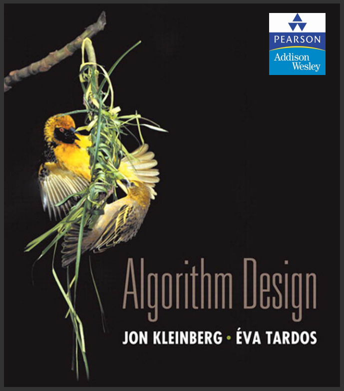


# 8. INTRACTABILITY II

- *P vs. NP*
- *NP-complete*
- *co-NP*
- *NP-hard*

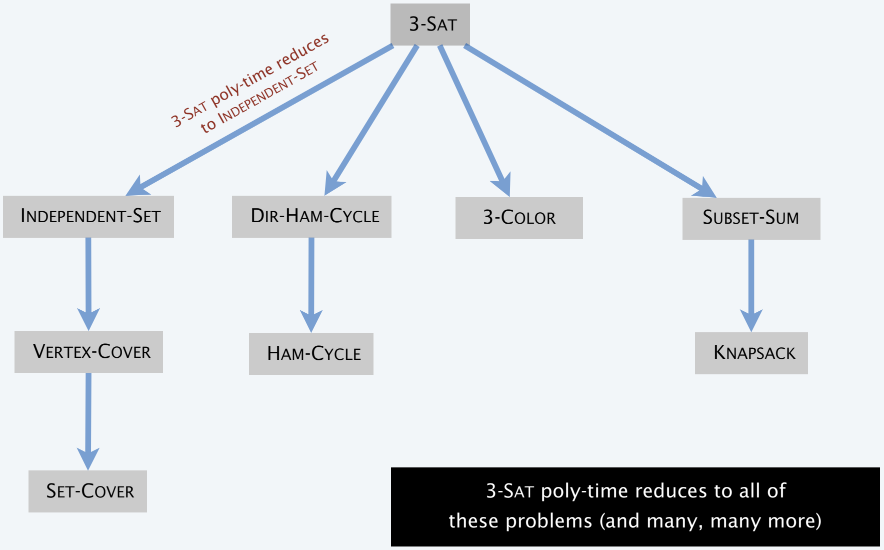
Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 2/16/20 10:57 AM

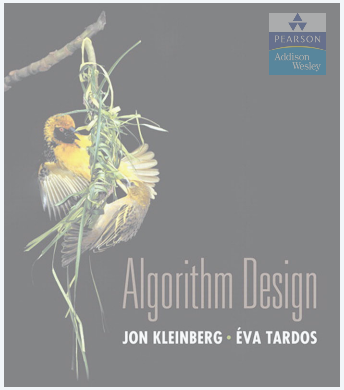
---

## Recap



3-SAT poly-time reduces to all of these problems (and many, many more)

---



SECTION 8.3

# 8. INTRACTABILITY II

- *P vs. NP*
- *NP-complete*
- *co-NP*
- *NP-hard*

---

## P

Decision problem.
- Problem $X$ is a set of strings.
- Instance $s$ is one string.
- Algorithm $A$ solves problem $X$: $A(s) = \begin{cases} yes & \text{if } s \in X \\ no & \text{if } s \notin X \end{cases}$

Def. Algorithm $A$ runs in polynomial time if for every string $s$, $A(s)$ terminates in $\leq p(|s|)$ "steps," where $p(\cdot)$ is some polynomial function.

length of $s$

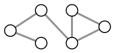
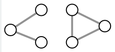
Def. **P** = set of decision problems for which there exists a poly-time algorithm.

on a deterministic Turing machine

| | |
|---|---|
| **problem PRIMES:** | $\{ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots \}$ |
| **instance s:** | 592335744548702854681 |
| **algorithm:** | Agrawal–Kayal–Saxena (2002) |

## Some problems in P

P. Decision problems for which there exists a poly-time algorithm.

| problem | description | poly-time algorithm | yes | no |
|---|---|---|---|---|
| MULTIPLE | Is $x$ a multiple of $y$ ? | grade-school division | `51, 17` | `51, 16` |
| REL-PRIME | Are $x$ and $y$ relatively prime ? | Euclid's algorithm | `34, 39` | `34, 51` |
| PRIMES | Is $x$ prime ? | Agrawal–Kayal–Saxena | `53` | `51` |
| EDIT-DISTANCE | Is the edit distance between $x$ and $y$ less than 5 ? | Needleman–Wunsch | `niether`<br>`neither` | `acgggt`<br>`ttttta` |
| L-SOLVE | Is there a vector $x$ that satisfies $Ax = b$ ? | Gauss–Edmonds elimination | $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ |
| U-CONN | Is an undirected graph $G$ connected? | depth-first search |  |  |

## NP

Def. Algorithm $C(s, t)$ is a certifier for problem $X$ if for every string $s$ : $s \in X$ iff there exists a string $t$ such that $C(s, t) = yes$.

Def. **NP** = set of decision problems for which there exists a poly-time certifier.

- $C(s, t)$ is a poly-time algorithm.
- Certificate $t$ is of polynomial size: $|t| \leq p(|s|)$ for some polynomial $p(\cdot)$.

"certificate" or "witness"

| | |
|---|---|
| **problem COMPOSITES:** | $\{ 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, \ldots. \}$ |
| **instance s:** | 437669 |
| **certificate t:** | 541 ⟵ 437,669 = 541 × 809 |
| **certifier C(s, t):** | grade-school division |

## Certifiers and certificates: satisfiability

SAT. Given a CNF formula Φ, does it have a satisfying truth assignment?
3-SAT. SAT where each clause contains exactly 3 literals.

Certificate. An assignment of truth values to the Boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

| | |
|---|---|
| **instance s** | $\Phi \;=\; \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$ |
| **certificate t** | $x_1 = true,\; x_2 = true,\; x_3 = false,\; x_4 = false$ |

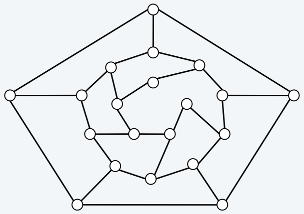
Conclusions. SAT ∈ **NP**, 3-SAT ∈ **NP**.

## Certifiers and certificates: Hamilton path

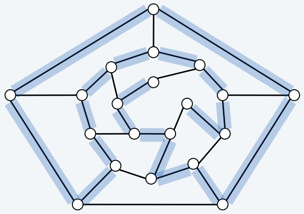
HAMILTON-PATH. Given an undirected graph $G = (V, E)$, does there exist a simple path $P$ that visits every node?

Certificate. A permutation $\pi$ of the $n$ nodes.

Certifier. Check that $\pi$ contains each node in $V$ exactly once, and that $G$ contains an edge between each pair of adjacent nodes.



**instance s**

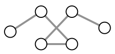
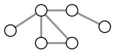


**certificate t**

Conclusion. HAMILTON-PATH ∈ **NP**.

## Some problems in NP

**NP.** Decision problems for which there exists a poly-time certifier.

| problem | description | poly-time algorithm | yes | no |
|---|---|---|---|---|
| L-SOLVE | Is there a vector $x$ that satisfies $Ax = b$ ? | Gauss–Edmonds elimination | $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ |
| COMPOSITES | Is $x$ composite ? | Agrawal–Kayal–Saxena | 51 | 53 |
| FACTOR | Does $x$ have a nontrivial factor less than $y$ ? | ??? | (56159, 50) | (55687, 50) |
| SAT | Given a CNF formula, does it have a satisfying truth assignment? | ??? | $\neg x_1 \vee x_2 \vee \neg x_3$<br>$x_1 \vee \neg x_2 \vee x_3$<br>$\neg x_1 \vee \neg x_2 \vee x_3$ | $\neg x_2$<br>$x_1 \vee x_2$<br>$\neg x_1 \vee x_2$ |
| HAMILTON-PATH | Is there a simple path between $u$ and $v$ that visits every node? | ??? |  |  |

---

## Intractability: quiz 1

**Which of the following graph problems are known to be in NP?**

- **A.** Is the length of the longest simple path $\le k$ ?
- **B.** Is the length of the longest simple path $\ge k$ ?
- **C.** Is the length of the longest simple path $= k$ ?
- **D.** Find the length of the longest simple path.
- **E.** All of the above.

---

## Intractability: quiz 2

**In complexity theory, the abbreviation NP stands for…**

- **A.** Nope.
- **B.** No problem.
- **C.** Not polynomial time.
- **D.** Not polynomial space.
- **E.** Nondeterministic polynomial time.

---

## Significance of NP

**NP.** Decision problems for which there exists a poly-time certifier.

> *" NP captures vast domains of computational, scientific, and mathematical endeavors, and seems to roughly delimit what mathematicians and scientists have been aspiring to compute feasibly. "* — *Christos Papadimitriou*

> *" In an ideal world it would be renamed P vs VP. "* — *Clyde Kruskal*

## P, NP, and EXP

P. Decision problems for which there exists a poly-time algorithm.
NP. Decision problems for which there exists a poly-time certifier.
EXP. Decision problems for which there exists an exponential-time algorithm.

Proposition. **P** $\subseteq$ **NP**.
Pf. Consider any problem $X \in$ **P**.
- By definition, there exists a poly-time algorithm $A(s)$ that solves $X$.
- Certificate $t = \varepsilon$, certifier $C(s, t) = A(s)$. ▪

Proposition. **NP** $\subseteq$ **EXP**.
Pf. Consider any problem $X \in$ **NP**.
- By definition, there exists a poly-time certifier $C(s, t)$ for $X$, where certificate $t$ satisfies $|t| \le p(|s|)$ for some polynomial $p(\cdot)$.
- To solve instance $s$, run $C(s, t)$ on all strings $t$ with $|t| \le p(|s|)$.
- Return *yes* iff $C(s, t)$ returns *yes* for any of these potential certificates. ▪

Fact. **P** ≠ **EXP** ⇒ either **P** ≠ **NP**, or **NP** ≠ **EXP**, or both.

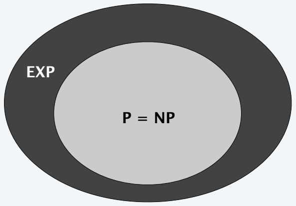
## The main question: P vs. NP

Q. How to solve an instance of 3-SAT with $n$ variables?
A. Exhaustive search: try all $2^n$ truth assignments.

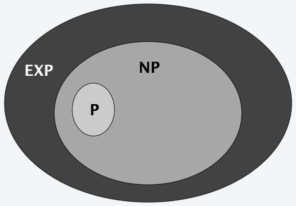
Q. Can we do anything substantially more clever?
Conjecture. $\underbrace{\text{No poly-time algorithm}}_{\text{“intractable”}}$ for 3-SAT.

## The main question: P vs. NP

Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
Is the decision problem as easy as the certification problem?



**If P = NP**



**If P ≠ NP**

If yes... Efficient algorithms for 3-SAT, TSP, VERTEX-COVER, FACTOR, ...
If no... No efficient algorithms possible for 3-SAT, TSP, VERTEX-COVER, ...

Consensus opinion. Probably no.

## Reductions: quiz 3

**Suppose P ≠ NP. Which of the following are still possible?**

- **A.** $O(n^3)$ algorithm for factoring $n$-bit integers.
- **B.** $O(1.657^n)$ time algorithm for HAMILTON-CYCLE.
- **C.** $O(n^{\log \log \log n})$ algorithm for 3-SAT.
- **D.** All of the above.

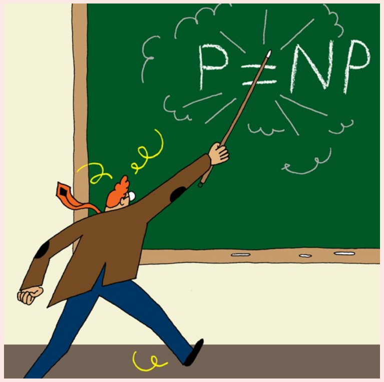
## Intractability: quiz 4

**Does P = NP?**

**A.** Yes.

**B.** No.

**C.** None of the above.

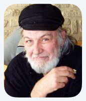


---

## Possible outcomes

**P ≠ NP**

> *" I conjecture that there is no good algorithm for the traveling salesman problem. My reasons are the same as for any mathematical conjecture: (i) It is a legitimate mathematical possibility and (ii) I do not know."*
>
> *— Jack Edmonds 1966*



> *" In my view, there is no way to even make intelligent guesses about the answer to any of these questions. If I had to bet now, I would bet that P is not equal to NP. I estimate the half-life of this problem at 25–50 more years, but I wouldn't bet on it being solved before 2100. "*
>
> *— Bob Tarjan (2002)*

---

## Possible outcomes

**P ≠ NP**

> *" We seem to be missing even the most basic understanding of the nature of its difficulty…. All approaches tried so far probably (in some cases, provably) have failed. In this sense P =NP is different from many other major mathematical problems on which a gradual progress was being constantly done (sometimes for centuries) whereupon they yielded, either completely or partially. "*
>
> *— Alexander Razborov (2002)*

---

## Possible outcomes

**P = NP**

> *" I think that in this respect I am on the loony fringe of the mathematical community: I think (not too strongly!) that P=NP and this will be proved within twenty years. Some years ago, Charles Read and I worked on it quite bit, and we even had a celebratory dinner in a good restaurant before we found an absolutely fatal mistake. "*
>
> *— Béla Bollobás (2002)*

> *" In my opinion this shouldn't really be a hard problem; it's just that we came late to this theory, and haven't yet developed any techniques for proving computations to be hard. Eventually, it will just be a footnote in the books. "* *— John Conway*

## Other possible outcomes

**P** = **NP**, but only $\Omega(n^{100})$ algorithm for 3-SAT.

**P** ≠ **NP**, but with $O(n^{\log^* n})$ algorithm for 3-SAT.

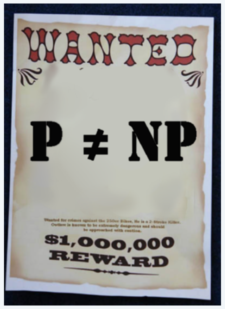
**P** = **NP** is independent (of ZFC axiomatic set theory).

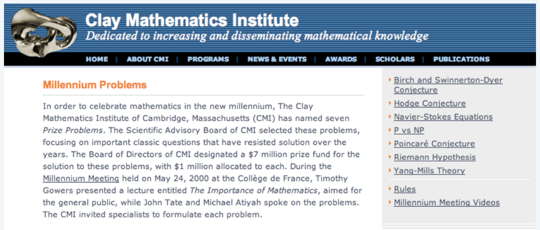
> " *It will be solved by either 2048 or 4096. I am currently somewhat pessimistic. The outcome will be the truly worst case scenario: namely that someone will prove P = NP because there are only finitely many obstructions to the opposite hypothesis; hence there exists a polynomial time solution to SAT but we will never know its complexity!* " — *Donald Knuth*

---

## Millennium prize

Millennium prize. $1 million for resolution of **P** ≠ **NP** problem.





**Clay Mathematics Institute**

*Dedicated to increasing and disseminating mathematical knowledge*

HOME | ABOUT CMI | PROGRAMS | NEWS & EVENTS | AWARDS | SCHOLARS | PUBLICATIONS

**Millennium Problems**

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven *Prize Problems*. The Scientific Advisory Board of CMI selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a $7 million prize fund for the solution to these problems, with $1 million allocated to each. During the Millennium Meeting held on May 24, 2000 at the Collège de France, Timothy Gowers presented a lecture entitled *The Importance of Mathematics*, aimed for the general public, while John Tate and Michael Atiyah spoke on the problems. The CMI invited specialists to formulate each problem.

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs NP
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills Theory

- Rules
- Millennium Meeting Videos

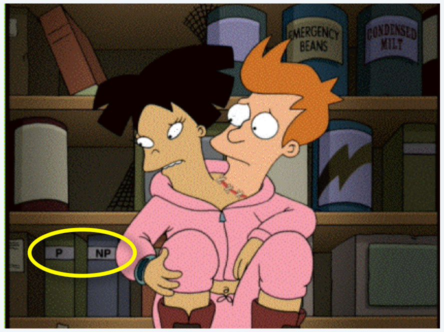
---

## P vs. NP and pop culture

Some writers for the Simpsons and Futurama.

- J. Steward Burns. *M.S. in mathematics* (*Berkeley '93*).
- David X. Cohen. *M.S. in computer science* (*Berkeley '92*).
- Al Jean. *B.S. in mathematics*. (*Harvard '81*).
- Ken Keeler. *Ph.D. in applied mathematics* (*Harvard '90*).
- Jeff Westbrook. *Ph.D. in computer science* (*Princeton '89*).

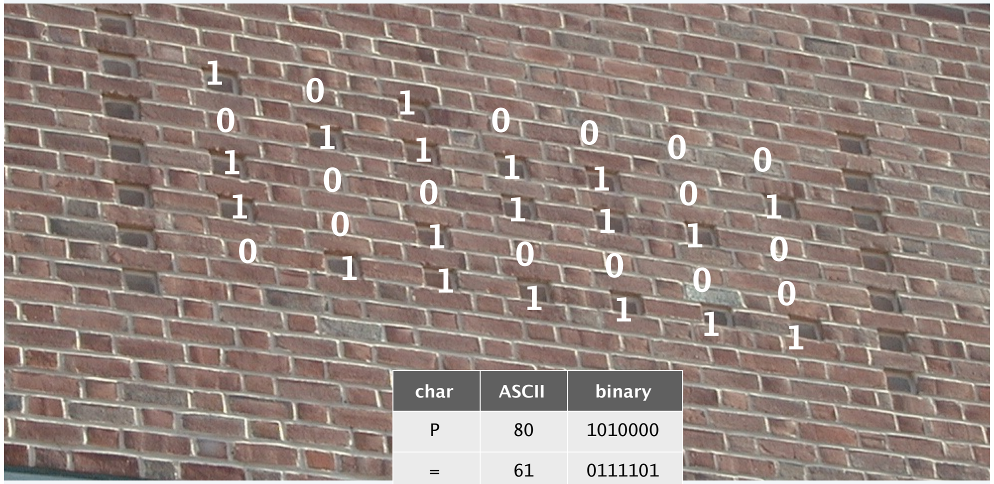
Copyright © 1990, Matt Groening



Copyright © 2000, Twentieth Century Fox

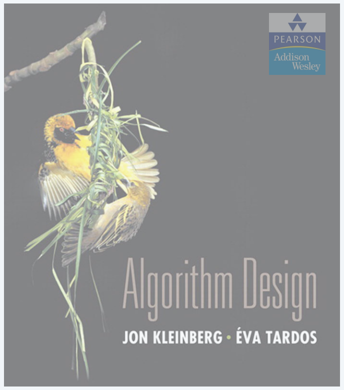
---

## Princeton CS Building, West Wall, Circa 2001



| char | ASCII | binary |
|---|---|---|
| P | 80 | 1010000 |
| = | 61 | 0111101 |
| N | 78 | 1001110 |
| P | 80 | 1010000 |
| ? | 63 | 0111111 |

---



SECTION 8.4

## 8. INTRACTABILITY II

- *P vs. NP*
- *NP-complete*
- *co-NP*
- *NP-hard*

---

## Polynomial transformations

Def. Problem $X$ polynomial (Cook) reduces to problem $Y$ if arbitrary instances of problem $X$ can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem $Y$.

Def. Problem $X$ polynomial (Karp) transforms to problem $Y$ if given any instance $x$ of $X$, we can construct an instance $y$ of $Y$ such that $x$ is a *yes* instance of $X$ iff $y$ is a *yes* instance of $Y$.

↑ we require $|y|$ to be of size polynomial in $|x|$

Note. Polynomial transformation is polynomial reduction with just one call to oracle for $Y$, exactly at the end of the algorithm for $X$. Almost all previous reductions were of this form.

Open question. Are these two concepts the same with respect to **NP**?

↑ we abuse notation $\le_P$ and blur distinction

---

## NP-complete

NP-complete. A problem $Y \in$ **NP** with the property that for every problem $X \in$ **NP**, $X \le_P Y$.

Proposition. Suppose $Y \in$ **NP**-complete. Then, $Y \in$ **P** iff **P** = **NP**.

Pf. $\Leftarrow$ If **P** = **NP**, then $Y \in$ **P** because $Y \in$ **NP**.

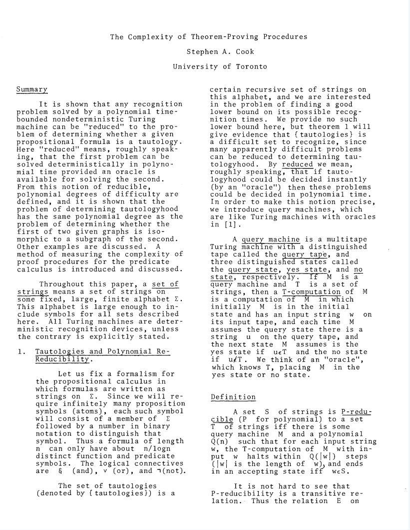
Pf. $\Rightarrow$ Suppose $Y \in$ **P**.

- Consider any problem $X \in$ **NP**. Since $X \le_P Y$, we have $X \in$ **P**.
- This implies **NP** $\subseteq$ **P**.
- We already know **P** $\subseteq$ **NP**. Thus **P** = **NP**. ▪

Fundamental question. Are there any "natural" **NP**-complete problems?

## The "first" NP-complete problem

Theorem. [Cook 1971, Levin 1973] SAT ∈ **NP**-complete.



The Complexity of Theorem-Proving Procedures

Stephen A. Cook

University of Toronto

Summary

It is shown that any recognition problem solved by a polynomial time-bounded nondeterministic Turing machine can be "reduced" to the problem of determining whether a given propositional formula is a tautology. Here "reduced" means, roughly speaking, that the first problem can be solved deterministically in polynomial time provided an oracle is available for solving the second. From this notion of reducible, polynomial degrees of difficulty are defined, and it is shown that the problem of determining tautologyhood has the same polynomial degree as the problem of determining whether the first of two given graphs is isomorphic to a subgraph of the second. Other examples are discussed. A method of measuring the complexity of proof procedures for the predicate calculus is introduced and discussed.

Throughout this paper, a set of strings means a set of strings on some fixed, large, finite alphabet Σ. This alphabet is large enough to include symbols for all sets described here. All Turing machines are deterministic recognition devices, unless the contrary is explicitly stated.

1. Tautologies and Polynomial Re-Reducibility.

Let us fix a formalism for the propositional calculus in which formulas are written as strings on Σ. Since we will require infinitely many proposition symbols (atoms), each such symbol will consist of a member of Σ followed by a number in binary notation to distinguish that symbol. Thus a formula of length n can only have about n/logn distinct function and predicate symbols. The logical connectives are & (and), ∨ (or), and ¬(not).

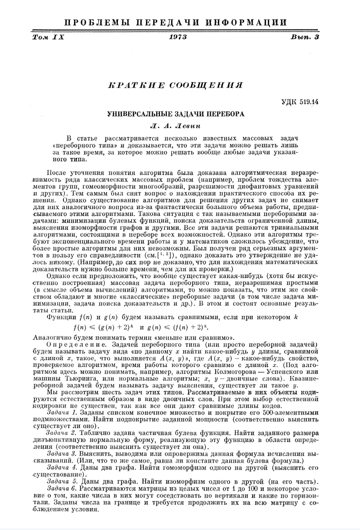
The set of tautologies (denoted by {tautologies}) is a certain recursive set of strings on this alphabet, and we are interested in the problem of finding a good lower bound on its possible recognition times. We provide no such lower bound here, but theorem 1 will give evidence that {tautologies} is a difficult set to recognize, since many apparently difficult problems can be reduced to determining tautologyhood. By reduced we mean, roughly speaking, that if tautologyhood could be decided instantly (by an "oracle") then these problems could be decided in polynomial time. In order to make this notion precise, we introduce query machines, which are like Turing machines with oracles in [1].

A query machine is a multitape Turing machine with a distinguished tape called the query tape, and three distinguished states called the query state, yes state, and no state, respectively. If M is a query machine and T is a set of strings, then a T-computation of M is a computation of M in which initially M is in the initial state and has an input string w on its input tape, and each time M assumes the query state there is a string u on the query tape, and the next state M assumes is the yes state if u∈T and the no state if u∉T. We think of an "oracle", which knows T, placing M in the yes state or no state.

Definition

A set S of strings is P-reducible (P for polynomial) to a set T of strings iff there is some query machine M and a polynomial Q(n) such that for each input string w, the T-computation of M with input w halts within Q(|w|) steps (|w| is the length of w), and ends in an accepting state iff w∈S.

It is not hard to see that P-reducibility is a transitive relation. Thus the relation E on



ПРОБЛЕМЫ ПЕРЕДАЧИ ИНФОРМАЦИИ

Том IX 1973 Вып. 3

КРАТКИЕ СООБЩЕНИЯ

УДК 519.14

УНИВЕРСАЛЬНЫЕ ЗАДАЧИ ПЕРЕБОРА

*Л. А. Левин*

В статье рассматривается несколько известных массовых задач «переборного типа» и доказывается, что эти задачи можно решать лишь за такое время, за которое можно решать вообще любые задачи указанного типа.

После уточнения понятия алгоритма была доказана алгоритмическая неразрешимость ряда классических массовых проблем (например, проблем тождества элементов групп, гомеоморфности многообразий, разрешимости диофантовых уравнений и других). Тем самым был снят вопрос о нахождении практического способа их решения. Однако существование алгоритмов для решения других задач не снимает для них аналогичного вопроса из-за фантастически большого объема работы, предписываемого этими алгоритмами. Такова ситуация с так называемыми переборными задачами: минимизации булевых функций, поиска доказательств ограниченной длины, выяснения изоморфности графов и другими. Все эти задачи решаются тривиальными алгоритмами, состоящими в переборе всех возможностей. Однако эти алгоритмы требуют экспоненциального времени работы и у математиков сложилось убеждение, что более простые алгоритмы для них невозможны. Был получен ряд серьезных аргументов в пользу его справедливости (см.[1, 2]), однако доказать это утверждение не удалось никому. (Например, до сих пор не доказано, что для нахождения математических доказательств нужно больше времени, чем для их проверки.)

Однако если предположить, что вообще существует какая-нибудь (хотя бы искусственно построенная) массовая задача переборного типа, неразрешимая простыми (в смысле объема вычислений) алгоритмами, то можно показать, что этим же свойством обладают и многие «классические» переборные задачи (в том числе задача минимизации, задача поиска доказательств и др.). В этом и состоят основные результаты статьи.

Функции $f(n)$ и $g(n)$ будем называть сравнимыми, если при некотором $k$

$$f(n) \leqslant (g(n)+2)^k \quad \text{и} \quad g(n) \leqslant (f(n)+2)^k.$$

Аналогично будем понимать термин «меньше или сравнимо».

Определение. Задачей переборного типа (или просто переборной задачей) будем называть задачу вида «по данному $x$ найти какое-нибудь $y$ длины, сравнимой с длиной $x$, такое, что выполняется $A(x, y)$», где $A(x, y)$ – какое-нибудь свойство, проверяемое алгоритмом, время работы которого сравнимо с длиной $x$. (Под алгоритмом здесь можно понимать, например, алгоритмы Колмогорова — Успенского или машины Тьюринга, или нормальные алгоритмы; $x$, $y$ – двоичные слова). Квазипереборной задачей будем называть задачу выяснения, существует ли такое $y$.

Мы рассмотрим шесть задач этих типов. Рассматриваемые в них объекты кодируются естественным образом в виде двоичных слов. При этом выбор естественной кодировки не существен, так как все они дают сравнимые длины кодов.

*Задача 1.* Заданы списком конечное множество и покрытие его 500-элементными подмножествами. Найти подпокрытие заданной мощности (соответственно выяснить существует ли оно).

*Задача 2.* Таблично задана частичная булева функция. Найти заданного размера дизъюнктивную нормальную форму, реализующую эту функцию в области определения (соответственно выяснить существует ли она).

*Задача 3.* Выяснить, выводима или опровержима данная формула исчисления высказываний. (Или, что то же самое, равна ли константе данная булева формула.)

*Задача 4.* Даны два графа. Найти гомоморфизм одного на другой (выяснить его существование).

*Задача 5.* Даны два графа. Найти изоморфизм одного в другой (на его часть).

*Задача 6.* Рассматриваются матрицы из целых чисел от 1 до 100 и некоторое условие о том, какие числа в них могут соседствовать по вертикали и какие по горизонтали. Заданы числа на границе и требуется продолжить их на всю матрицу с соблюдением условия.

---

## Establishing NP-completeness

Remark. Once we establish first "natural" **NP**-complete problem, others fall like dominoes.

Recipe. To prove that $Y \in$ **NP**-complete:

- Step 1. Show that $Y \in$ **NP**.
- Step 2. Choose an **NP**-complete problem $X$.
- Step 3. Prove that $X \leq_P Y$.

Proposition. If $X \in$ **NP**-complete, $Y \in$ **NP**, and $X \leq_P Y$, then $Y \in$ **NP**-complete.

Pf. Consider any problem $W \in$ **NP**. Then, both $W \leq_P X$ and $X \leq_P Y$.

- By transitivity, $W \leq_P Y$.
- Hence $Y \in$ **NP**-complete. ▪

by definition of **NP**-complete

by assumption

---

## Reductions: quiz 4

**Suppose that $X \in$ NP-COMPLETE, $Y \in$ NP, and $X \leq_P Y$. Which can you infer?**

**A.** $Y$ is **NP**-complete.

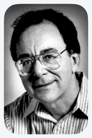
**B.** If $Y \notin$ **P**, then **P** ≠ **NP**.

**C.** If **P** ≠ **NP**, then neither $X$ nor $Y$ is in **P**.

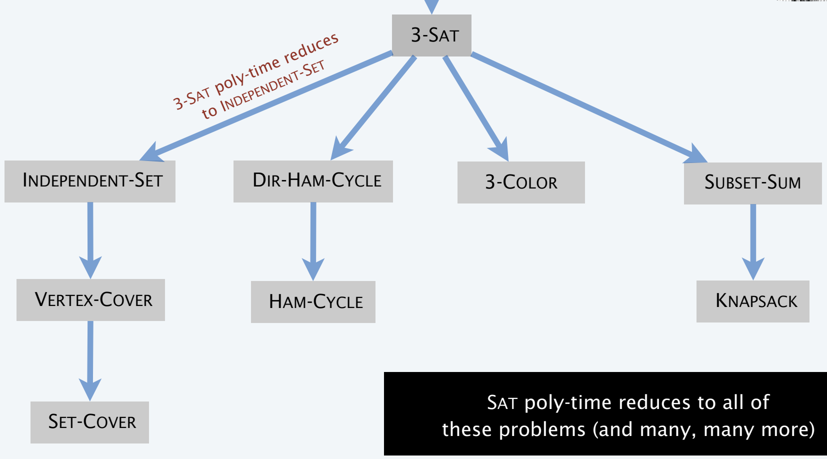
**D.** All of the above.

---

## Implications of Karp

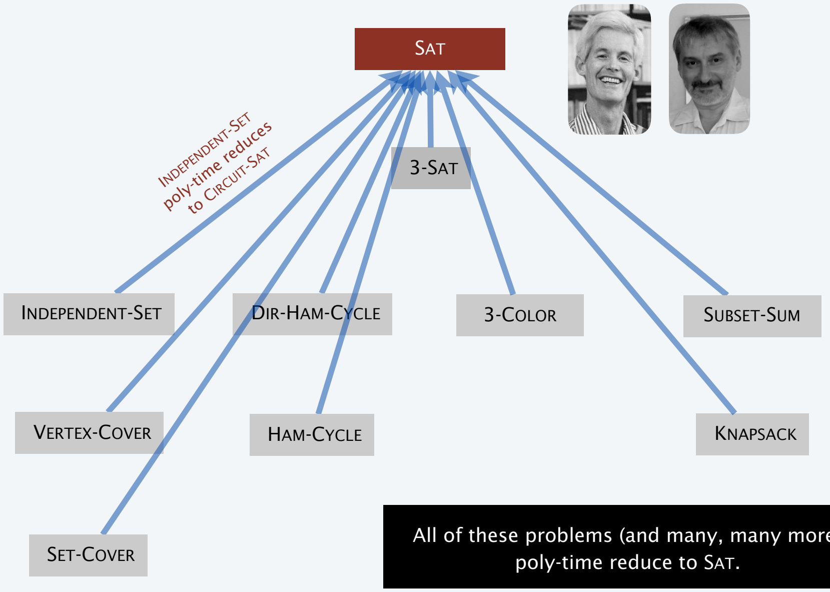


SAT



SAT poly-time reduces to all of these problems (and many, many more)

## Implications of Cook–Levin



INDEPENDENT-SET poly-time reduces to CIRCUIT-SAT

SAT

3-SAT

INDEPENDENT-SET

DIR-HAM-CYCLE

3-COLOR

SUBSET-SUM

VERTEX-COVER

HAM-CYCLE

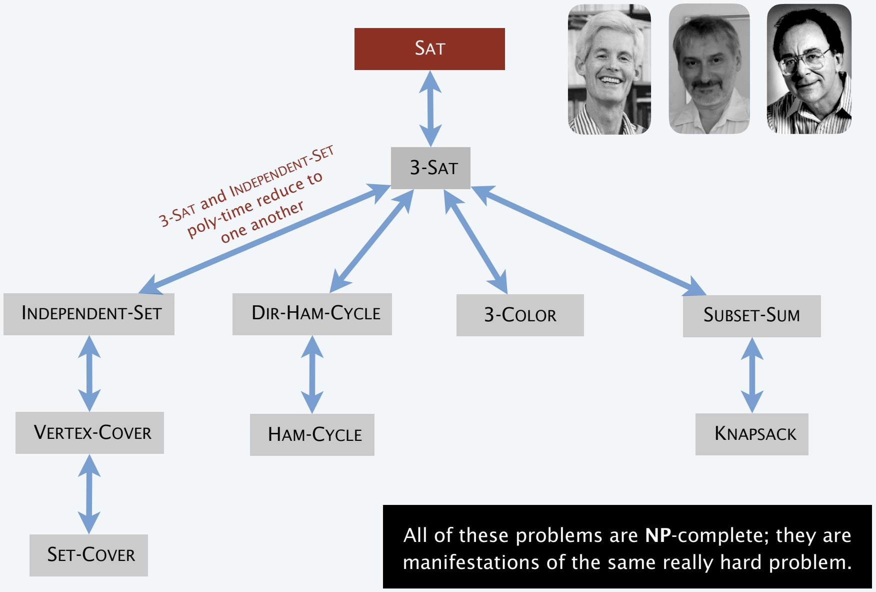
KNAPSACK

SET-COVER

All of these problems (and many, many more) poly-time reduce to SAT.

---

## Implications of Karp + Cook–Levin



SAT

3-SAT

3-SAT and INDEPENDENT-SET poly-time reduce to one another

INDEPENDENT-SET

DIR-HAM-CYCLE

3-COLOR

SUBSET-SUM

VERTEX-COVER

HAM-CYCLE

KNAPSACK

SET-COVER

All of these problems are **NP**-complete; they are manifestations of the same really hard problem.

---

# I'D TELL YOU ANOTHER NP-COMPLETE JOKE, BUT ONCE YOU'VE HEARD ONE,

# YOU'VE HEARD THEM ALL.

---

## Some NP-complete problems

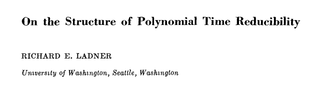
Basic genres of NP-complete problems and paradigmatic examples.

- Packing/covering problems: SET-COVER, VERTEX-COVER,INDEPENDENT-SET.
- Constraint satisfaction problems: CIRCUIT-SAT, SAT, 3-SAT.
- Sequencing problems: HAMILTON-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Practice. Most **NP** problems are known to be either in **P** or **NP**-complete.

NP-intermediate? FACTOR, DISCRETE-LOG, GRAPH-ISOMORPHISM, ….

Theorem. [Ladner 1975] Unless **P** = **NP**, there exist problems in **NP** that are neither in **P** nor **NP**-complete.



**On the Structure of Polynomial Time Reducibility**

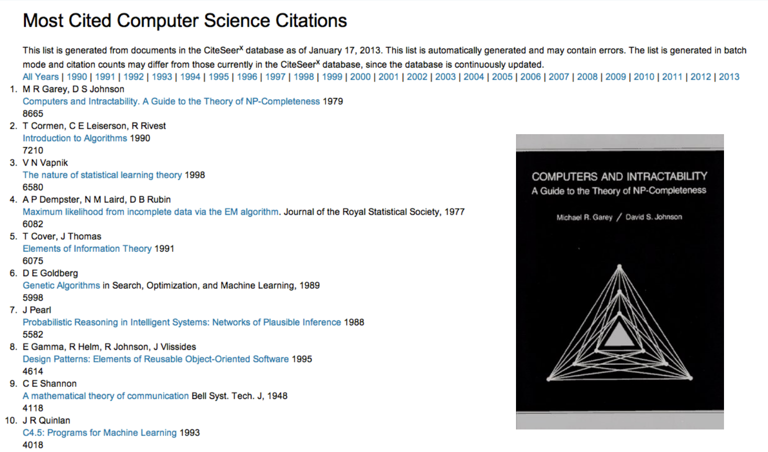
RICHARD E. LADNER

*University of Washington, Seattle, Washington*

## More hard computational problems

Garey and Johnson. Computers and Intractability.

- Appendix includes over 300 **NP**-complete problems.
- Most cited reference in computer science literature.



Most Cited Computer Science Citations

This list is generated from documents in the CiteSeer$^x$ database as of January 17, 2013. This list is automatically generated and may contain errors. The list is generated in batch mode and citation counts may differ from those currently in the CiteSeer$^x$ database, since the database is continuously updated.

All Years | 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2013

1. M R Garey, D S Johnson
   Computers and Intractability. A Guide to the Theory of NP-Completeness 1979
   8665
2. T Cormen, C E Leiserson, R Rivest
   Introduction to Algorithms 1990
   7210
3. V N Vapnik
   The nature of statistical learning theory 1998
   6580
4. A P Dempster, N M Laird, D B Rubin
   Maximum likelihood from incomplete data via the EM algorithm. Journal of the Royal Statistical Society, 1977
   6082
5. T Cover, J Thomas
   Elements of Information Theory 1991
   6075
6. D E Goldberg
   Genetic Algorithms in Search, Optimization, and Machine Learning, 1989
   5998
7. J Pearl
   Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference 1988
   5582
8. E Gamma, R Helm, R Johnson, J Vlissides
   Design Patterns: Elements of Reusable Object-Oriented Software 1995
   4614
9. C E Shannon
   A mathematical theory of communication Bell Syst. Tech. J, 1948
   4118
10. J R Quinlan
    C4.5: Programs for Machine Learning 1993
    4018

---

## More hard computational problems

Aerospace engineering. Optimal mesh partitioning for finite elements.
Biology. Phylogeny reconstruction.
Chemical engineering. Heat exchanger network synthesis.
Chemistry. Protein folding.
Civil engineering. Equilibrium of urban traffic flow.
Economics. Computation of arbitrage in financial markets with friction.
Electrical engineering. VLSI layout.
Environmental engineering. Optimal placement of contaminant sensors.
Financial engineering. Minimum risk portfolio of given return.
Game theory. Nash equilibrium that maximizes social welfare.
Mathematics. Given integer $a_1$, …, $a_n$, compute $\int_0^{2\pi} \cos(a_1\theta) \times \cos(a_2\theta) \times \cdots \times \cos(a_n\theta)\, d\theta$
Mechanical engineering. Structure of turbulence in sheared flows.
Medicine. Reconstructing 3d shape from biplane angiocardiogram.
Operations research. Traveling salesperson problem.
Physics. Partition function of 3d Ising model.
Politics. Shapley–Shubik voting power.
Recreation. Versions of Sudoku, Checkers, Minesweeper, Tetris, Rubik's Cube.
Statistics. Optimal experimental design.

---

## Extent and impact of NP-completeness

Extent of NP-completeness. [Papadimitriou 1995]

- Prime intellectual export of CS to other disciplines.
- 6,000 citations per year (more than "compiler", "OS", "database").
- Broad applicability and classification power.

NP-completeness can guide scientific inquiry.

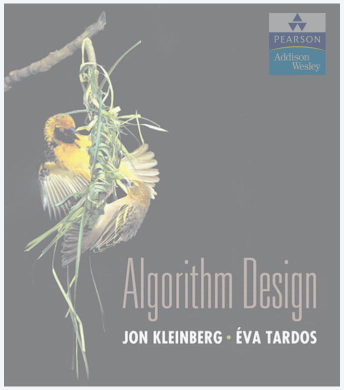
- 1926: Ising introduces simple model for phase transitions.
- 1944: Onsager finds closed-form solution to 2D-ISING in tour de force.
- 19xx: Feynman and other top minds seek solution to 3D-ISING.
- 2000: Istrail proves 3D-ISING ∈ **NP**-complete.

a holy grail of
statistical mechanics

search for closed formula appears doomed

---

## You NP-complete me



SECTION 8.9

# 8. INTRACTABILITY II

- *P vs. NP*
- *NP-complete*
- *co-NP*
- *NP-hard*

## Asymmetry of NP

Asymmetry of NP. We need short certificates only for *yes* instances.

Ex 1. SAT vs. UN-SAT.

- Can prove a CNF formula is satisfiable by specifying an assignment.
- How could we prove that a formula is not satisfiable?

SAT. Given a CNF formula $\Phi$, is there a satisfying truth assignment?

UN-SAT. Given a CNF formula $\Phi$, is there no satisfying truth assignment?

## Asymmetry of NP

Asymmetry of NP. We need short certificates only for *yes* instances.

Ex 2. HAMILTON-CYCLE vs. NO-HAMILTON-CYCLE.

- Can prove a graph is Hamiltonian by specifying a permutation.
- How could we prove that a graph is not Hamiltonian?

HAMILTON-CYCLE. Given a graph $G = (V, E)$, is there a simple cycle $\Gamma$ that contains every node in $V$ ?

NO-HAMILTON-CYCLE. Given a graph $G = (V, E)$, is there no simple cycle $\Gamma$ that contains every node in $V$ ?

## Asymmetry of NP

Asymmetry of NP. We need short certificates only for *yes* instances.

Q. How to classify UN-SAT and NO-HAMILTON-CYCLE ?

- SAT $\in$ **NP**-complete and SAT $\equiv_P$ UN-SAT.
- HAMILTON-CYCLE $\in$ **NP**-complete and HAMILTON-CYCLE $\equiv_P$ NO-HAMILTON-CYCLE.
- But neither UN-SAT nor NO-HAMILTON-CYCLE are known to be in **NP**.

# NP and co-NP

NP. Decision problems for which there is a poly-time certifier.

Ex. SAT, HAMILTON-CYCLE, and COMPOSITES.

Def. Given a decision problem $X$, its complement $\overline{X}$ is the same problem with the *yes* and *no* answers reversed.

Ex. $X = \{ 4, 6, 8, 9, 10, 12, 14, 15, \ldots \}$

$\overline{X} = \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, \ldots \}$ ←

ignore 0 and 1
(neither prime nor composite)

co-NP. Complements of decision problems in **NP**.

Ex. UN-SAT, NO-HAMILTON-CYCLE, and PRIMES.

# NP = co-NP ?

Fundamental open question. Does **NP = co-NP**?

- Do *yes* instances have succinct certificates iff *no* instances do?
- Consensus opinion: no.

Theorem. If **NP ≠ co-NP**, then **P ≠ NP**.

Pf idea.

- **P** is closed under complementation.
- If **P = NP**, then **NP** is closed under complementation.
- In other words, **NP = co-NP**.
- This is the contrapositive of the theorem.

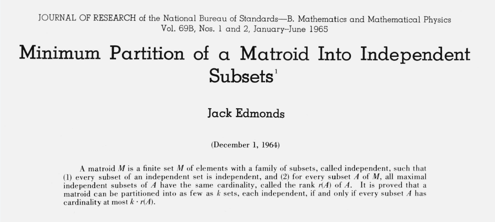
# Good characterizations

Good characterization. [Edmonds 1965] **NP ∩ co-NP**.

- If problem $X$ is in both **NP** and **co-NP**, then:
  - for *yes* instance, there is a succinct certificate
  - for *no* instance, there is a succinct disqualifier
- Provides conceptual leverage for reasoning about a problem.

Ex. Given a bipartite graph, is there a perfect matching?

- If yes, can exhibit a perfect matching.
- If no, can exhibit a set of nodes $S$ such that $|neighbors(S)| < |S|$.



JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 69B, Nos. 1 and 2, January–June 1965

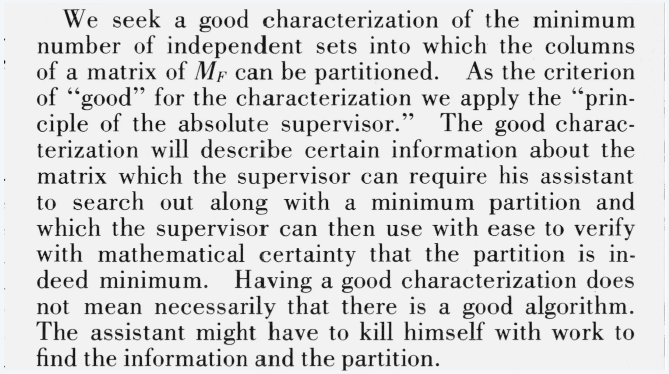
Minimum Partition of a Matroid Into Independent Subsets[1]

Jack Edmonds

(December 1, 1964)

A matroid $M$ is a finite set $M$ of elements with a family of subsets, called independent, such that (1) every subset of an independent set is independent, and (2) for every subset $A$ of $M$, all maximal independent subsets of $A$ have the same cardinality, called the rank $r(A)$ of $A$. It is proved that a matroid can be partitioned into as few as $k$ sets, each independent, if and only if every subset $A$ has cardinality at most $k \cdot r(A)$.

# Good characterizations



We seek a good characterization of the minimum number of independent sets into which the columns of a matrix of $M_F$ can be partitioned. As the criterion of "good" for the characterization we apply the "principle of the absolute supervisor." The good characterization will describe certain information about the matrix which the supervisor can require his assistant to search out along with a minimum partition and which the supervisor can then use with ease to verify with mathematical certainty that the partition is indeed minimum. Having a good characterization does not mean necessarily that there is a good algorithm. The assistant might have to kill himself with work to find the information and the partition.

## Good characterizations

Observation. **P** ⊆ **NP** ∩ **co-NP**.

- Proof of max-flow min-cut theorem led to stronger result that max-flow and min-cut are in **P**.
- Sometimes finding a good characterization seems easier than finding an efficient algorithm.

Fundamental open question. Does **P** = **NP** ∩ **co-NP**?

- Mixed opinions.
- Many examples where problem found to have a nontrivial good characterization, but only years later discovered to be in **P**.

---

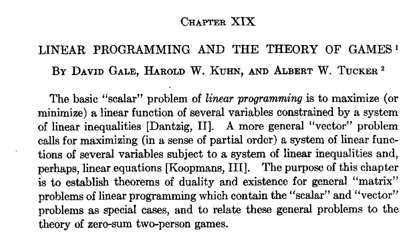
## Linear programming is in NP ∩ co-NP

LINEAR-PROGRAMMING. Given $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, and $\alpha \in \Re$, does there exist $x \in \Re^n$ such that $Ax \le b$, $x \ge 0$ and $c^T x \ge \alpha$?

Theorem. [Gale–Kuhn–Tucker 1948] LINEAR-PROGRAMMING ∈ **NP** ∩ **co-NP**.

Pf sketch. If (P) and (D) are nonempty, then max = min.

$$\begin{array}{llrcl} \text{(P)} & \max c^T x & & & \\ & \text{s.t.} & Ax & \le & b \\ & & x & \ge & 0 \end{array} \qquad \begin{array}{llrcl} \text{(D)} & \min y^T b & & & \\ & \text{s.t.} & A^T y & \ge & c \\ & & y & \ge & 0 \end{array}$$



CHAPTER XIX

LINEAR PROGRAMMING AND THE THEORY OF GAMES[1]

BY DAVID GALE, HAROLD W. KUHN, AND ALBERT W. TUCKER[2]

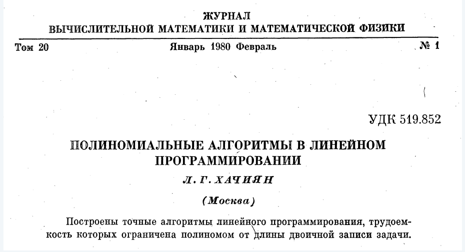
The basic "scalar" problem of *linear programming* is to maximize (or minimize) a linear function of several variables constrained by a system of linear inequalities [Dantzig, II]. A more general "vector" problem calls for maximizing (in a sense of partial order) a system of linear functions of several variables subject to a system of linear inequalities and, perhaps, linear equations [Koopmans, III]. The purpose of this chapter is to establish theorems of duality and existence for general "matrix" problems of linear programming which contain the "scalar" and "vector" problems as special cases, and to relate these general problems to the theory of zero-sum two-person games.

---

## Linear programming is in NP ∩ co-NP

LINEAR-PROGRAMMING. Given $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, and $\alpha \in \Re$, does there exist $x \in \Re^n$ such that $Ax \le b$, $x \ge 0$ and $c^T x \ge \alpha$?

Theorem. [Khachiyan 1979] LINEAR-PROGRAMMING ∈ **P**.



ЖУРНАЛ
ВЫЧИСЛИТЕЛЬНОЙ МАТЕМАТИКИ И МАТЕМАТИЧЕСКОЙ ФИЗИКИ

Том 20 Январь 1980 Февраль № 1

УДК 519.852

ПОЛИНОМИАЛЬНЫЕ АЛГОРИТМЫ В ЛИНЕЙНОМ ПРОГРАММИРОВАНИИ

*Л. Г. ХАЧИЯН*

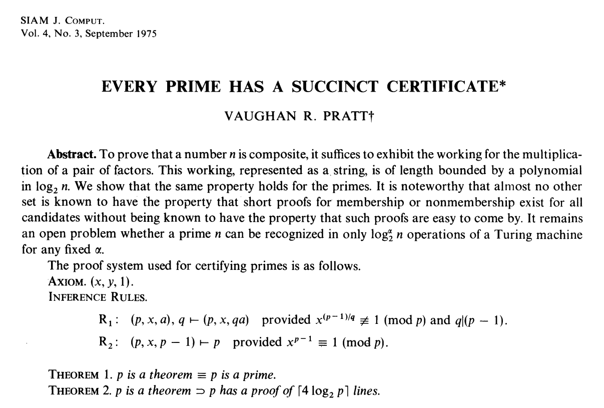
*(Москва)*

Построены точные алгоритмы линейного программирования, трудоемкость которых ограничена полиномом от длины двоичной записи задачи.

---

## Primality testing is in NP ∩ co-NP

Theorem. [Pratt 1975] PRIMES ∈ **NP** ∩ **co-NP**.



SIAM J. COMPUT.
Vol. 4, No. 3, September 1975

EVERY PRIME HAS A SUCCINCT CERTIFICATE*

VAUGHAN R. PRATT†

**Abstract.** To prove that a number $n$ is composite, it suffices to exhibit the working for the multiplication of a pair of factors. This working, represented as a string, is of length bounded by a polynomial in $\log_2 n$. We show that the same property holds for the primes. It is noteworthy that almost no other set is known to have the property that short proofs for membership or nonmembership exist for all candidates without being known to have the property that such proofs are easy to come by. It remains an open problem whether a prime $n$ can be recognized in only $\log_2^\alpha n$ operations of a Turing machine for any fixed $\alpha$.

The proof system used for certifying primes is as follows.

AXIOM. $(x, y, 1)$.

INFERENCE RULES.

$R_1$: $(p, x, a), q \vdash (p, x, qa)$ provided $x^{(p-1)/q} \not\equiv 1 \pmod{p}$ and $q | (p-1)$.

$R_2$: $(p, x, p-1) \vdash p$ provided $x^{p-1} \equiv 1 \pmod{p}$.

THEOREM 1. *$p$ is a theorem $\equiv$ $p$ is a prime.*

THEOREM 2. *$p$ is a theorem $\supset$ $p$ has a proof of $\lceil 4 \log_2 p \rceil$ lines.*

## Primality testing is in NP ∩ co-NP

**Theorem.** [Pratt 1975] PRIMES ∈ **NP** ∩ **co-NP**.

**Pf sketch.** An odd integer $s$ is prime iff there exists an integer $1 < t < s$ s.t.

$$\begin{aligned} t^{s-1} &\equiv 1 \pmod{s} \\ t^{(s-1)/p} &\not\equiv 1 \pmod{s} \\ &\text{for all prime divisors } p \text{ of } s\text{-}1 \end{aligned}$$

| | |
|---|---|
| **instance s** | 437677 |
| **certificate t** | 17, $2^2 \times 3 \times 36473$ |

prime factorization of s–1
also need a recursive certificate
to assert that 3 and 36,473 are prime

CERTIFIER ($s$)

CHECK $s - 1 = 2 \times 2 \times 3 \times 36473$.

CHECK $17^{s-1} = 1 \pmod{s}$.

CHECK $17^{(s-1)/2} \equiv 437676 \pmod{s}$.

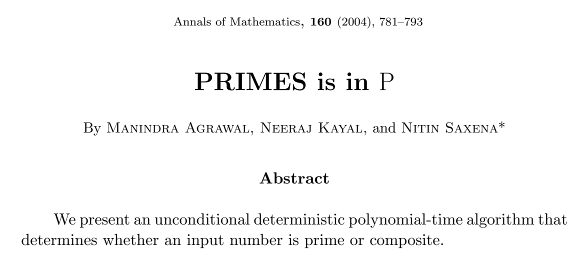
CHECK $17^{(s-1)/3} \equiv 329415 \pmod{s}$.

CHECK $17^{(s-1)/36{,}473} \equiv 305452 \pmod{s}$.

use repeated squaring

## Primality testing is in P

**Theorem.** [Agrawal–Kayal–Saxena 2004] PRIMES ∈ **P**.



Annals of Mathematics, **160** (2004), 781–793

### PRIMES is in P

By MANINDRA AGRAWAL, NEERAJ KAYAL, and NITIN SAXENA*

**Abstract**

We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.

## Factoring is in NP ∩ co-NP

FACTORIZE. Given an integer $x$, find its prime factorization.

FACTOR. Given two integers $x$ and $y$, does $x$ have a nontrivial factor $< y$?

**Theorem.** FACTOR $\equiv_P$ FACTORIZE.

**Pf.**

- $\le_P$ trivial.
- $\ge_P$ binary search to find a factor; divide out the factor and repeat. ▪

**Theorem.** FACTOR ∈ **NP** ∩ **co-NP**.

**Pf.**

- Certificate: a factor $p$ of $x$ that is less than $y$.
- Disqualifier: the prime factorization of $x$ (where each prime factor is less than $y$), along with a Pratt certificate that each factor is prime. ▪

## Is factoring in P ?

**Fundamental question.** Is FACTOR ∈ **P** ?

**Challenge.** Factor this number.

7403756347956171282804679609742957314259318888923128908493623263897276503402826627689199641962511784399589433050212758537011896809828673317327310893090055250511687706329907239638078671008609696253793465056379635

**RSA-704**
**($30,000 prize if you can factor)**

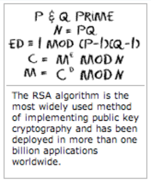
## Exploiting intractability

Modern cryptography.

- Ex. Send your credit card number to Amazon.
- Ex. Digitally sign an e-document.
- Enables freedom of privacy, speech, press, political association.

RSA. Based on dichotomy between complexity of two problems.

- To use: generate two random $n$-bit primes and multiply.
- To break: suffices to factor a $2n$-bit integer.



P & Q PRIME
N = PQ
ED ≡ 1 MOD (P-1)(Q-1)
C = M^E MOD N
M = C^D MOD N

The RSA algorithm is the most widely used method of implementing public key cryptography and has been deployed in more than one billion applications worldwide.

RSA algorithm

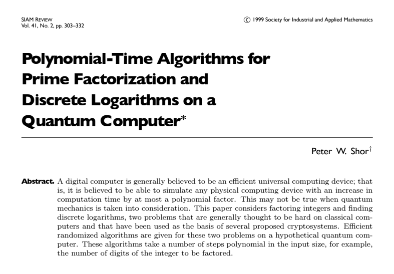
RSA sold
for $2.1 billion

or design a t-shirt

---

## Factoring on a quantum computer

Theorem. [Shor 1994] Can factor an $n$-bit integer in $O(n^3)$ steps on a "quantum computer."



SIAM Review
Vol. 41, No. 2, pp. 303–332

© 1999 Society for Industrial and Applied Mathematics

**Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer***

Peter W. Shor[†]

**Abstract.** A digital computer is generally believed to be an efficient universal computing device; that is, it is believed to be able to simulate any physical computing device with an increase in computation time by at most a polynomial factor. This may not be true when quantum mechanics is taken into consideration. This paper considers factoring integers and finding discrete logarithms, two problems that are generally thought to be hard on classical computers and that have been used as the basis of several proposed cryptosystems. Efficient randomized algorithms are given for these two problems on a hypothetical quantum computer. These algorithms take a number of steps polynomial in the input size, for example, the number of digits of the integer to be factored.

2001. Factored $15 = 3 \times 5$ (with high probability) on a quantum computer.

2012. Factored $21 = 3 \times 7$.

Fundamental question. Does **P** = **BQP** ?

quantum analog of **P**
(bounded error quantum polynomial time)

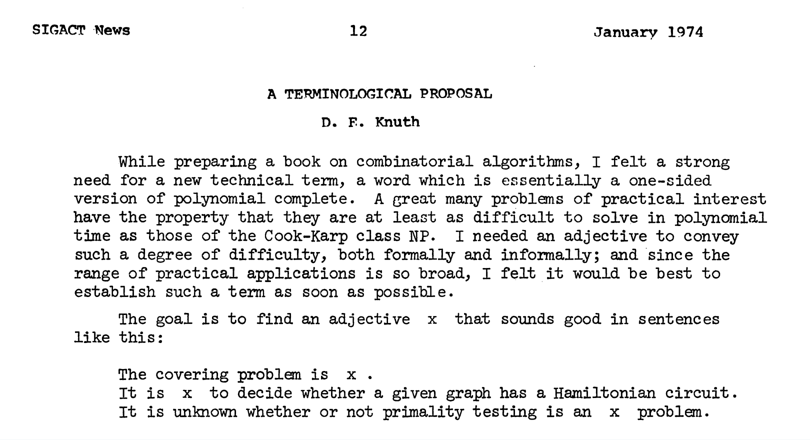
---

## 8. INTRACTABILITY II

- P vs. NP
- NP-complete
- co-NP
- *NP-hard*

---

## A note on terminology



SIGACT News 12 January 1974

A TERMINOLOGICAL PROPOSAL

D. E. Knuth

While preparing a book on combinatorial algorithms, I felt a strong need for a new technical term, a word which is essentially a one-sided version of polynomial complete. A great many problems of practical interest have the property that they are at least as difficult to solve in polynomial time as those of the Cook-Karp class NP. I needed an adjective to convey such a degree of difficulty, both formally and informally; and since the range of practical applications is so broad, I felt it would be best to establish such a term as soon as possible.

The goal is to find an adjective x that sounds good in sentences like this:

The covering problem is x .
It is x to decide whether a given graph has a Hamiltonian circuit.
It is unknown whether or not primality testing is an x problem.

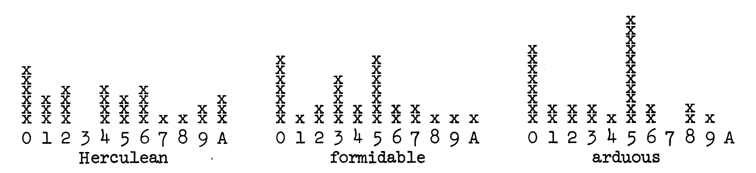
Note. The term $x$ does not necessarily imply that a problem is in **NP**, just that every problem in **NP** poly-time reduces to $x$.

## A note on terminology

Knuth's original suggestions.

- Hard.
- Tough.
- Herculean.
- Formidable.
- Arduous.

so common that it is unclear whether it is being used in a technical sense



**assign a real number between 0 and 1 to each choice**

## A note on terminology

Some English word write-ins.

- Impractical.
- Bad.
- Heavy.
- Tricky.
- Intricate.
- Prodigious.
- Difficult.
- Intractable.
- Costly.
- Obdurate.
- Obstinate.
- Exorbitant.
- Interminable.

## A note on terminology

Hard-boiled. [Ken Steiglitz] In honor of Cook.

Hard-ass. [Al Meyer] Hard as satisfiability.

Sisyphean. [Bob Floyd] Problem of Sisyphus was time-consuming.

Ulyssean. [Donald Knuth] Ulysses was known for his persistence.

> *" creative research workers are as full of ideas for new terminology as they are empty of enthusiasm for adopting it. "*
>
> *— Donald Knuth*

## A note on terminology: acronyms

PET. [Shen Lin] Probably exponential time.

- If **P** ≠ **NP**, provably exponential time.
- If **P** = **NP**, previously exponential time.

GNP. [Al Meyer] Greater than or equal to **NP** in difficulty.

- And costing more than the GNP to solve.

## A note on terminology: made-up words

Exparent. [Mike Paterson] Exponential + apparent.

Perarduous. [Mike Paterson] Throughout (in space or time) + completely.

Supersat. [Al Meyer] Greater than or equal to satisfiability.

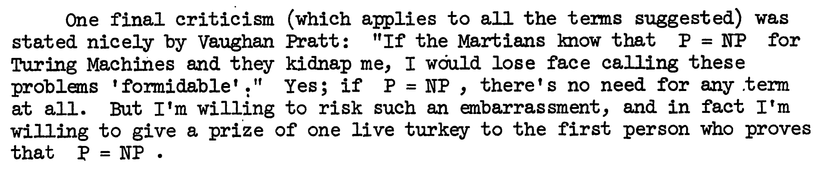
Polychronious. [Ed Reingold] Enduringly long; chronic.

---

## A note on terminology: consensus

NP-complete. A problem in **NP** such that every problem in **NP** poly-time reduces to it.

NP-hard. [Bell Labs, Steve Cook, Ron Rivest, Sartaj Sahni]
A problem such that every problem in **NP** poly-time reduces to it.



One final criticism (which applies to all the terms suggested) was stated nicely by Vaughan Pratt: "If the Martians know that P = NP for Turing Machines and they kidnap me, I would lose face calling these problems 'formidable'." Yes; if P = NP , there's no need for any term at all. But I'm willing to risk such an embarrassment, and in fact I'm willing to give a prize of one live turkey to the first person who proves that P = NP .