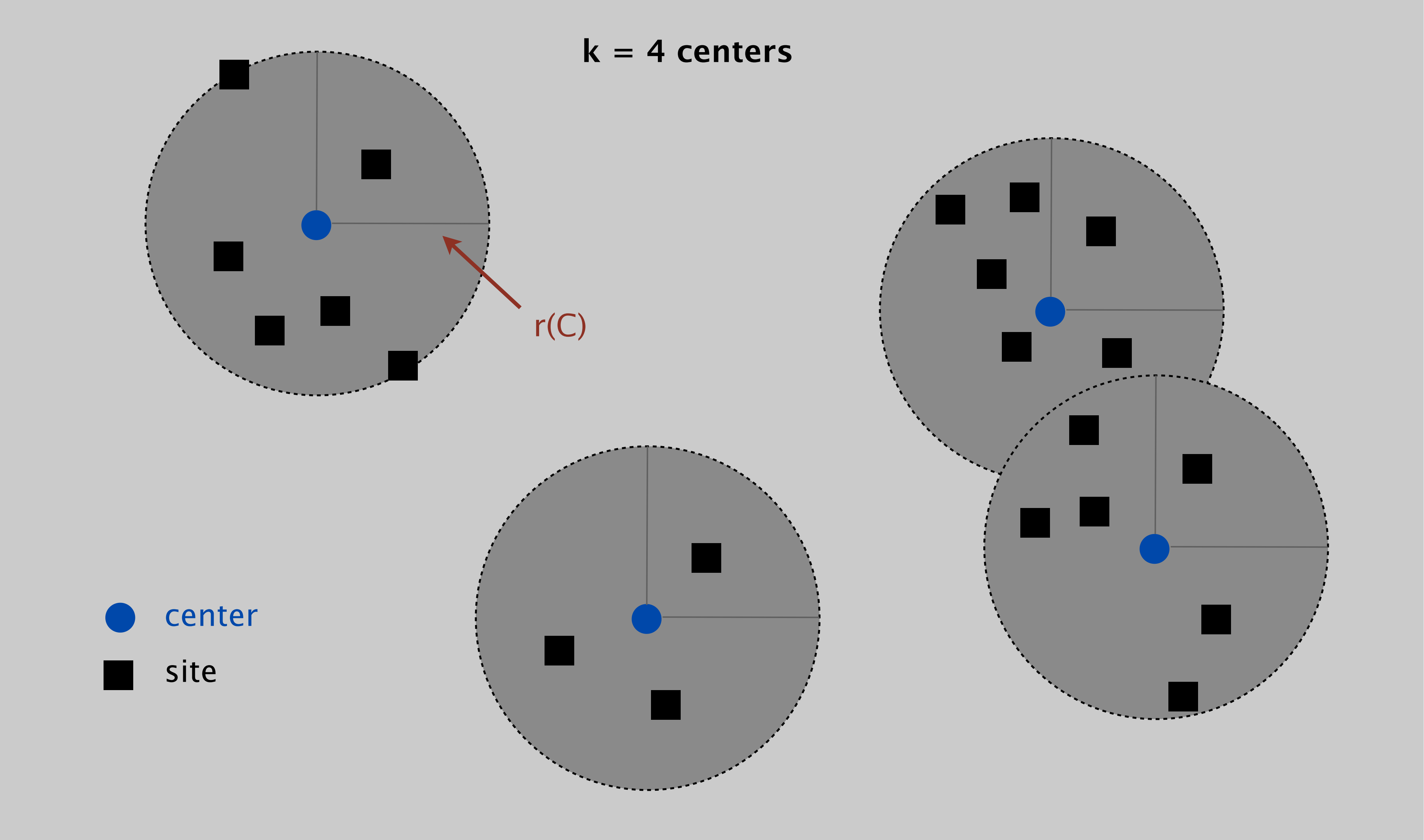
11. APPROXIMATION ALGORITHMS

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

Center selection problem

Input. Set of n sites $s_1, \ldots, s_n$ and an integer $k > 0$.

Center selection problem. Select set of k centers C so that maximum distance $r(C)$ from a site to nearest center is minimized.



Center selection problem

Input. Set of n sites $s_1, \ldots, s_n$ and an integer $k > 0$.

Center selection problem. Select set of k centers C so that maximum distance $r(C)$ from a site to nearest center is minimized.

Notation.

- $dist(x, y)$ = distance between sites x and y.
- $dist(s_i, C) = \min_{c \in C} dist(s_i, c)$ = distance from s_i to closest center.
- $r(C) = \max_i dist(s_i, C)$ = smallest covering radius.

Goal. Find set of centers C that minimizes $r(C)$, subject to $|C| = k$.

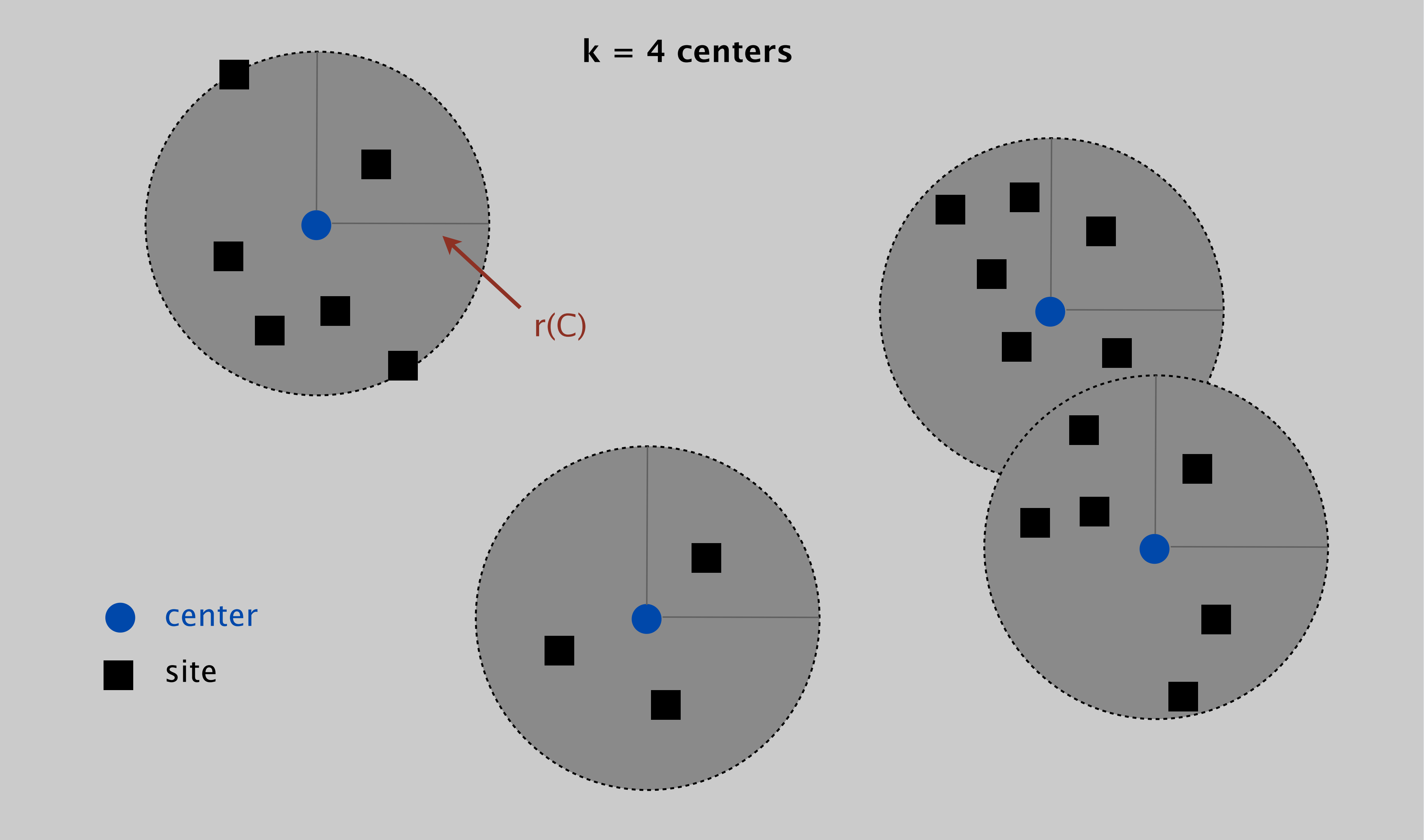
Distance function properties.

- $dist(x, x) = 0$ [identity]
- $dist(x, y) = dist(y, x)$ [symmetry]
- $dist(x, y) \leq dist(x, z) + dist(z, y)$ [triangle inequality]

Center selection example

Ex: each site is a point in the plane, a center can be any point in the plane, $dist(x, y)$ = Euclidean distance.

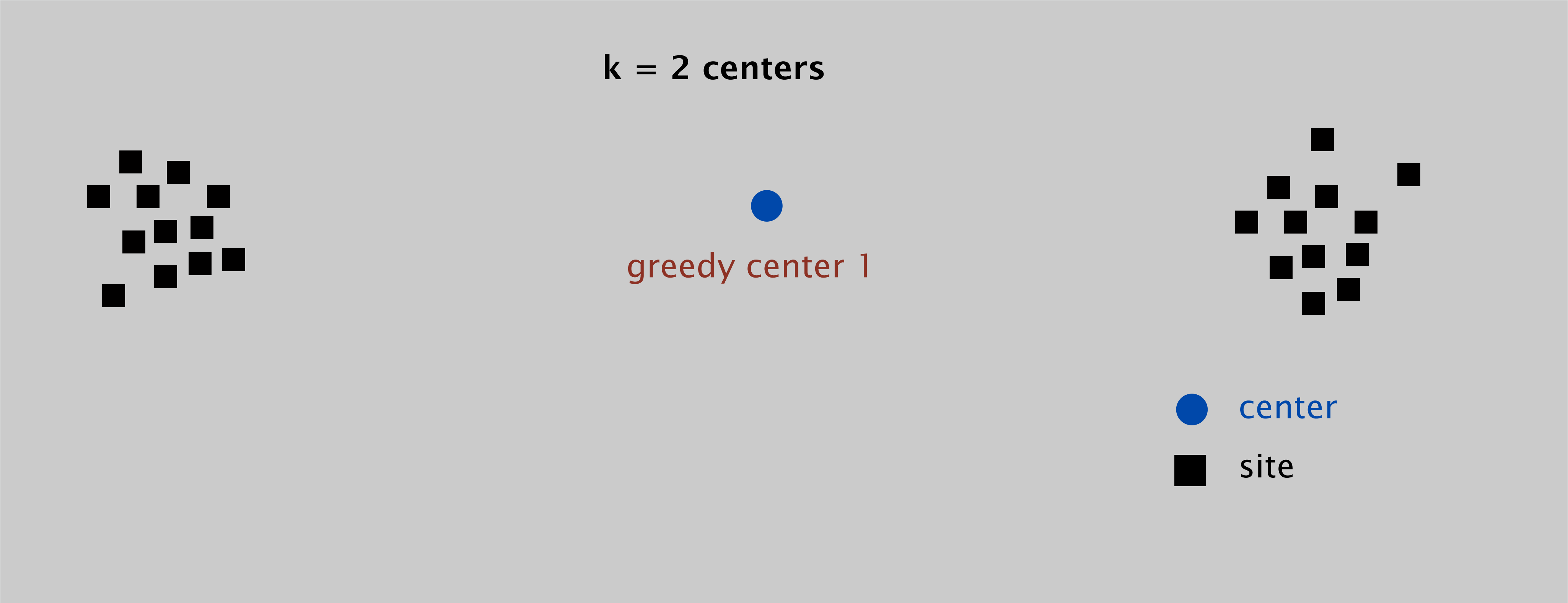
Remark: search can be infinite!



Greedy algorithm: a false start

Greedy algorithm. Put the first center at the best possible location for a single center, and then keep adding centers so as to reduce the covering radius each time by as much as possible.

Remark: arbitrarily bad!



Center selection: greedy algorithm

Repeatedly choose next center to be site farthest from any existing center.

```
GREEDY-CENTER-SELECTION $(k, n, s_1, s_2, \ldots, s_n)$

$C \leftarrow \varnothing$.
REPEAT k times
    Select a site $s_i$ with maximum distance $dist(s_i, C)$.
    $C \leftarrow C \cup s_i$.
RETURN $C$.
```

site farthest from any center

Property. Upon termination, all centers in C are pairwise at least $r(C)$ apart.

Pf. By construction of algorithm.

Center selection: analysis of greedy algorithm

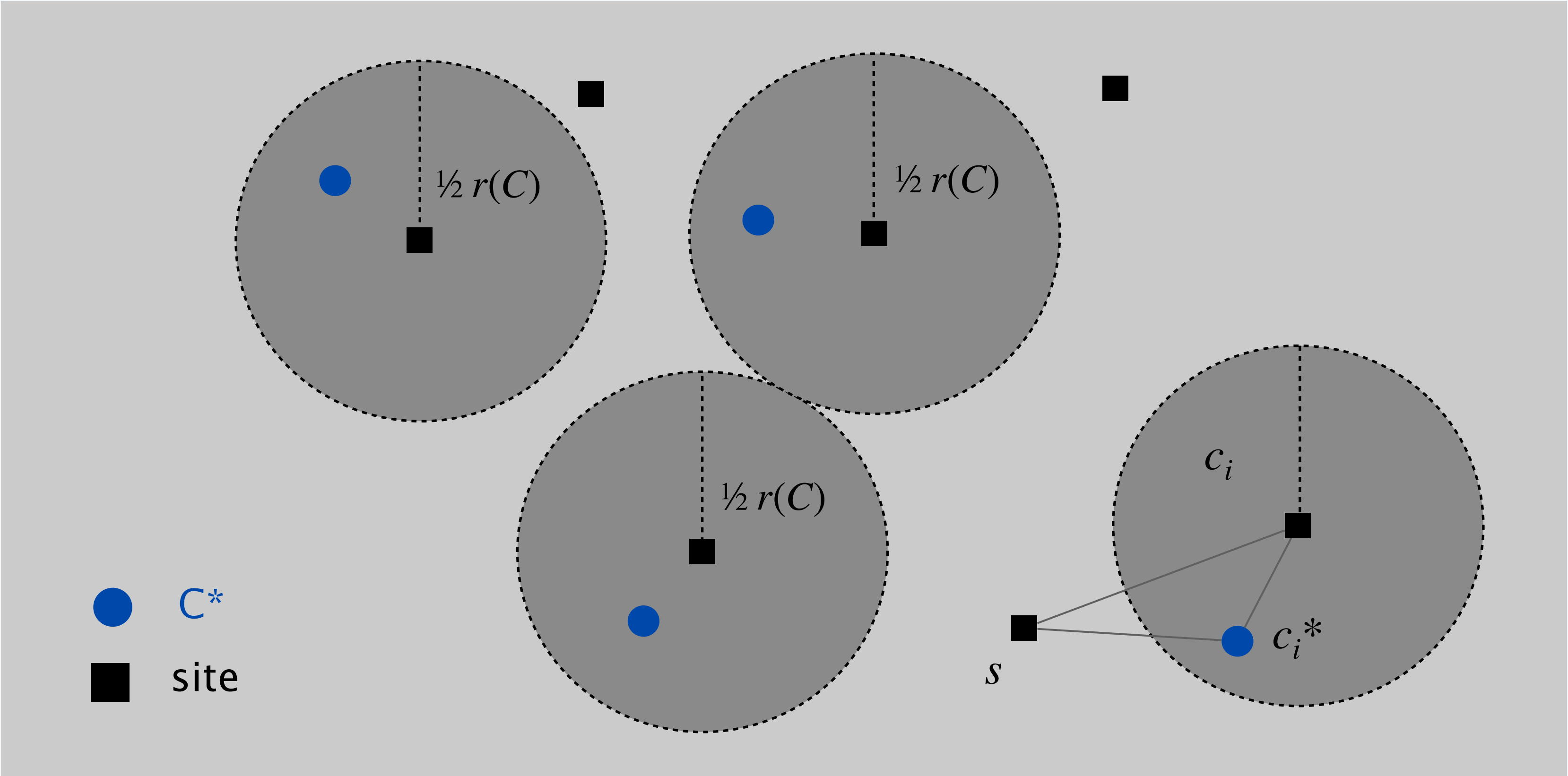
Lemma. Let C^* be an optimal set of centers. Then $r(C) \le 2r(C^*)$.

Pf. [by contradiction] Assume $r(C^*) < \frac{1}{2}\, r(C)$.

- For each site $c_i \in C$, consider ball of radius $\frac{1}{2}\, r(C)$ around it.
- Exactly one c_i^* in each ball; let c_i be the site paired with c_i^*.
- Consider any site s and its closest center $c_i^* \in C^*$.
- $dist(s, C) \le dist(s, c_i) \le dist(s, c_i^*) + dist(c_i^*, c_i) \le 2r(C^*)$.
- Thus, $r(C) \le 2r(C^*)$. ▪

Δ-inequality

≤ r(C*) since c_i* is closest center



Center selection

Lemma. Let C^* be an optimal set of centers. Then $r(C) \le 2r(C^*)$.

Theorem. Greedy algorithm is a 2-approximation for center selection problem.

Remark. Greedy algorithm always places centers at sites, but is still within a factor of 2 of best solution that is allowed to place centers anywhere.

e.g., points in the plane

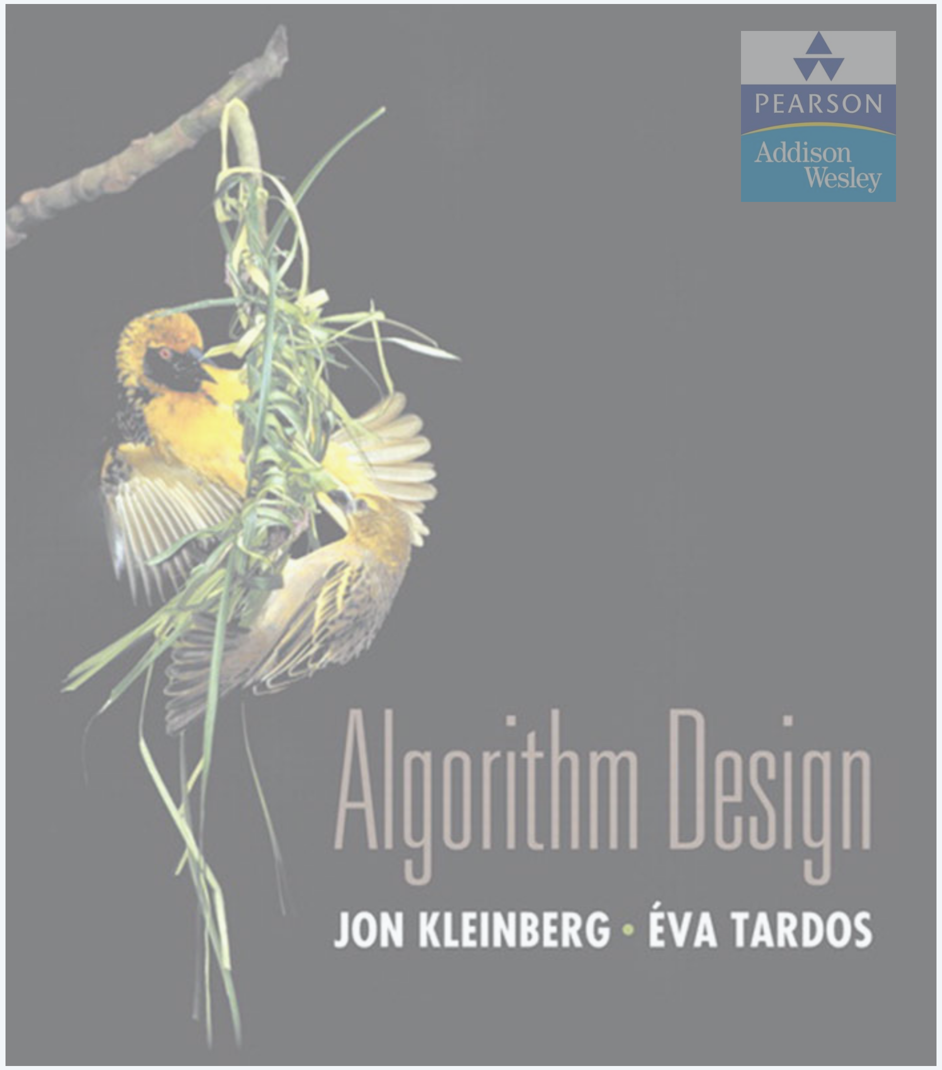
Question. Is there hope of a 3/2-approximation? 4/3?

Dominating set reduces to center selection

Theorem. Unless **P** = **NP**, there no ρ-approximation for center selection problem for any ρ < 2.

Pf. We show how we could use a $(2 - \varepsilon)$ approximation algorithm for CENTER-SELECTION selection to solve DOMINATING-SET in poly-time.

- Let $G = (V, E)$, k be an instance of DOMINATING-SET.
- Construct instance G' of CENTER-SELECTION with sites V and distances
  - $dist(u, v) = 1$ if $(u, v) \in E$
  - $dist(u, v) = 2$ if $(u, v) \notin E$
- Note that G' satisfies the triangle inequality.
- G has dominating set of size k iff there exists k centers C^* with $r(C^*) = 1$.
- Thus, if G has a dominating set of size k, a $(2 - \varepsilon)$-approximation algorithm for CENTER-SELECTION would find a solution C^* with $r(C^*) = 1$ since it cannot use any edge of distance 2. ▪



SECTION 11.4

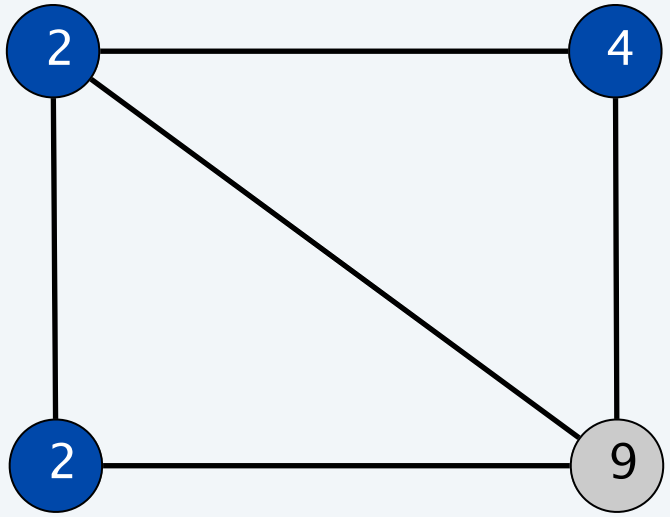
11. APPROXIMATION ALGORITHMS

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

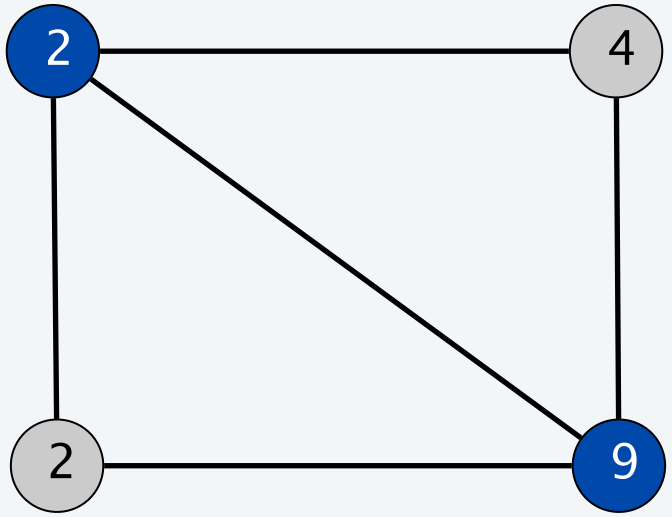
Weighted vertex cover

Definition. Given a graph $G = (V, E)$, a vertex cover is a set $S \subseteq V$ such that each edge in E has at least one end in S.

Weighted vertex cover. Given a graph G with vertex weights, find a vertex cover of minimum weight.



weight = 2 + 2 + 4



weight = 11

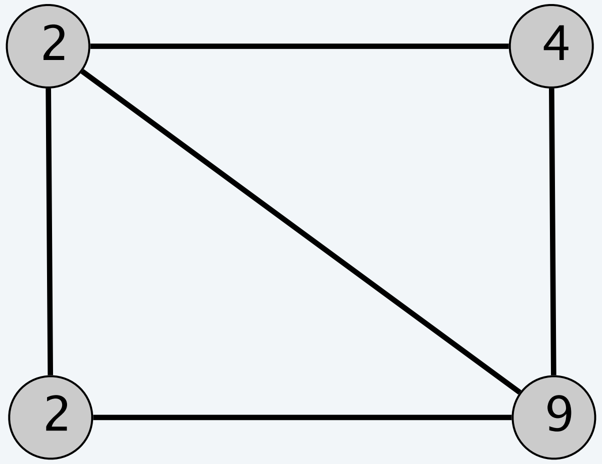
Pricing method

Pricing method. Each edge must be covered by some vertex.

Edge $e = (i, j)$ pays price $p_e \geq 0$ to use both vertex i and j.

Fairness. Edges incident to vertex i should pay $\leq w_i$ in total.

$$\text{for each vertex } i : \sum_{e=(i,j)} p_e \leq w_i$$



Fairness lemma. For any vertex cover S and any fair prices p_e : $\sum_e p_e \leq w(S)$.

Pf.

$$\sum_{e \in E} p_e \underset{\substack{\uparrow \\ \text{each edge e covered by} \\ \text{at least one node in S}}}{\leq} \sum_{i \in S} \sum_{e=(i,j)} p_e \underset{\substack{\uparrow \\ \text{sum fairness inequalities} \\ \text{for each node in S}}}{\leq} \sum_{i \in S} w_i = w(S). \blacksquare$$

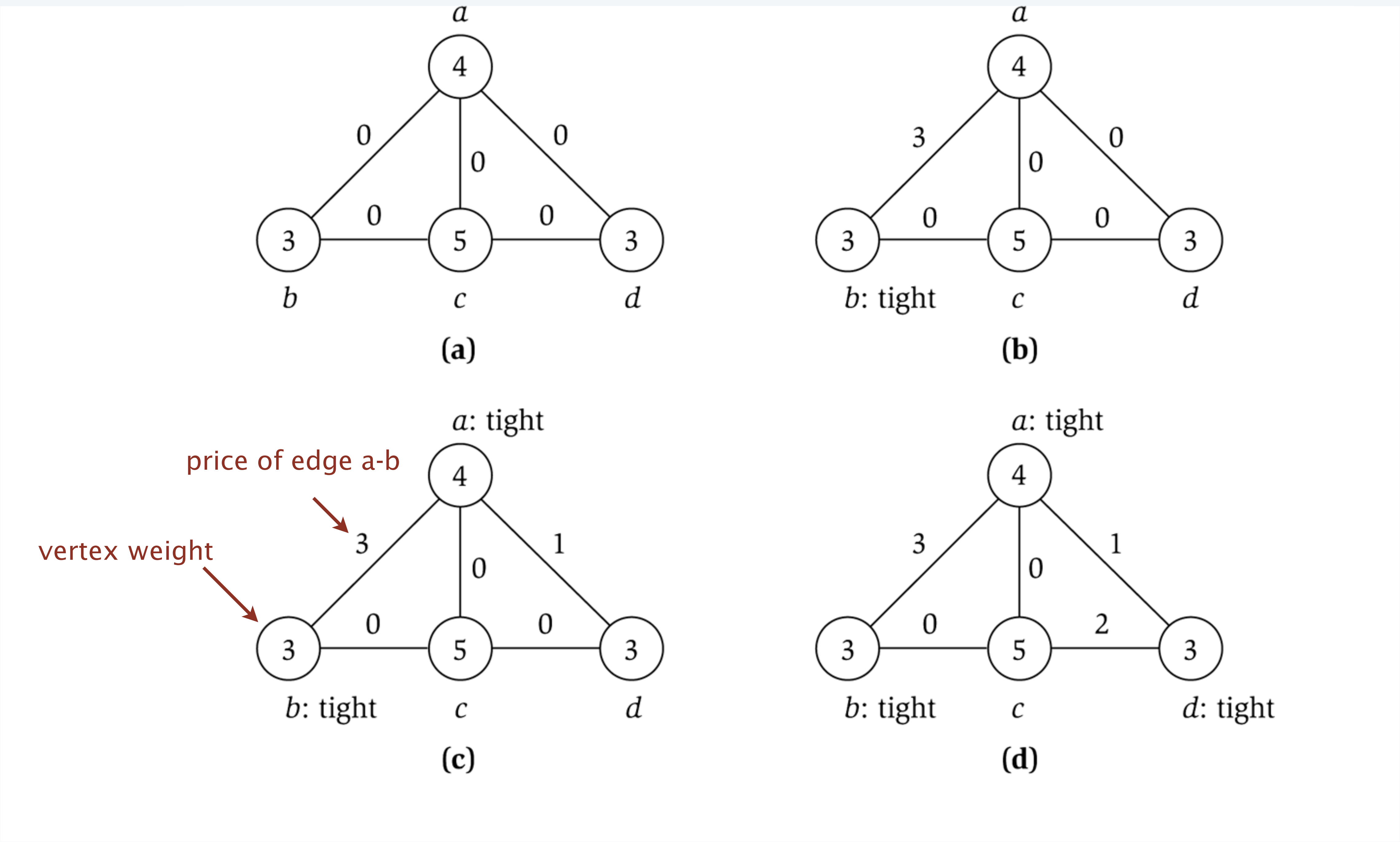
Pricing method

Set prices and find vertex cover simultaneously.

```
WEIGHTED-VERTEX-COVER $(G, w)$

$S \leftarrow \varnothing$.
FOREACH $e \in E$
    $p_e \leftarrow 0$.

WHILE (there exists an edge $(i, j)$ such that neither $i$ nor $j$ is tight)
    Select such an edge $e = (i, j)$.
    Increase $p_e$ as much as possible until $i$ or $j$ tight.

$S \leftarrow$ set of all tight nodes.
RETURN $S$.
```

$$\sum_{e = (i, j)} p_e = w_i$$

Pricing method example



Pricing method: analysis

Theorem. Pricing method is a 2-approximation for Weighted-Vertex-Cover.

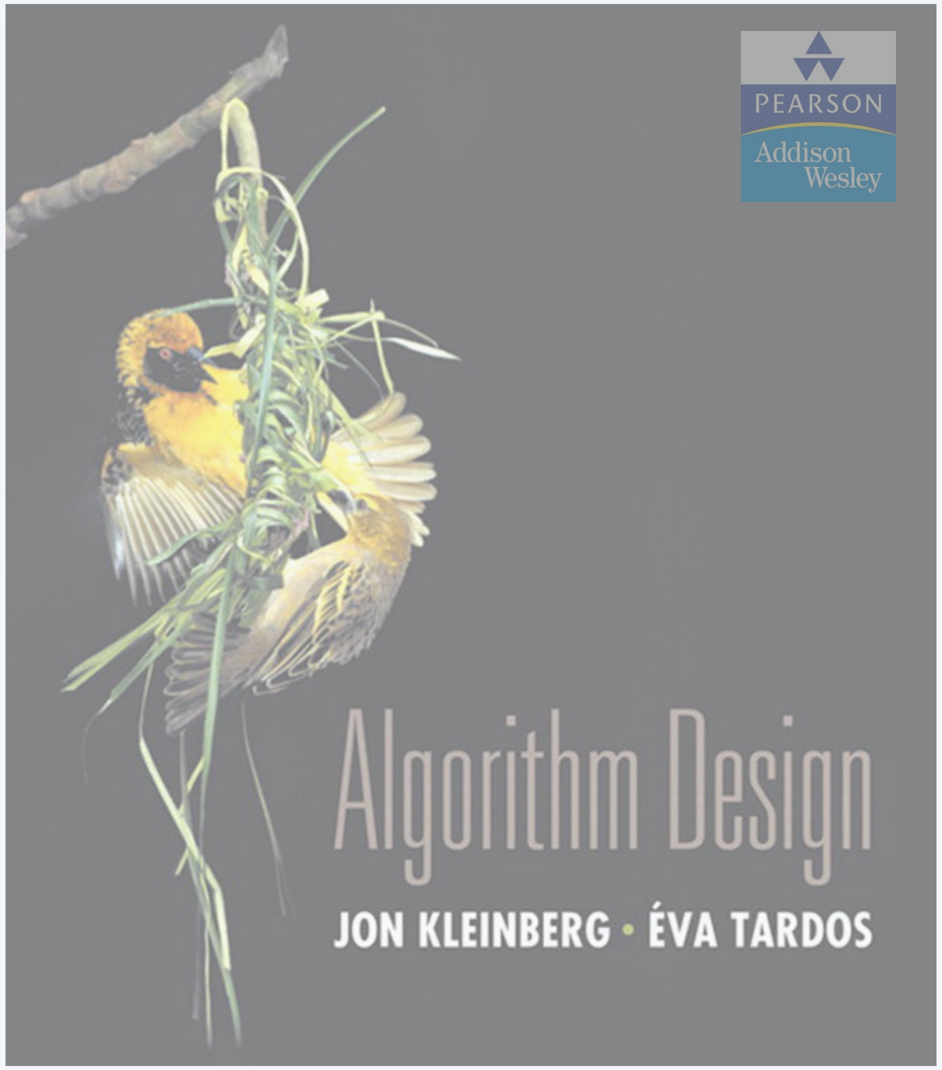
Pf.

- Algorithm terminates since at least one new node becomes tight after each iteration of while loop.

- Let S = set of all tight nodes upon termination of algorithm. S is a vertex cover: if some edge (i, j) is uncovered, then neither i nor j is tight. But then while loop would not terminate.

- Let S^* be optimal vertex cover. We show $w(S) \le 2\, w(S^*)$.

$$w(S) = \sum_{i \in S} w_i \underset{\substack{\uparrow \\ \text{all nodes in S are tight}}}{=} \sum_{i \in S} \; \sum_{e=(i,j)} p_e \underset{\substack{\uparrow \\ S \subseteq V, \\ \text{prices} \ge 0}}{\le} \sum_{i \in V} \; \sum_{e=(i,j)} p_e \underset{\substack{\uparrow \\ \text{each edge counted twice}}}{=} 2 \sum_{e \in E} p_e \underset{\substack{\uparrow \\ \text{fairness lemma}}}{\le} 2w(S^*). \quad ▪$$



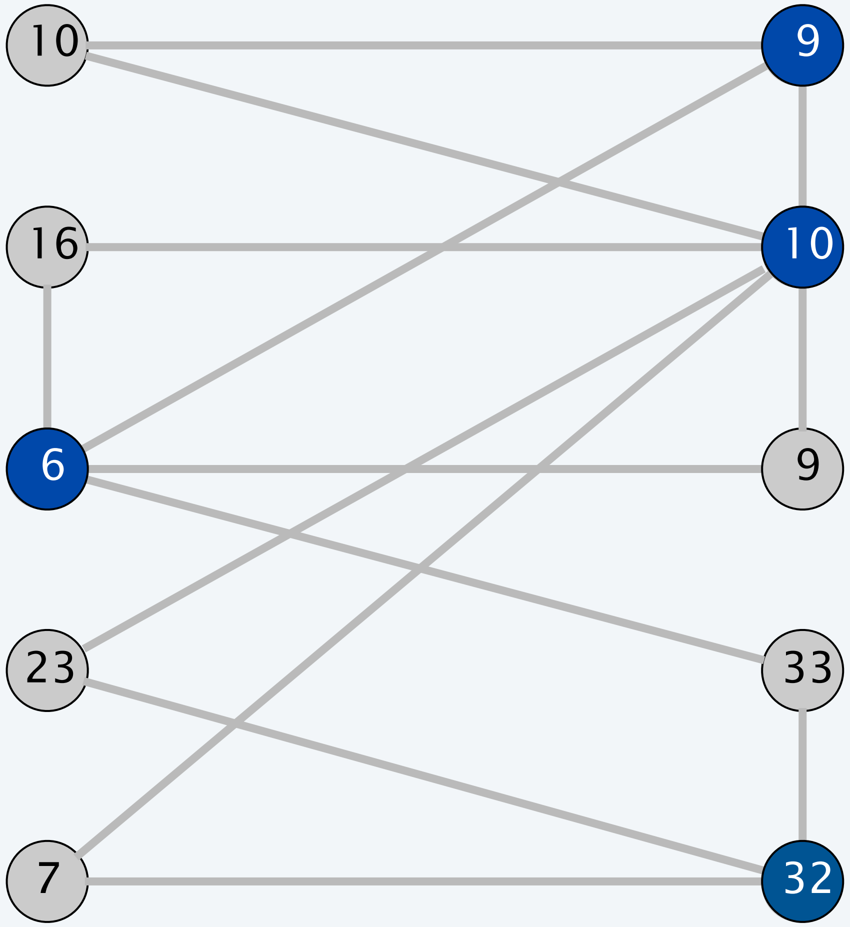
SECTION 11.6

11. APPROXIMATION ALGORITHMS

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

Weighted vertex cover

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S.



total weight = 6 + 9 + 10 + 32 = 57

Weighted vertex cover: ILP formulation

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S.

Integer linear programming formulation.

- Model inclusion of each vertex i using a 0/1 variable x_i.

$$x_i = \begin{cases} 0 & \text{if vertex } i \text{ is not in vertex cover} \\ 1 & \text{if vertex } i \text{ is in vertex cover} \end{cases}$$

  Vertex covers in 1–1 correspondence with 0/1 assignments:
  $S = \{ i \in V : x_i = 1 \}$.

- Objective function: minimize $\Sigma_i w_i x_i$.

- For every edge (i, j), must take either vertex i or j (or both): $x_i + x_j \geq 1$.

Weighted vertex cover: ILP formulation

Weighted vertex cover. Integer linear programming formulation.

$$
\begin{array}{llrcll}
(ILP) & \min & \displaystyle\sum_{i \in V} w_i \, x_i & & & \\
 & \text{s.t.} & x_i + x_j & \geq & 1 & (i,j) \in E \\
 & & x_i & \in & \{0,\ 1\} & i \in V
\end{array}
$$

Observation. If x^* is optimal solution to ILP, then $S = \{\, i \in V : x_i^* = 1\}$ is a min-weight vertex cover.

Integer linear programming

Given integers a_{ij}, b_i, and c_j, find integers x_j that satisfy:

$$\begin{array}{rrcl} \min & c^{\mathsf{T}} x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \\ & x & & \text{integral} \end{array}$$

$$\begin{array}{rrcll} \min & \displaystyle\sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & 1 \leq i \leq m \\ & x_j & \geq & 0 & 1 \leq j \leq n \\ & x_j & & \text{integral} & 1 \leq j \leq n \end{array}$$

Observation. Vertex cover formulation proves that INTEGER-PROGRAMMING is an **NP**-hard optimization problem.

Linear programming

Given integers a_{ij}, b_i, and c_j, find real numbers x_j that satisfy:

$$\begin{array}{lrcl} \min & c^{\mathsf{T}} x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrclr} \min & \displaystyle\sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & 1 \leq i \leq m \\ & x_j & \geq & 0 & 1 \leq j \leq n \end{array}$$

Linear. No x^2, xy, $\arccos(x)$, $x(1-x)$, etc.

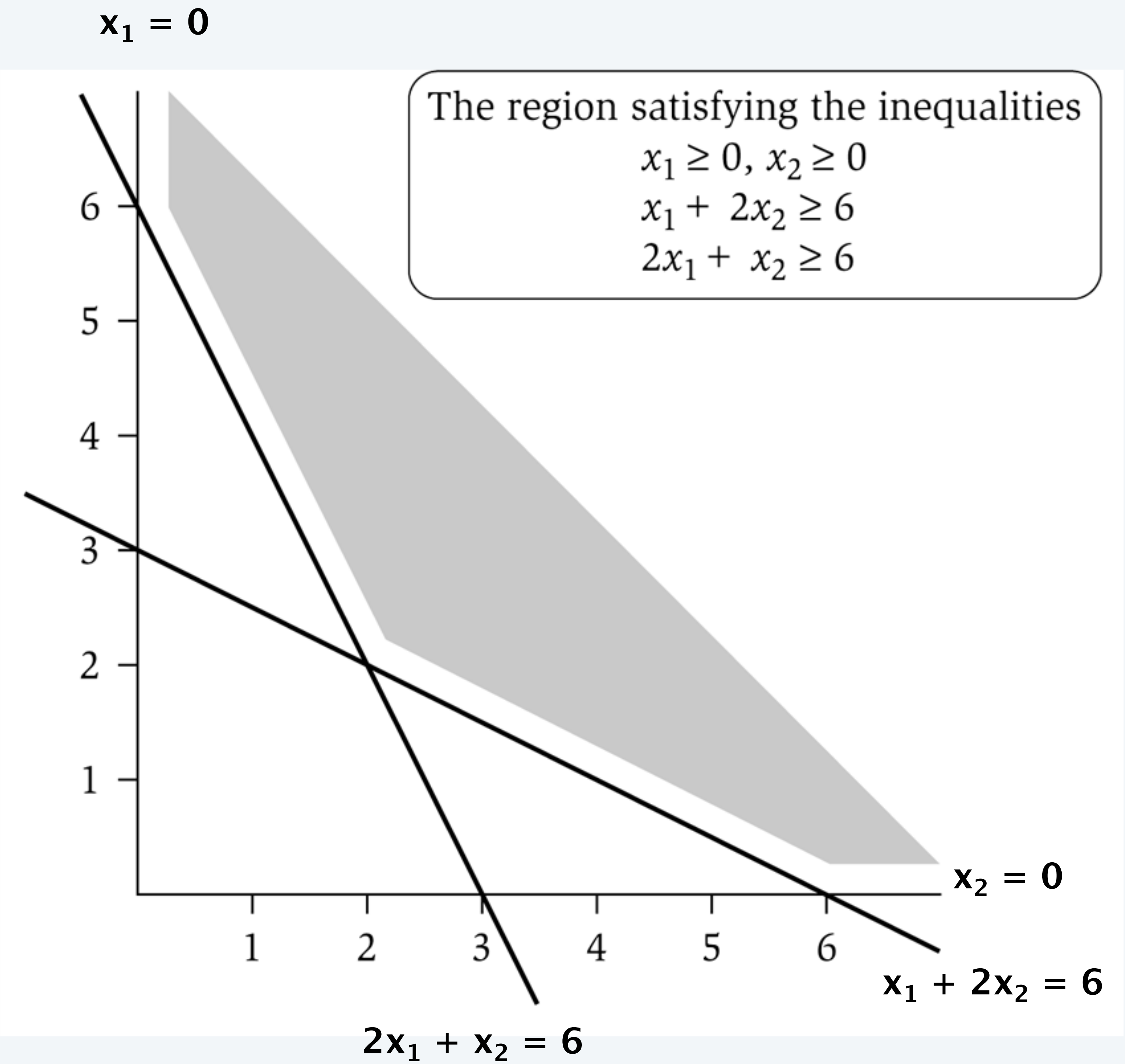
Simplex algorithm. [Dantzig 1947] Can solve LP in practice.

Ellipsoid algorithm. [Khachiyan 1979] Can solve LP in poly-time.

Interior point algorithms. [Karmarkar 1984, Renegar 1988, …]
Can solve LP both in poly-time and in practice.

LP feasible region

LP geometry in 2D.



The region satisfying the inequalities

$$x_1 \geq 0,\ x_2 \geq 0$$
$$x_1 + 2x_2 \geq 6$$
$$2x_1 + x_2 \geq 6$$

Weighted vertex cover: LP relaxation

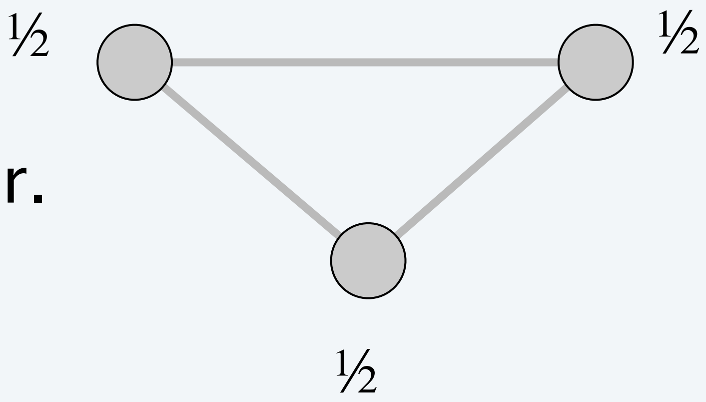
Linear programming relaxation.

$$
\begin{array}{llrcll}
(LP) & \min & \displaystyle\sum_{i \in V} w_i \, x_i & & & \\
 & \text{s.t.} & x_i + x_j & \geq & 1 & (i,j) \in E \\
 & & x_i & \geq & 0 & i \in V
\end{array}
$$

Observation. Optimal value of *LP* is ≤ optimal value of *ILP*.

Pf. *LP* has fewer constraints.

Note. *LP* solution x^* may not correspond to a vertex cover.
(even if all weights are 1)



Q. How can solving *LP* help us find a low-weight vertex cover?

A. Solve *LP* and round fractional values in x^*.

Weighted vertex cover: LP rounding algorithm

Lemma. If x^* is optimal solution to LP, then $S = \{ i \in V : x_i^* \geq ½ \}$ is a vertex cover whose weight is at most twice the min possible weight.

Pf. [S is a vertex cover]

- Consider an edge $(i, j) \in E$.
- Since $x_i^* + x_j^* \geq 1$, either $x_i^* \geq ½$ or $x_j^* \geq ½$ (or both) $\Rightarrow$ (i, j) covered.

Pf. [S has desired weight]

- Let S^* be optimal vertex cover. Then

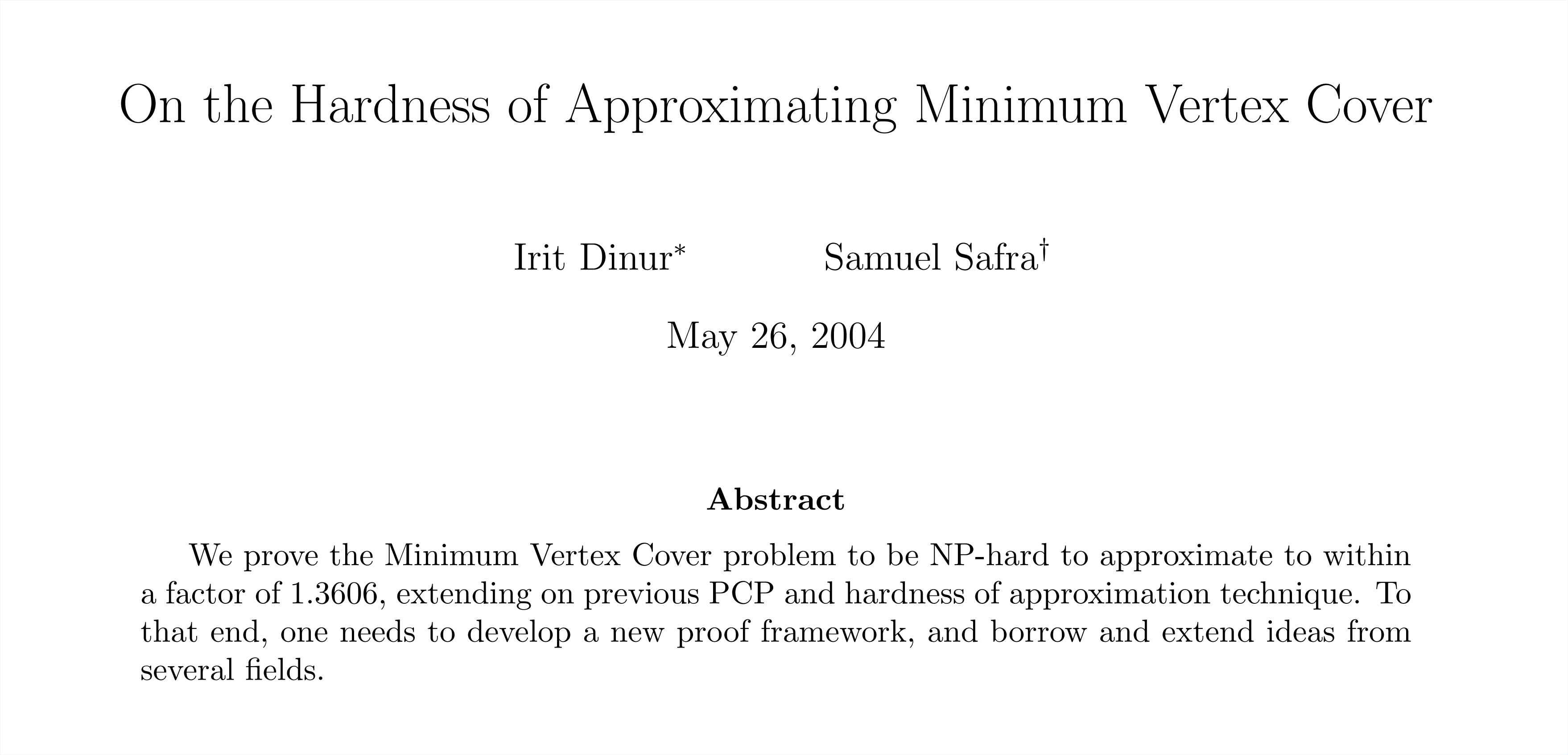
$$\sum_{i \in S^*} w_i \underset{\substack{\uparrow \\ \text{LP is a relaxation}}}{\geq} \sum_{i \in S} w_i \, x_i^* \underset{\substack{\uparrow \\ x_i^* \, \geq \, ½}}{\geq} \frac{1}{2} \sum_{i \in S} w_i$$

Theorem. The rounding algorithm is a 2-approximation algorithm.

Pf. Lemma + fact that LP can be solved in poly-time.

Weighted vertex cover inapproximability

Theorem. [Dinur–Safra 2004] If **P** ≠ **NP**, then no ρ-approximation algorithm for Weighted-Vertex-Cover for any $\rho < 1.3606$ (even if all weights are 1).



On the Hardness of Approximating Minimum Vertex Cover

Irit Dinur[*] Samuel Safra[†]

May 26, 2004

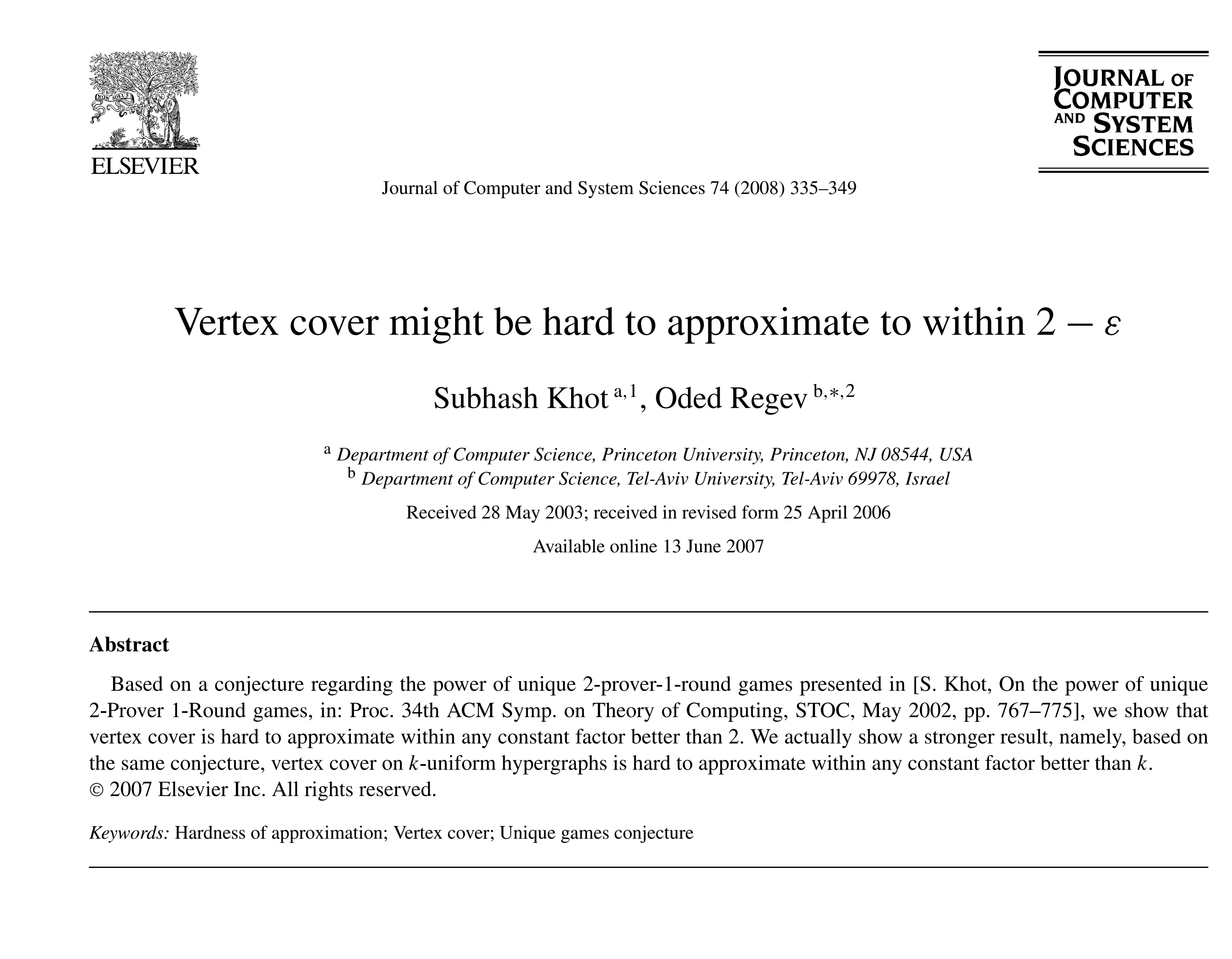
Abstract

We prove the Minimum Vertex Cover problem to be NP-hard to approximate to within a factor of 1.3606, extending on previous PCP and hardness of approximation technique. To that end, one needs to develop a new proof framework, and borrow and extend ideas from several fields.

Open research problem. Close the gap.

Weighted vertex cover inapproximability

Theorem. [Kohot–Regev 2008] If Unique Games Conjecture is true, then no $2 - \varepsilon$ approximation algorithm for WEIGHTED-VERTEX-COVER for any $\varepsilon > 0$.



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Vertex cover might be hard to approximate to within $2 - \varepsilon$

Subhash Khot [a,1], Oded Regev [b,*,2]

[a] *Department of Computer Science, Princeton University, Princeton, NJ 08544, USA*
[b] *Department of Computer Science, Tel-Aviv University, Tel-Aviv 69978, Israel*

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Abstract

Based on a conjecture regarding the power of unique 2-prover-1-round games presented in [S. Khot, On the power of unique 2-Prover 1-Round games, in: Proc. 34th ACM Symp. on Theory of Computing, STOC, May 2002, pp. 767–775], we show that vertex cover is hard to approximate within any constant factor better than 2. We actually show a stronger result, namely, based on the same conjecture, vertex cover on k-uniform hypergraphs is hard to approximate within any constant factor better than k.

Keywords: Hardness of approximation; Vertex cover; Unique games conjecture

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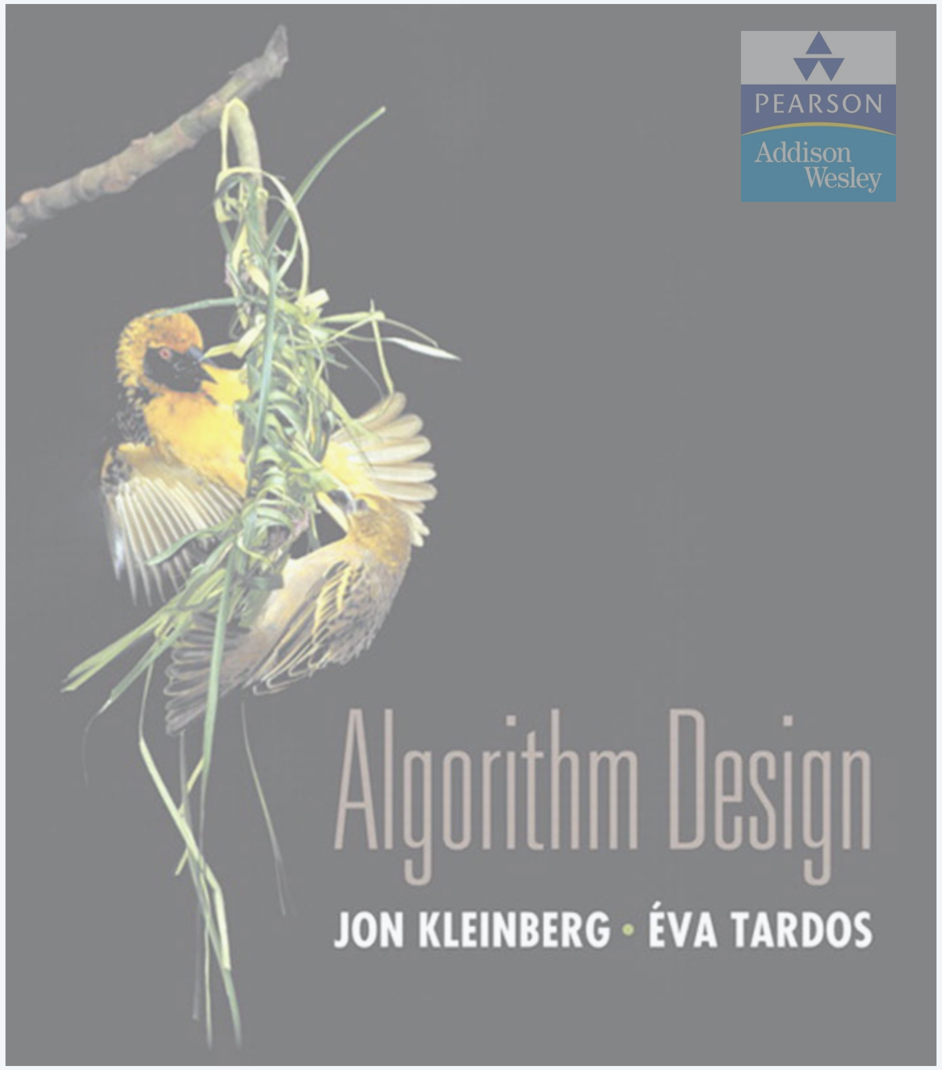
Open research problem. Prove the Unique Games Conjecture.



SECTION 11.7

11. APPROXIMATION ALGORITHMS

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

Generalized load balancing

Input. Set of m machines M; set of n jobs J.

- Job $j \in J$ must run contiguously on an **authorized machine** in $M_j \subseteq M$.
- Job $j \in J$ has processing time t_j.
- Each machine can process at most one job at a time.

Def. Let J_i be the subset of jobs assigned to machine i.
The load of machine i is $L_i = \Sigma_{j \in J_i} t_j$.

Def. The makespan is the maximum load on any machine $= \max_i L_i$.

Generalized load balancing. Assign each job to an authorized machine to minimize makespan.

Generalized load balancing: integer linear program and relaxation

ILP formulation. x_{ij} = time machine i spends processing job j.

$$
\begin{array}{llllll}
(IP) & \min & L & & & \\
& \text{s. t.} & \sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
& & \sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
& & x_{ij} & \in & \{0,\ t_j\} & \text{for all } j \in J \text{ and } i \in M_j \\
& & x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}
$$

LP relaxation.

$$
\begin{array}{llllll}
(LP) & \min & L & & & \\
& \text{s. t.} & \sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
& & \sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
& & x_{ij} & \geq & 0 & \text{for all } j \in J \text{ and } i \in M_j \\
& & x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}
$$

Generalized load balancing: lower bounds

Lemma 1. The optimal makespan $L^* \geq \max_j t_j$.

Pf. Some machine must process the most time-consuming job. ▪

Lemma 2. Let L be optimal value to the LP. Then, optimal makespan $L^* \geq L$.

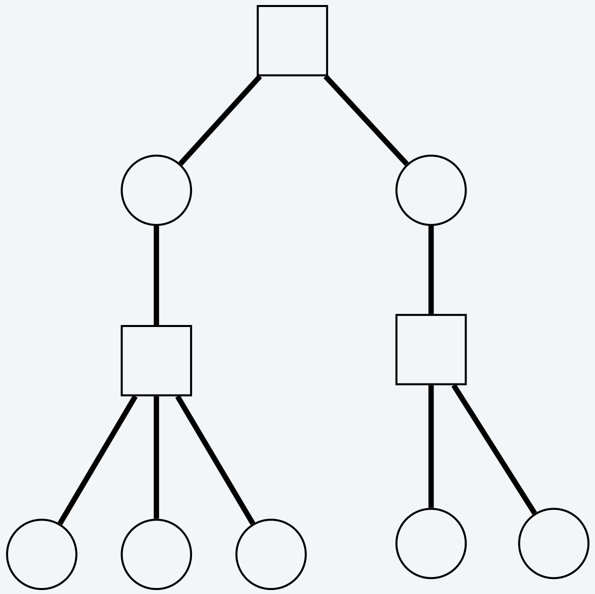
Pf. LP has fewer constraints than ILP formulation. ▪

Generalized load balancing: structure of LP solution

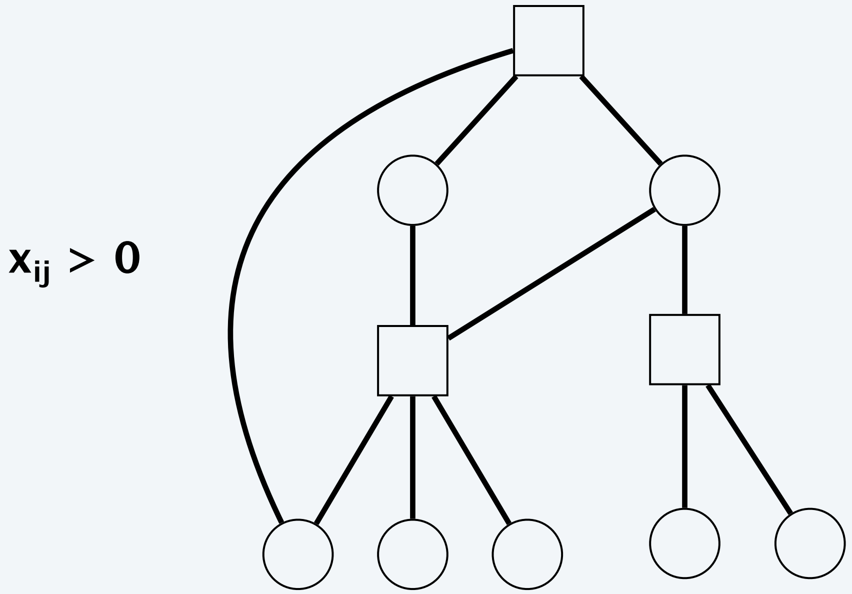
Lemma 3. Let x be solution to LP. Let $G(x)$ be the graph with an edge between machine i and job j if $x_{ij} > 0$. Then $G(x)$ is acyclic.

can transform x into another LP solution where G(x) is acyclic if LP solver doesn't return such an x

Pf. (deferred)



G(x) acyclic



G(x) cyclic

○ job

□ machine

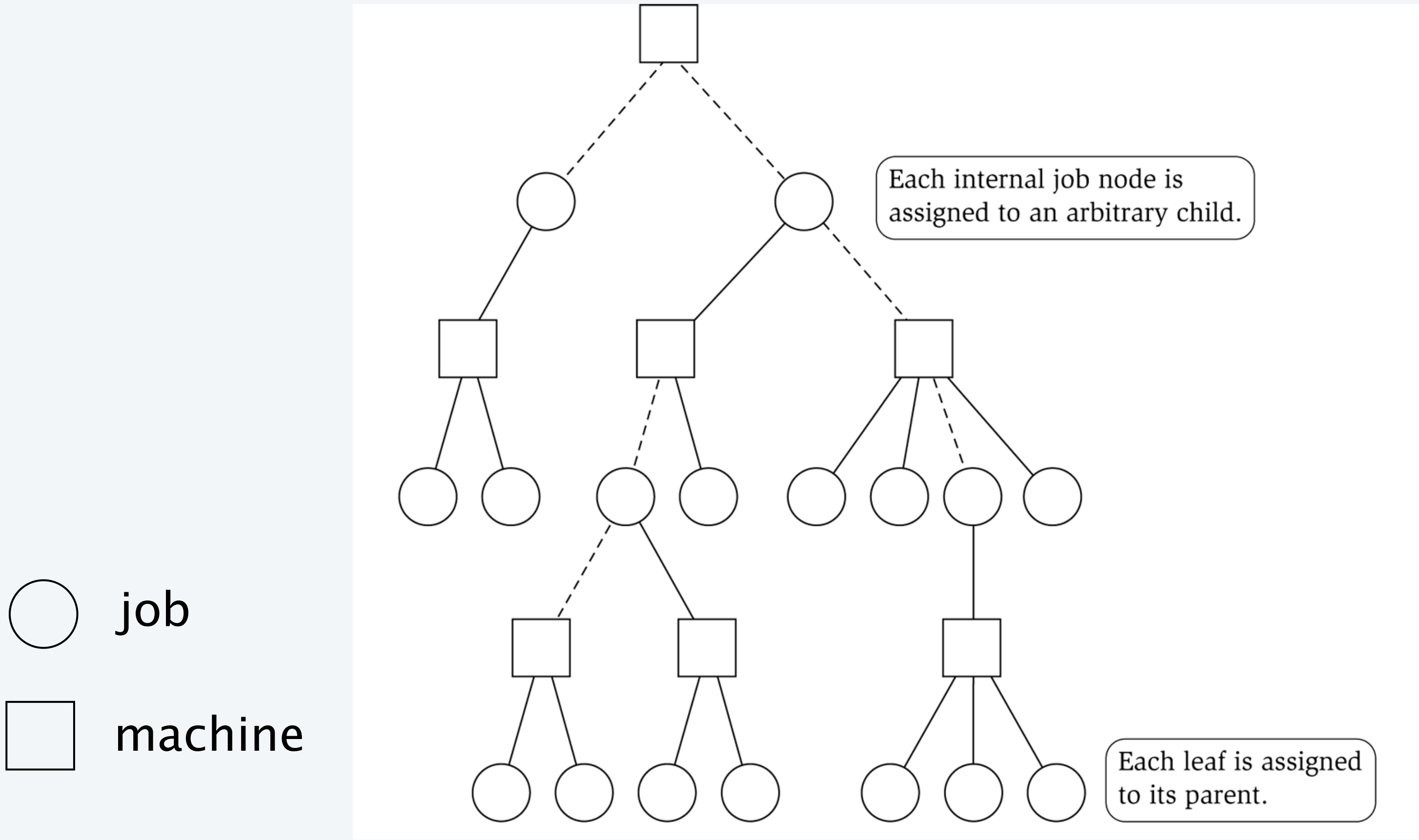
Generalized load balancing: rounding

Rounded solution. Find LP solution x where $G(x)$ is a forest. Root forest $G(x)$ at some arbitrary machine node r.

- If job j is a leaf node, assign j to its parent machine i.
- If job j is not a leaf node, assign j to any one of its children.

Lemma 4. Rounded solution only assigns jobs to authorized machines.

Pf. If job j is assigned to machine i, then $x_{ij} > 0$. LP solution can only assign positive value to authorized machines. ▪



Each internal job node is assigned to an arbitrary child.

Each leaf is assigned to its parent.

Generalized load balancing: analysis

Lemma 5. If job j is a leaf node and machine $i = parent(j)$, then $x_{ij} = t_j$.

Pf.

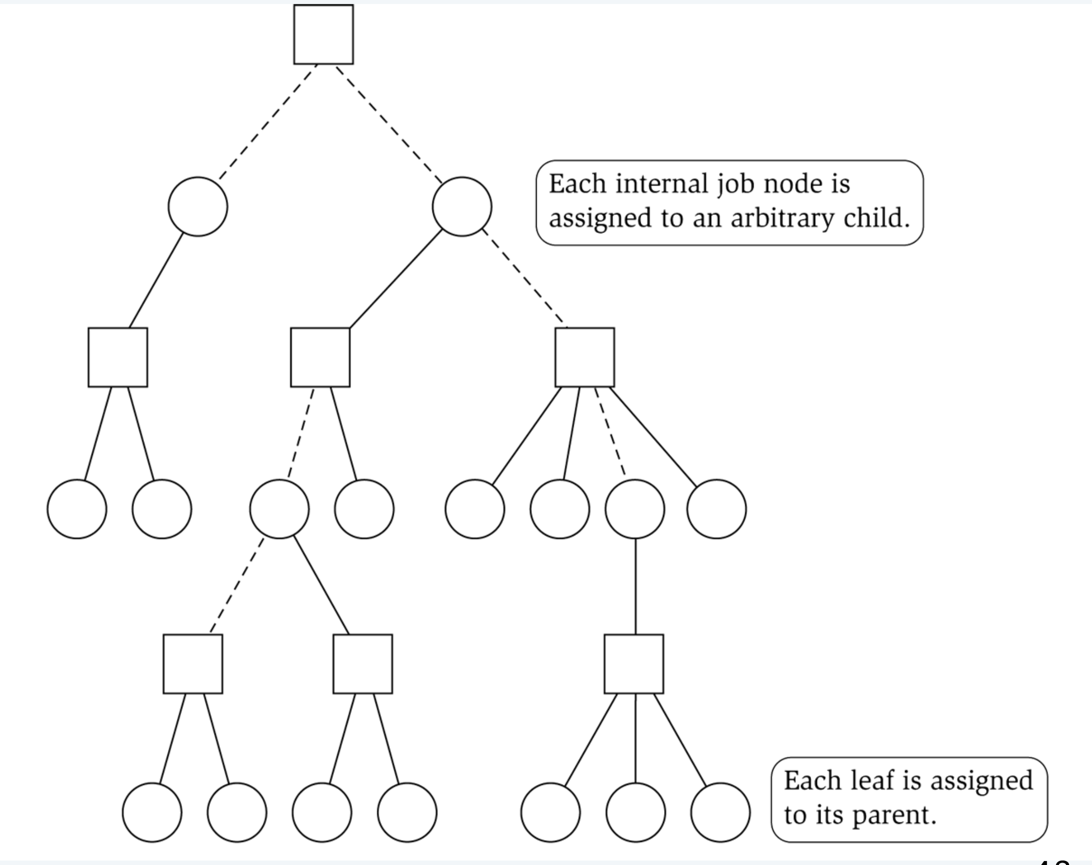
- Since i is a leaf, $x_{ij} = 0$ for all $j \neq parent(i)$.
- LP constraint guarantees $\Sigma_i \, x_{ij} = t_j$. ▪

Lemma 6. At most one non-leaf job is assigned to a machine.

Pf. The only possible non-leaf job assigned to machine i is $parent(i)$. ▪

○ job

□ machine



Each internal job node is assigned to an arbitrary child.

Each leaf is assigned to its parent.

Generalized load balancing: analysis

Theorem. Rounded solution is a 2-approximation.

Pf.

- Let $J(i)$ be the jobs assigned to machine i.
- By Lemma 6, the load L_i on machine i has two components:
  - leaf nodes:

$$\sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} t_j \overset{\text{Lemma 5}}{\downarrow =} \sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} x_{ij} \le \sum_{j \in J} x_{ij} \overset{\text{LP}}{\downarrow \le} \underset{\text{optimal value of LP}}{\overset{}{\underset{\uparrow}{L}}} \overset{\text{Lemma 2 (LP is a relaxation)}}{\downarrow \le} L^*$$

  - parent:

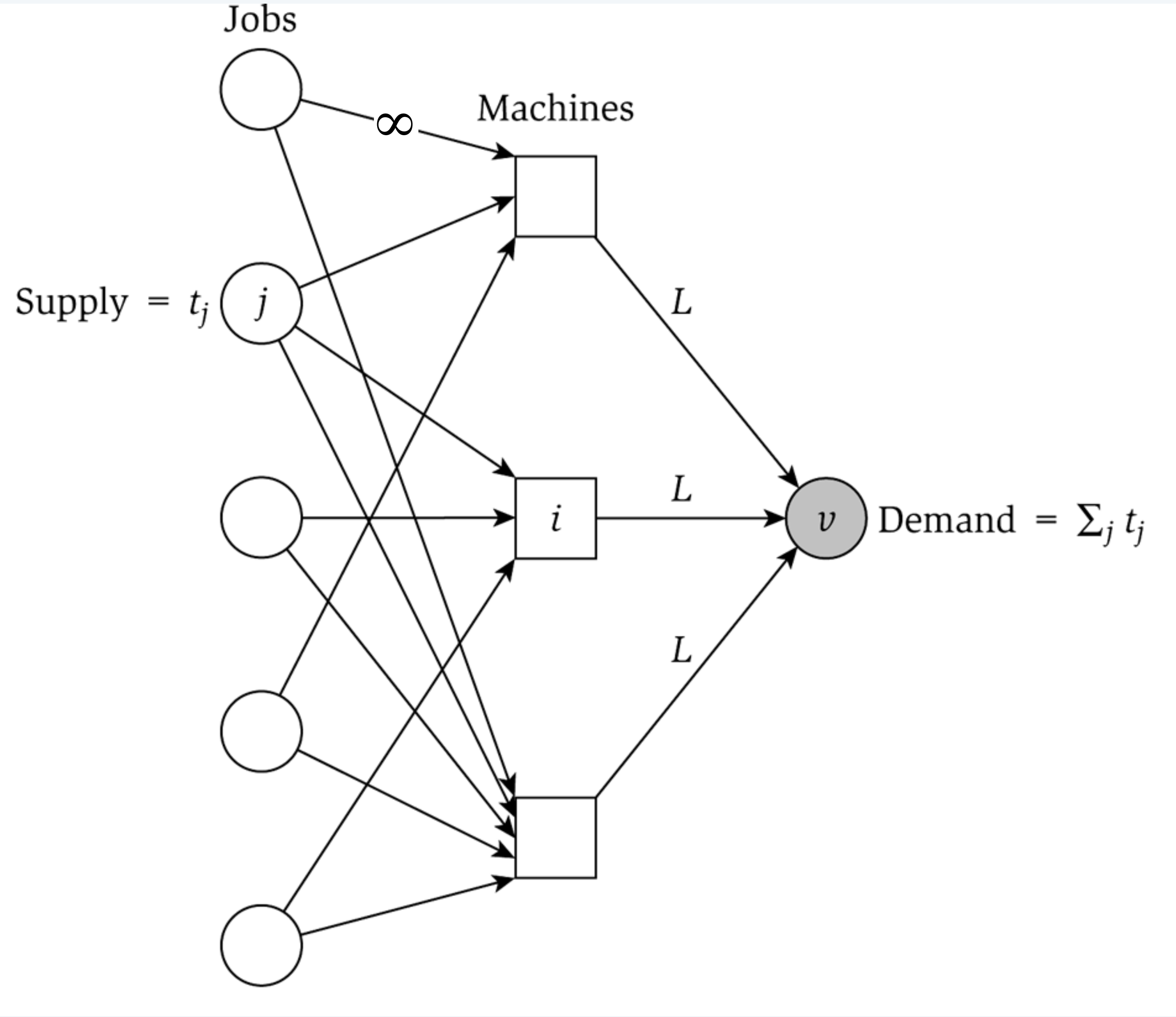
$$t_{\text{parent}(i)} \overset{\text{Lemma 1}}{\downarrow \le} L^*$$

- Thus, the overall load $L_i \le 2L^*$. ▪

Generalized load balancing: flow formulation

Flow formulation of LP.

$$
\begin{array}{llll}
\sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
\sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
x_{ij} & \geq & 0 & \text{for all } j \in J \text{ and } i \in M_j \\
x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}
$$



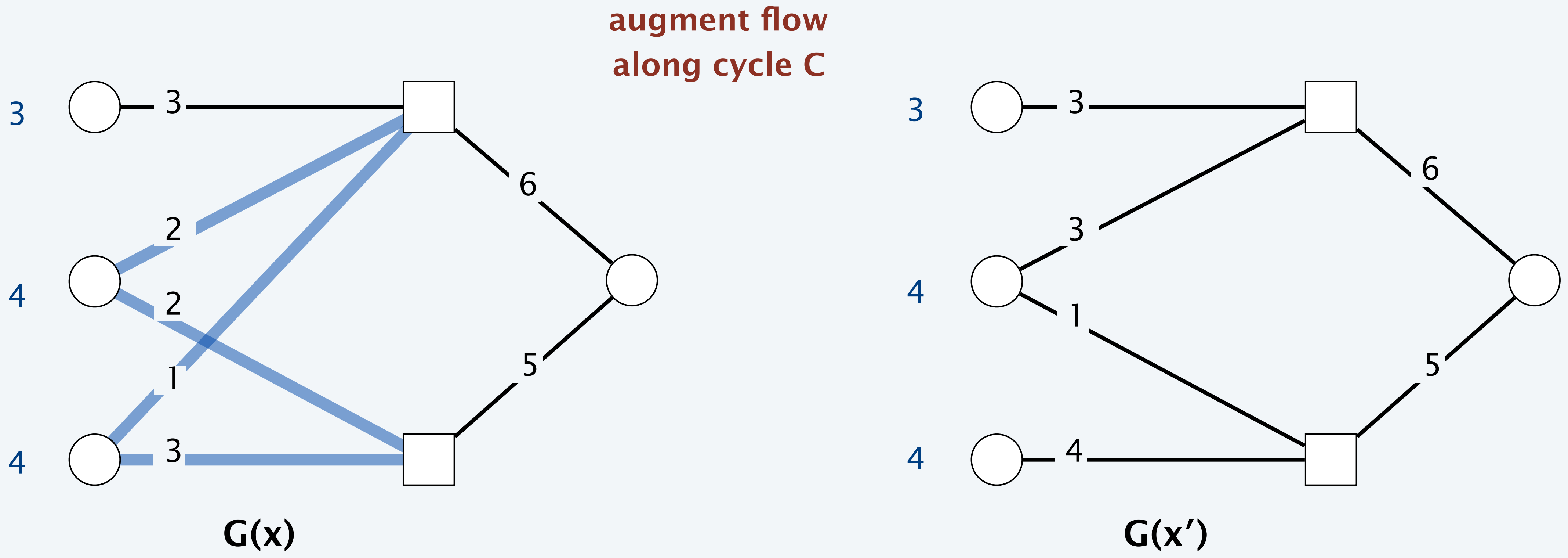
Observation. Solution to feasible flow problem with value L are in 1-to-1 correspondence with LP solutions of value L.

Generalized load balancing: structure of solution

Lemma 3. Let (x, L) be solution to LP. Let $G(x)$ be the graph with an edge from machine i to job j if $x_{ij} > 0$. We can find another solution (x', L) such that $G(x')$ is acyclic.

Pf. Let C be a cycle in $G(x)$.

- Augment flow along the cycle C. ← flow conservation maintained
- At least one edge from C is removed (and none are added).
- Repeat until $G(x')$ is acyclic. ▪



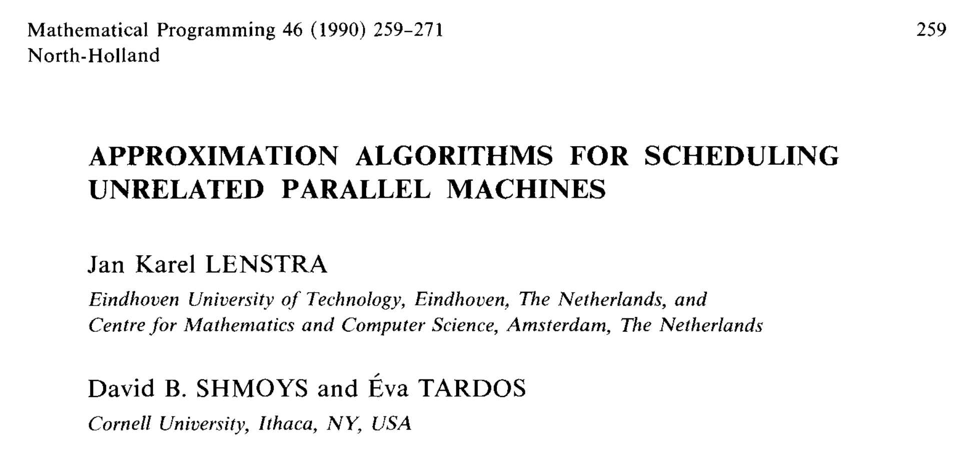
Conclusions

Running time. The bottleneck operation in our 2-approximation is solving one LP with $mn+1$ variables.

Remark. Can solve LP using flow techniques on a graph with $m+n+1$ nodes: given L, find feasible flow if it exists. Binary search to find L^*.

Extensions: unrelated parallel machines. [Lenstra–Shmoys–Tardos 1990]

- Job j takes t_{ij} time if processed on machine i.
- 2-approximation algorithm via LP rounding.
- If **P** ≠ **NP**, then no no ρ-approximation exists for any $\rho < 3/2$.

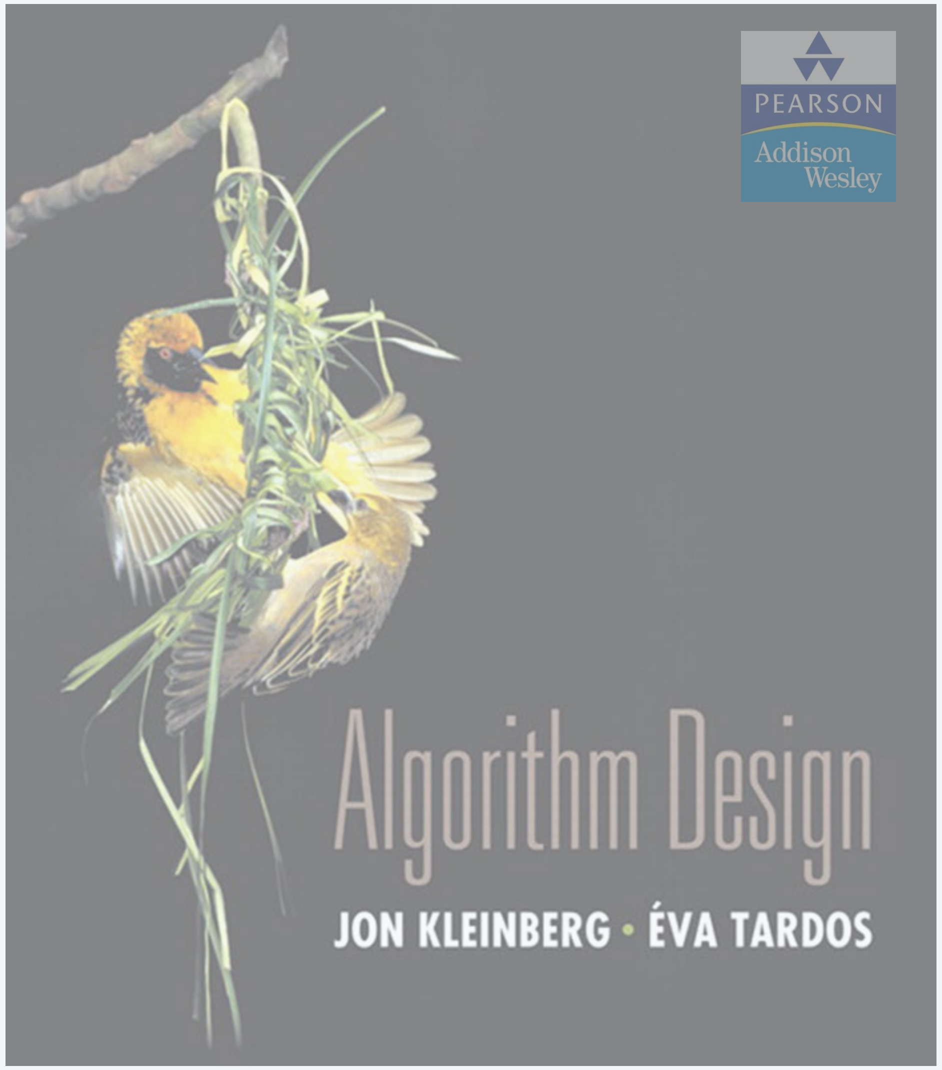


Mathematical Programming 46 (1990) 259–271 259
North-Holland

APPROXIMATION ALGORITHMS FOR SCHEDULING UNRELATED PARALLEL MACHINES

Jan Karel LENSTRA
Eindhoven University of Technology, Eindhoven, The Netherlands, and
Centre for Mathematics and Computer Science, Amsterdam, The Netherlands

David B. SHMOYS and Éva TARDOS
Cornell University, Ithaca, NY, USA



SECTION 11.8

11. APPROXIMATION ALGORITHMS

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- ***knapsack problem***

Polynomial-time approximation scheme

PTAS. $(1 + \varepsilon)$-approximation algorithm for any constant $\varepsilon > 0$.

- Load balancing. [Hochbaum–Shmoys 1987]
- Euclidean TSP. [Arora, Mitchell 1996]

Consequence. PTAS produces arbitrarily high quality solution, but trades off accuracy for time.

This section. PTAS for knapsack problem via rounding and scaling.

Knapsack problem

Knapsack problem.

- Given n objects and a knapsack.
- Item i has value $v_i > 0$ and weighs $w_i > 0$. ← we assume $w_i \leq W$ for each i
- Knapsack has weight limit W.
- Goal: fill knapsack so as to maximize total value.

Ex: $\{ 3, 4 \}$ has value 40.

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

original instance (W = 11)

Knapsack is NP-complete

KNAPSACK. Given a set X, weights $w_i \geq 0$, values $v_i \geq 0$, a weight limit W, and a target value V, is there a subset $S \subseteq X$ such that:

$$\begin{aligned} \sum_{i \in S} w_i &\leq W \\ \sum_{i \in S} v_i &\geq V \end{aligned}$$

SUBSET-SUM. Given a set X, values $u_i \geq 0$, and an integer U, is there a subset $S \subseteq X$ whose elements sum to exactly U ?

Theorem. SUBSET-SUM $\leq_P$ KNAPSACK.

Pf. Given instance $(u_1, \ldots, u_n, U)$ of SUBSET-SUM, create KNAPSACK instance:

$$\begin{aligned} v_i = w_i = u_i \qquad & \sum_{i \in S} u_i \leq U \\ V = W = U \qquad & \sum_{i \in S} u_i \geq U \end{aligned}$$

Knapsack problem: dynamic programming I

Def. $OPT(i, w)$ = max value subset of items $1,\ldots, i$ with weight limit w.

Case 1. OPT does not select item i.

- OPT selects best of $1, \ldots, i - 1$ using up to weight limit w.

Case 2. OPT selects item i.

- New weight limit = $w - w_i$.
- OPT selects best of $1, \ldots, i - 1$ using up to weight limit $w - w_i$.

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max\{ OPT(i-1, w), \quad v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$

Theorem. Computes the optimal value in $O(n\,W)$ time.

- Not polynomial in input size.
- Polynomial in input size if weights are small integers.

Knapsack problem: dynamic programming II

Def. $OPT(i, v)$ = min weight of a knapsack for which we can obtain a solution of value ≥ v using a subset of items $1,..., i$.

Note. Optimal value is the largest value v such that $OPT(n, v) \leq W$.

Case 1. OPT does not select item i.

- OPT selects best of $1, \ldots, i-1$ that achieves value ≥ v.

Case 2. OPT selects item i.

- Consumes weight w_i, need to achieve value ≥ $v - v_i$.
- OPT selects best of $1, \ldots, i-1$ that achieves value ≥ $v - v_i$.

$$OPT(i,v) = \begin{cases} 0 & \text{if } v \leq 0 \\ \infty & \text{if } i = 0 \text{ and } \ v > 0 \\ \min \{OPT(i-1, v),\ w_i + OPT(i-1, v - v_i)\} & \text{otherwise} \end{cases}$$

Knapsack problem: dynamic programming II

Theorem. Dynamic programming algorithm II computes the optimal value in $O(n^2\, v_{\max})$ time, where $v_{\max}$ is the maximum of any value.

Pf.

- The optimal value $V^* \le\ n\, v_{\max}$.
- There is one subproblem for each item and for each value $v \le V^*$.
- It takes $O(1)$ time per subproblem. ▪

Remark 1. Not polynomial in input size!

Remark 2. Polynomial time if values are small integers.

Knapsack problem: polynomial-time approximation scheme

Intuition for approximation algorithm.

- Round all values up to lie in smaller range.
- Run dynamic programming algorithm II on rounded/scaled instance.
- Return optimal items in rounded instance.

item	value	weight
1	934221	1
2	5956342	2
3	17810013	5
4	21217800	6
5	27343199	7

original instance (W = 11)

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

rounded instance (W = 11)

Knapsack problem: polynomial-time approximation scheme

Round up all values:

- $0 < \varepsilon \leq 1$ = precision parameter.
- $v_{\max}$ = largest value in original instance.
- θ = scaling factor = $\varepsilon\, v_{\max} / 2n$.

$$\bar{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil \theta\,, \quad \hat{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil$$

Observation. Optimal solutions to problem with $\bar{v}$ are equivalent to optimal solutions to problem with $\hat{v}$.

Intuition. $\bar{v}$ close to v so optimal solution using $\bar{v}$ is nearly optimal; $\hat{v}$ small and integral so dynamic programming algorithm II is fast.

Knapsack problem: polynomial-time approximation scheme

Theorem. If S is solution found by rounding algorithm and S^* is any other feasible solution, then $(1+\epsilon)\sum_{i\in S} v_i \;\ge\; \sum_{i\in S^*} v_i$

Pf. Let S^* be any feasible solution satisfying weight constraint.

$$\begin{aligned}
\sum_{i\in S^*} v_i &\le \sum_{i\in S^*} \bar{v}_i && \text{always round up}\\
&\le \sum_{i\in S} \bar{v}_i && \text{solve rounded instance optimally}\\
&\le \sum_{i\in S} (v_i+\theta) && \text{never round up by more than } \theta\\
&\le \sum_{i\in S} v_i \;+\; n\theta && |S| \le n\\
&= \sum_{i\in S} v_i \;+\; \tfrac{1}{2}\,\epsilon\, v_{\max} && \theta = \varepsilon\, v_{\max} / 2n\\
&\le (1+\epsilon)\sum_{i\in S} v_i && v_{\max} \le 2\, \textstyle\sum_{i\in S} v_i
\end{aligned}$$

subset containing
only the item
of largest value

choosing $S^* = \{\max\}$

$$\begin{aligned}
v_{\max} &\le \sum_{i\in S} v_i \;+\; \tfrac{1}{2}\,\epsilon\, v_{\max}\\
&\le \sum_{i\in S} v_i \;+\; \tfrac{1}{2}\, v_{\max}
\end{aligned}$$

thus

$$v_{\max} \le 2\sum_{i\in S} v_i$$

Knapsack problem: polynomial-time approximation scheme

Theorem. For any $\varepsilon > 0$, the rounding algorithm computes a feasible solution whose value is within a $(1 + \varepsilon)$ factor of the optimum in $O(n^3 / \varepsilon)$ time.

Pf.

- We have already proved the accuracy bound.
- Dynamic program II running time is $O(n^2\, \hat{v}_{\max})$, where

$$\hat{v}_{\max} = \left\lceil \frac{v_{\max}}{\theta} \right\rceil = \left\lceil \frac{2n}{\epsilon} \right\rceil$$