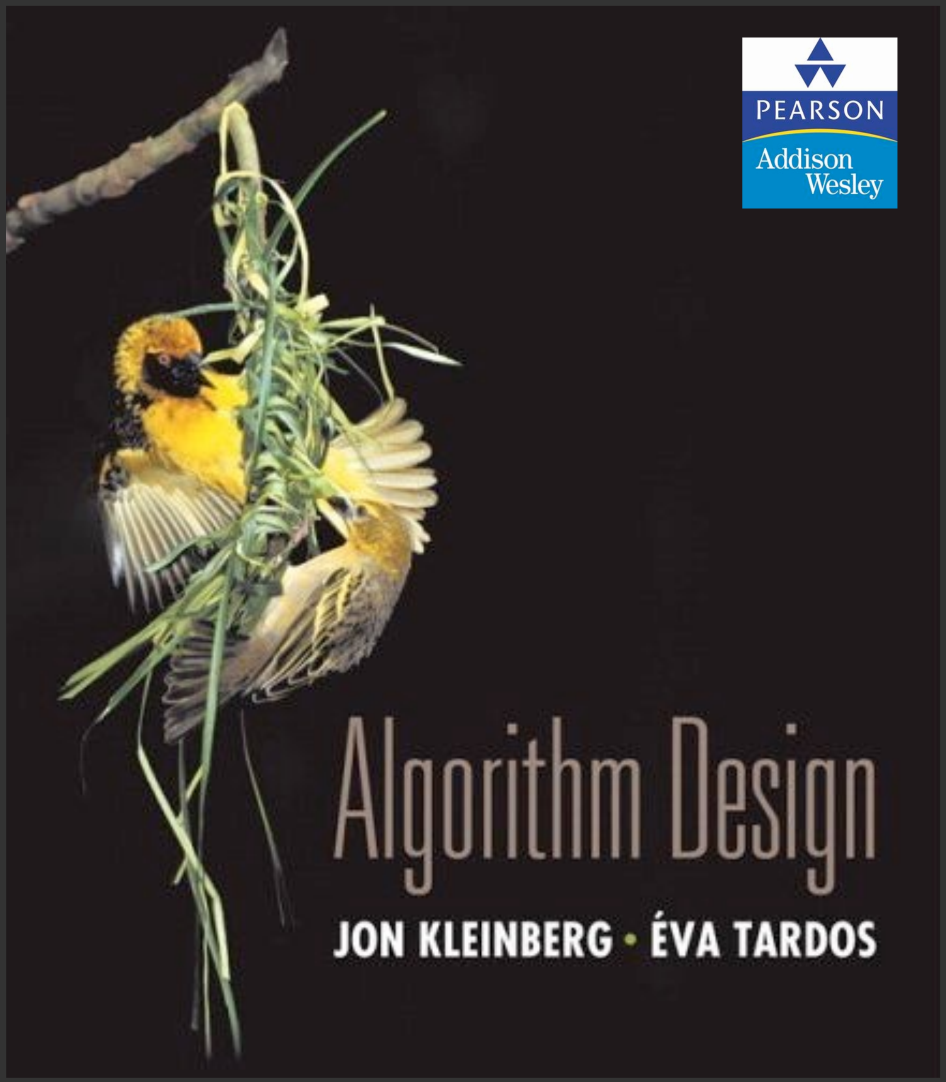


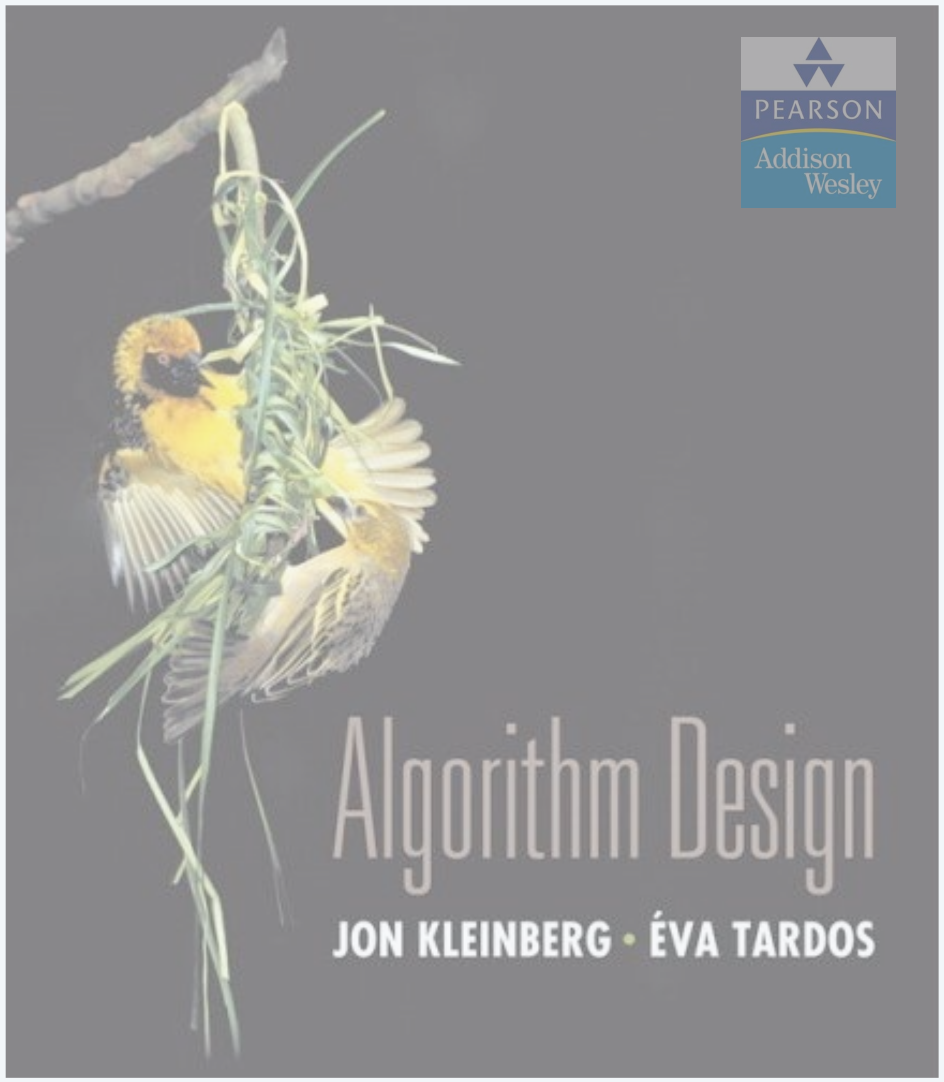
Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

1. Stable Matching

- *stable matching problem*
- *Gale–Shapley algorithm*
- *hospital optimality*
- *context*

Last updated on 2/5/18 3:54 PM



SECTION 1.1

1. STABLE MATCHING

- *stable matching problem*
- *Gale–Shapley algorithm*
- *hospital optimality*
- *context*

Matching med-school students to hospitals

Goal. Given a set of preferences among hospitals and med-school students, design a self-reinforcing admissions process.

Unstable pair. Hospital h and student s form an unstable pair if both:

- h prefers s to one of its admitted students.
- s prefers h to assigned hospital.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest prevents any hospital–student side deal.

Stable matching problem: input

Input. A set of n hospitals H and a set of n students S.

- Each hospital $h \in H$ ranks students.
- Each student $s \in S$ ranks hospitals.

one student per hospital (for now)

favorite — least favorite

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

hospitals' preference lists

favorite — least favorite

	1st	2nd	3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

students' preference lists

Perfect matching

Def. A matching M is a set of ordered pairs $h\text{–}s$ with $h \in H$ and $s \in S$ s.t.

- Each hospital $h \in H$ appears in at most one pair of M.
- Each student $s \in S$ appears in at most one pair of M.

Def. A matching M is perfect if $|M| = |H| = |S| = n$.

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

	1st	2nd	3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

a perfect matching M = { A–Z, B–Y, C–X }

Unstable pair

Def. Given a perfect matching M, hospital h and student s form an unstable pair if both:

- h prefers s to matched student.
- s prefers h to matched hospital.

Key point. An unstable pair h–s could each improve by joint action.

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

	1st	2nd	3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

A–Y is an unstable pair for matching M = { A–Z, B–Y, C–X }

Stable matching: quiz 1

Which pair is unstable in the matching { A–X, B–Z, C–Y } ?

- **A.** A–Y.
- **B.** B–X.
- **C.** B–Z.
- **D.** None of the above.

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

	1st	2nd	3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

Stable matching problem

Def. A stable matching is a perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n hospitals and n students, find a stable matching (if one exists).

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

	1st	2nd	3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

a stable matching M = { A–X, B–Y, C–Z }

Stable roommate problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

Stable roommate problem.

- $2n$ people; each person ranks others from 1 to $2n - 1$.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C

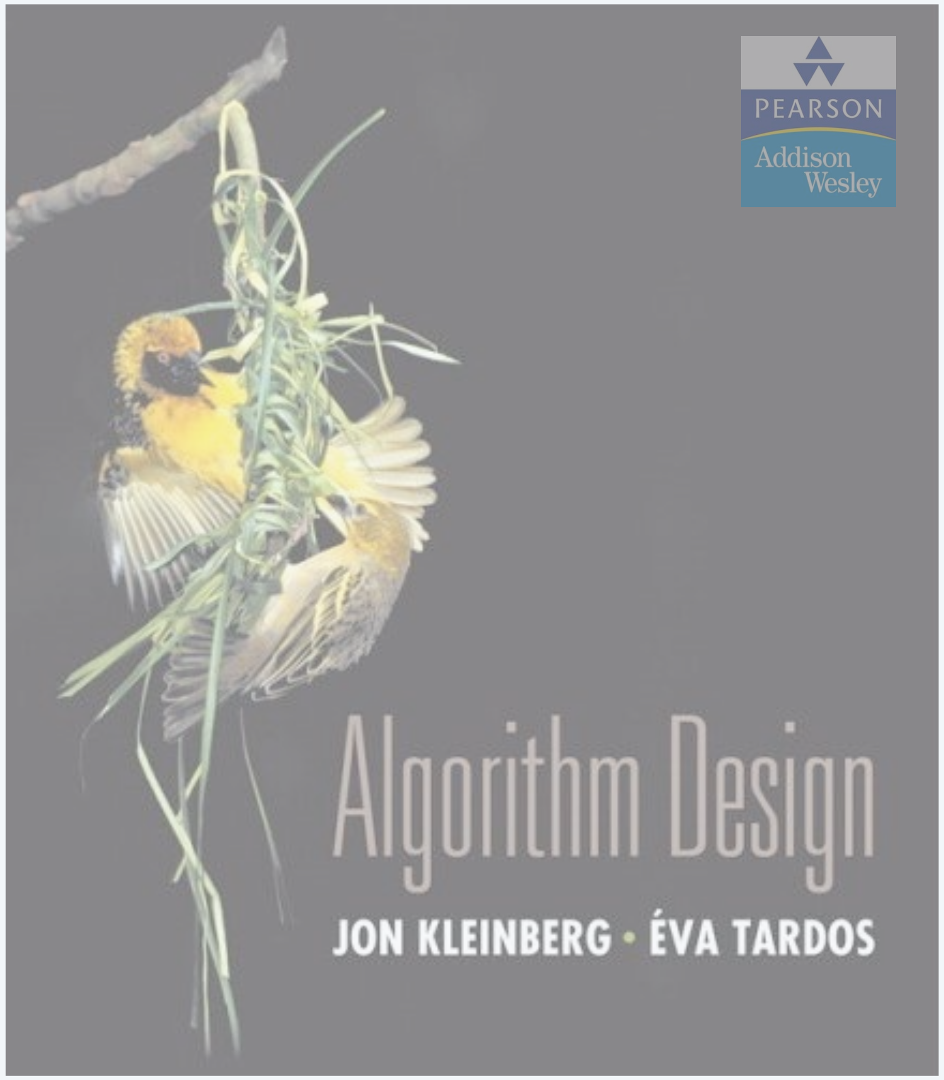
no perfect matching is stable

A–B, C–D $\Rightarrow$ B–C unstable

A–C, B–D $\Rightarrow$ A–B unstable

A–D, B–C $\Rightarrow$ A–C unstable

Observation. Stable matchings need not exist.



SECTION 1.1

1. STABLE MATCHING

- *stable matching problem*
- *Gale–Shapley algorithm*
- *hospital optimality*
- *context*

Gale–Shapley deferred acceptance algorithm

An intuitive method that guarantees to find a stable matching.

GALE–SHAPLEY (*preference lists for hospitals and students*)

INITIALIZE M to empty matching.

WHILE (some hospital h is unmatched and hasn't proposed to every student)

  $s \leftarrow$ first student on h's list to whom h has not yet proposed.

  IF (s is unmatched)

    Add h–s to matching M.

  ELSE IF (s prefers h to current partner h')

    Replace h'–s with h–s in matching M.

  ELSE

    s rejects h.

RETURN stable matching M.

Proof of correctness: termination

Observation 1. Hospitals propose to students in decreasing order of preference.

Observation 2. Once a student is matched, the student never becomes unmatched; only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of WHILE loop.
Pf. Each time through the WHILE loop, a hospital proposes to a new student. Thus, there are at most n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
A	V	W	X	Y	Z
B	W	X	Y	V	Z
C	X	Y	V	W	Z
D	Y	V	W	X	Z
E	V	W	X	Y	Z

	1st	2nd	3rd	4th	5th
V	B	C	D	E	A
W	C	D	E	A	B
X	D	E	A	B	C
Y	E	A	B	C	D
Z	A	B	C	D	E

n(n-1) + 1 proposals

Proof of correctness: perfect matching

Claim. Gale–Shapley outputs a matching.

Pf.

- Hospital proposes only if unmatched. $\Rightarrow$ matched to $\leq$ 1 student
- Student keeps only best hospital. $\Rightarrow$ matched to $\leq$ 1 hospital

Claim. In Gale–Shapley matching, all hospitals get matched.

Pf. [by contradiction]

- Suppose, for sake of contradiction, that some hospital $h \in H$ is unmatched upon termination of Gale–Shapley algorithm.
- Then some student, say $s \in S$, is unmatched upon termination.
- By Observation 2, s was never proposed to.
- But, h proposes to every student, since h ends up unmatched. ※

Claim. In Gale–Shapley matching, all students get matched.

Pf. [by counting]

- By previous claim, all n hospitals get matched.
- Thus, all n students get matched. ▪

Proof of correctness: stability

Claim. In Gale–Shapley matching M^*, there are no unstable pairs.

Pf. Consider any pair h–s that is not in M^*.

- Case 1: h never proposed to s.
  - $\Rightarrow$ h prefers its Gale–Shapley partner s' to s.
  - $\Rightarrow$ h–s is not unstable.

hospitals propose in decreasing order of preference

- Case 2: h proposed to s.
  - $\Rightarrow$ s rejected h (either right away or later)
  - $\Rightarrow$ s prefers Gale–Shapley partner h' to h.
  - $\Rightarrow$ h–s is not unstable.

students only trade up

- In either case, the pair h–s is not unstable. ▪

$h - s'$

$h' - s$

$\vdots$

Gale–Shapley matching M*

Summary

Stable matching problem. Given n hospitals and n students, and their preference lists, find a stable matching if one exists.

Theorem. [Gale–Shapley 1962] The Gale–Shapley algorithm guarantees to find a stable matching for any problem instance.

COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

1. Introduction. The problem with which we shall be concerned relates to the following typical situation: A college is considering a set of n applicants of which it can admit a quota of only q. Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the q best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept. Accordingly, in order for a college to receive q acceptances, it will generally have to offer to admit more than q applicants. The problem of determining how many and which ones to admit requires some rather involved guesswork. It may not be known (a) whether a given applicant has also applied elsewhere; if this is known it may not be known (b) how he ranks the colleges to which he has applied; even if this is known it will not be known (c) which of the other colleges will offer to admit him. A result of all this uncertainty is that colleges can expect only that the entering class will come reasonably close in numbers to the desired quota, and be reasonably close to the attainable optimum in quality.

Stable matching: quiz 2

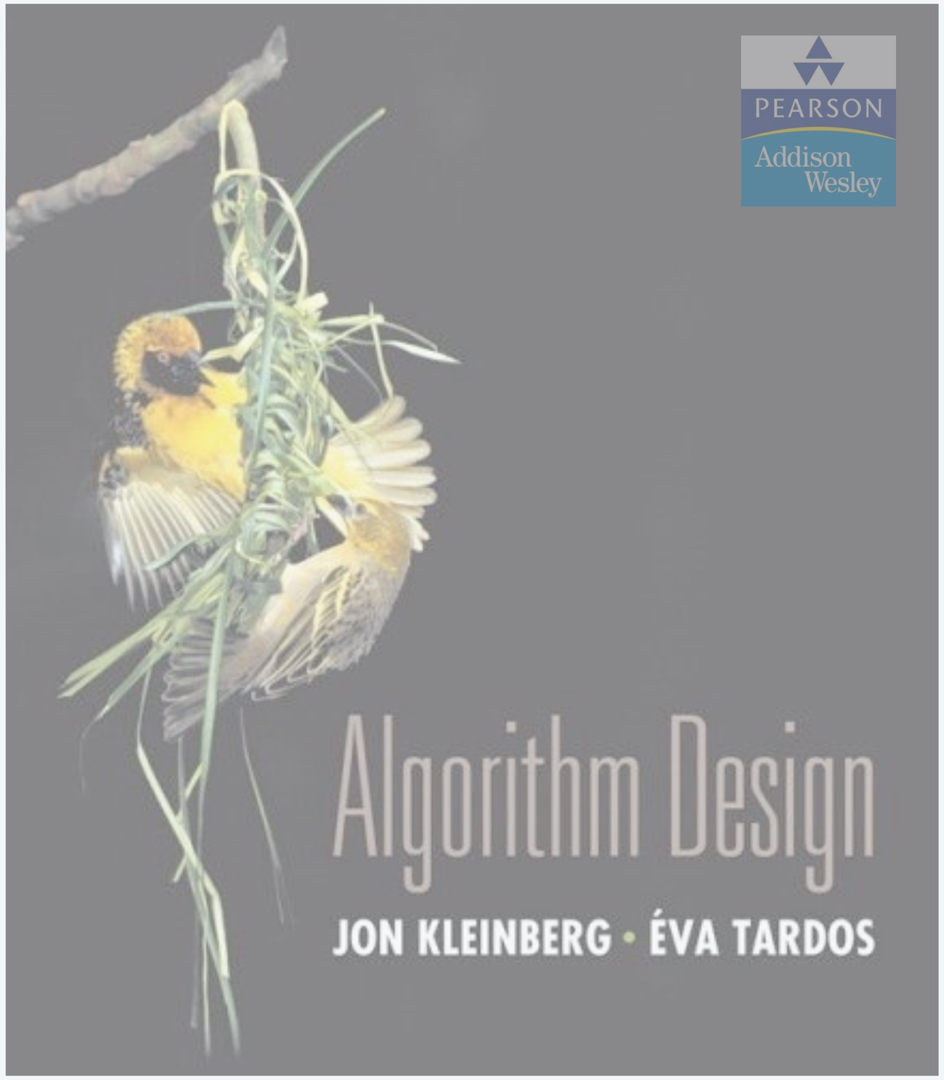
Do all executions of Gale–Shapley lead to the same stable matching?

A. No, because the algorithm is nondeterministic.

B. No, because an instance can have several stable matchings.

C. Yes, because each instance has a unique stable matching.

D. Yes, even though an instance can have several stable matchings and the algorithm is nondeterministic.



SECTION 1.1

1. STABLE MATCHING

- *stable matching problem*
- *Gale–Shapley algorithm*
- *hospital optimality*
- *context*

Understanding the solution

For a given problem instance, there may be several stable matchings.

	1st	2nd	3rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1st	2nd	3rd
X	B	A	C
Y	A	B	C
Z	A	B	C

an instance with two stable matchings: S = { A–X, B–Y, C–Z } and S′ = { A–Y, B–X, C–Z }

Understanding the solution

Def. Student s is a valid partner for hospital h if there exists any stable matching in which h and s are matched.

Ex.

- Both X and Y are valid partners for A.
- Both X and Y are valid partners for B.
- Z is the only valid partner for C.

	1st	2nd	3rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1st	2nd	3rd
X	B	A	C
Y	A	B	C
Z	A	B	C

an instance with two stable matchings: S = { A–X, B–Y, C–Z } and S′ = { A–Y, B–X, C–Z }

Stable matching: quiz 3

Who is the best valid partner for W in the following instance?

A.

B.

C.

D.

6 stable matchings
{ A–W, B–X, C–Y, D–Z }
{ A–X, B–W, C–Y, D–Z }
{ A–X, B–Y, C–W, D–Z }
{ A–Z, B–W, C–Y, D–X }
{ A–Z, B–Y, C–W, D–X }
{ A–Y, B–Z, C–W, D–X }

	1st	2nd	3rd	4th
A	Y	Z	X	W
B	Z	Y	W	X
C	W	Y	X	Z
D	X	Z	W	Y

	1st	2nd	3rd	4th
W	D	A	B	C
X	C	B	A	D
Y	C	B	A	D
Z	D	A	B	C

Understanding the solution

Def. Student s is a valid partner for hospital h if there exists any stable matching in which h and s are matched.

Hospital-optimal assignment. Each hospital receives best valid partner.

- Is it a perfect matching?
- Is it stable?

Claim. All executions of Gale–Shapley yield hospital-optimal assignment.
Corollary. Hospital-optimal assignment is a stable matching!

Hospital optimality

Claim. Gale–Shapley matching M^* is hospital-optimal.

Pf. [by contradiction]

- Suppose a hospital is matched with student other than best valid partner.
- Hospitals propose in decreasing order of preference.
  $\Rightarrow$ some hospital is rejected by a valid partner during Gale–Shapley
- Let h be first such hospital, and let s be the first valid partner that rejects h.
- Let M be a stable matching where h and s are matched.
- When s rejects h in Gale–Shapley, s forms (or re-affirms) commitment to a hospital, say h'.
  $\Rightarrow$ $\boxed{s \text{ prefers } h' \text{ to } h.}$
- Let s' be partner of h' in M.
- h' had not been rejected by any valid partner (including s') at the point when h is rejected by s. ← because this is the first rejection by a valid partner
- Thus, h' had not yet proposed to s' when h' proposed to s.
  $\Rightarrow$ $\boxed{h' \text{ prefers } s \text{ to } s'.}$
- Thus, h'–s is unstable in M, a contradiction. ▪

$h - s$

$h' - s'$

$\vdots$

stable matching M

Student pessimality

Q. Does hospital-optimality come at the expense of the students?

A. Yes.

Student-pessimal assignment. Each student receives worst valid partner.

Claim. Gale–Shapley finds student-pessimal stable matching M^*.

Pf. [by contradiction]

- Suppose h–s matched in M^* but h is not the worst valid partner for s.
- There exists stable matching M in which s is paired with a hospital, say h', whom s prefers less than h.
  $\Rightarrow$ $\boxed{s \text{ prefers } h \text{ to } h'.}$
- Let s' be the partner of h in M.
- By hospital-optimality, s is the best valid partner for h.
  $\Rightarrow$ $\boxed{h \text{ prefers } s \text{ to } s'.}$
- Thus, h–s is an unstable pair in M, a contradiction. ▪

$h' - s$

$h - s'$

$\vdots$

stable matching M

Stable matching: quiz 4

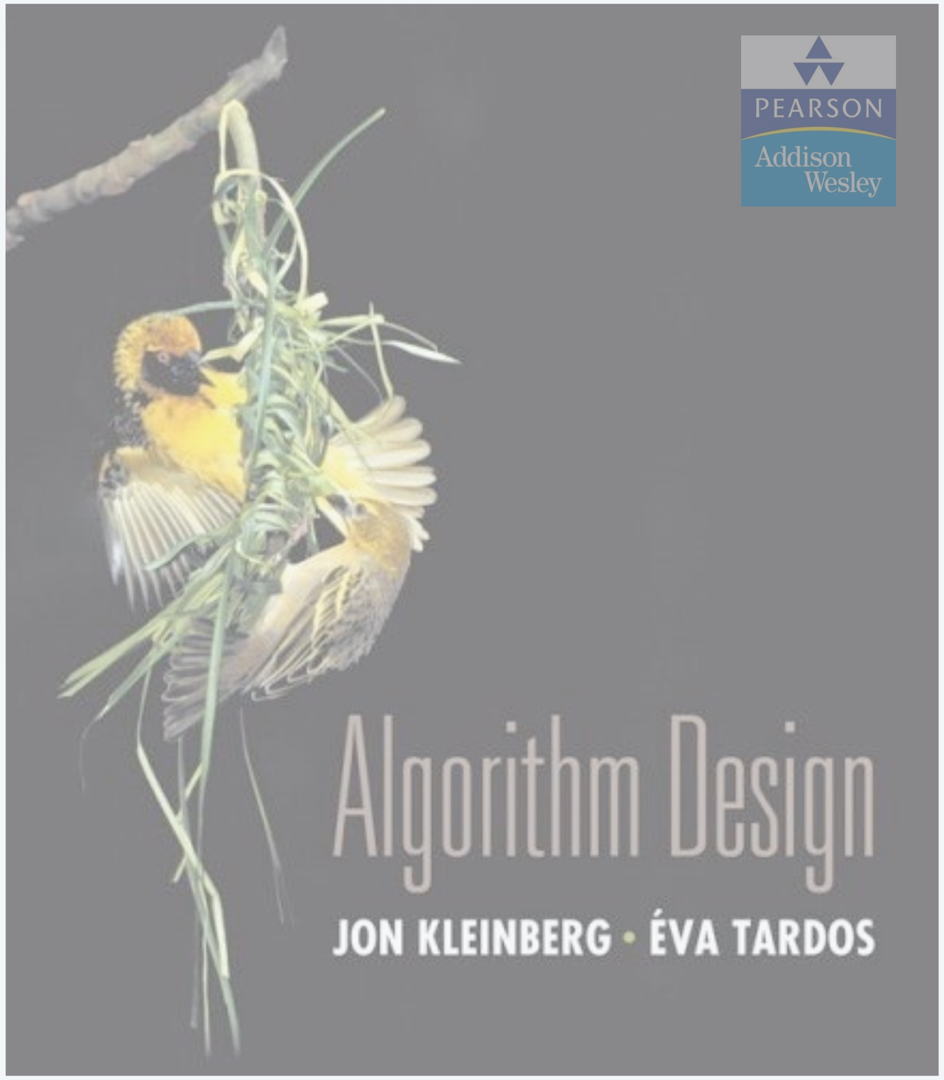
Suppose each agent knows the preference lists of every other agent before the hospital propose-and-reject algorithm is executed. Which is true?

A. No hospital can improve by falsifying its preference list.

B. No student can improve by falsifying their preference list.

C. Both A and B.

D. Neither A nor B.



SECTION 1.1

1. STABLE MATCHING

- *stable matching problem*
- *Gale–Shapley algorithm*
- *hospital optimality*
- *context*

Extensions

Extension 1. Some agents declare others as unacceptable.

Extension 2. Some hospitals have more than one position.

Extension 3. Unequal number of positions and students.

med-school student unwilling to work in Cleveland

≥ 43K med-school students; only 31K positions

Def. Matching M is unstable if there is a hospital h and student s such that:

- h and s are acceptable to each other; and
- Either s is unmatched, or s prefers h to assigned hospital; and
- Either h does not have all its places filled, or h prefers s to at least one of its assigned students.

Theorem. There exists a stable matching.

Pf. Straightforward generalization of Gale–Shapley algorithm.

Historical context

National resident matching program (NRMP).

- Centralized clearinghouse to match med-school students to hospitals.
- Began in 1952 to fix unraveling of offer dates.
- Originally used the "Boston Pool" algorithm.
- Algorithm overhauled in 1998.
  - med-school student optimal
  - deals with various side constraints (e.g., allow couples to match together)

hospitals began making offers earlier and earlier, up to 2 years in advance

stable matching no longer guaranteed to exist

The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design

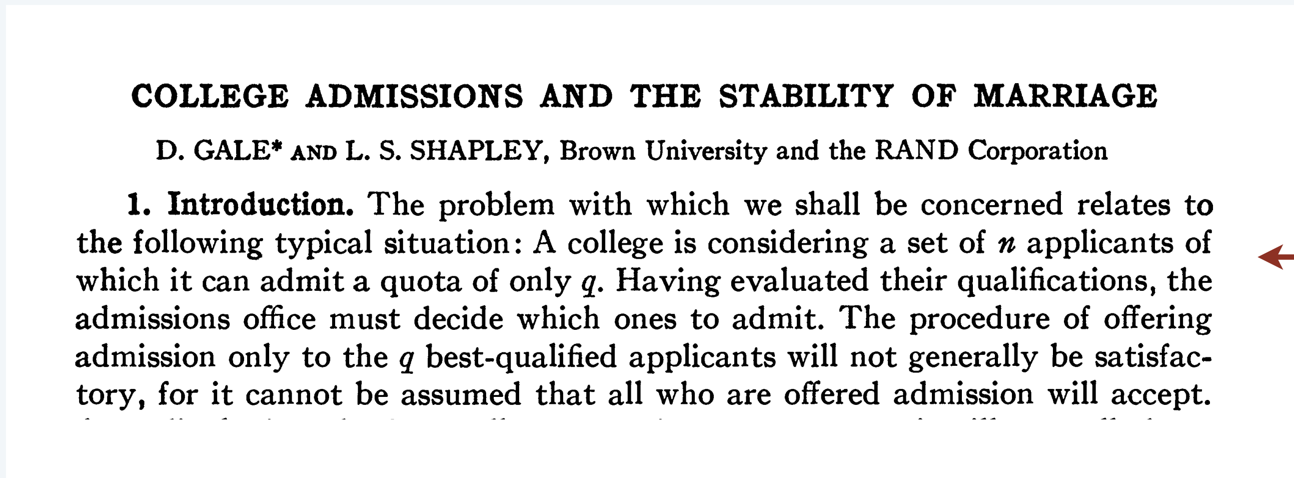
By Alvin E. Roth and Elliott Peranson*

We report on the design of the new clearinghouse adopted by the National Resident Matching Program, which annually fills approximately 20,000 jobs for new physicians. Because the market has complementarities between applicants and between positions, the theory of simple matching markets does not apply directly. However, computational experiments show the theory provides good approximations. Furthermore, the set of stable matchings, and the opportunities for strategic manipulation, are surprisingly small. A new kind of "core convergence" result explains this; that each applicant interviews only a small fraction of available positions is important. We also describe engineering aspects of the design process. (JEL C78, B41, J44)

THE MATCH
NATIONAL RESIDENT MATCHING PROGRAM®

2012 Nobel Prize in Economics

Lloyd Shapley. Stable matching theory and Gale–Shapley algorithm.



COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

1. Introduction. The problem with which we shall be concerned relates to the following typical situation: A college is considering a set of n applicants of which it can admit a quota of only q. Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the q best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept.

original applications: college admissions and opposite-sex marriage

Alvin Roth. Applied Gale–Shapley to matching med-school students with hospitals, students with schools, and organ donors with patients.

Lloyd Shapley Alvin Roth

New York City high school match

8th grader. Ranks top-5 high schools.

High school. Ranks students (and limit).

Goal. Match 90K students to 500 high school programs.

How Game Theory Helped Improve New York City's High School Application Process

By TRACY TULLIS DEC. 5, 2014

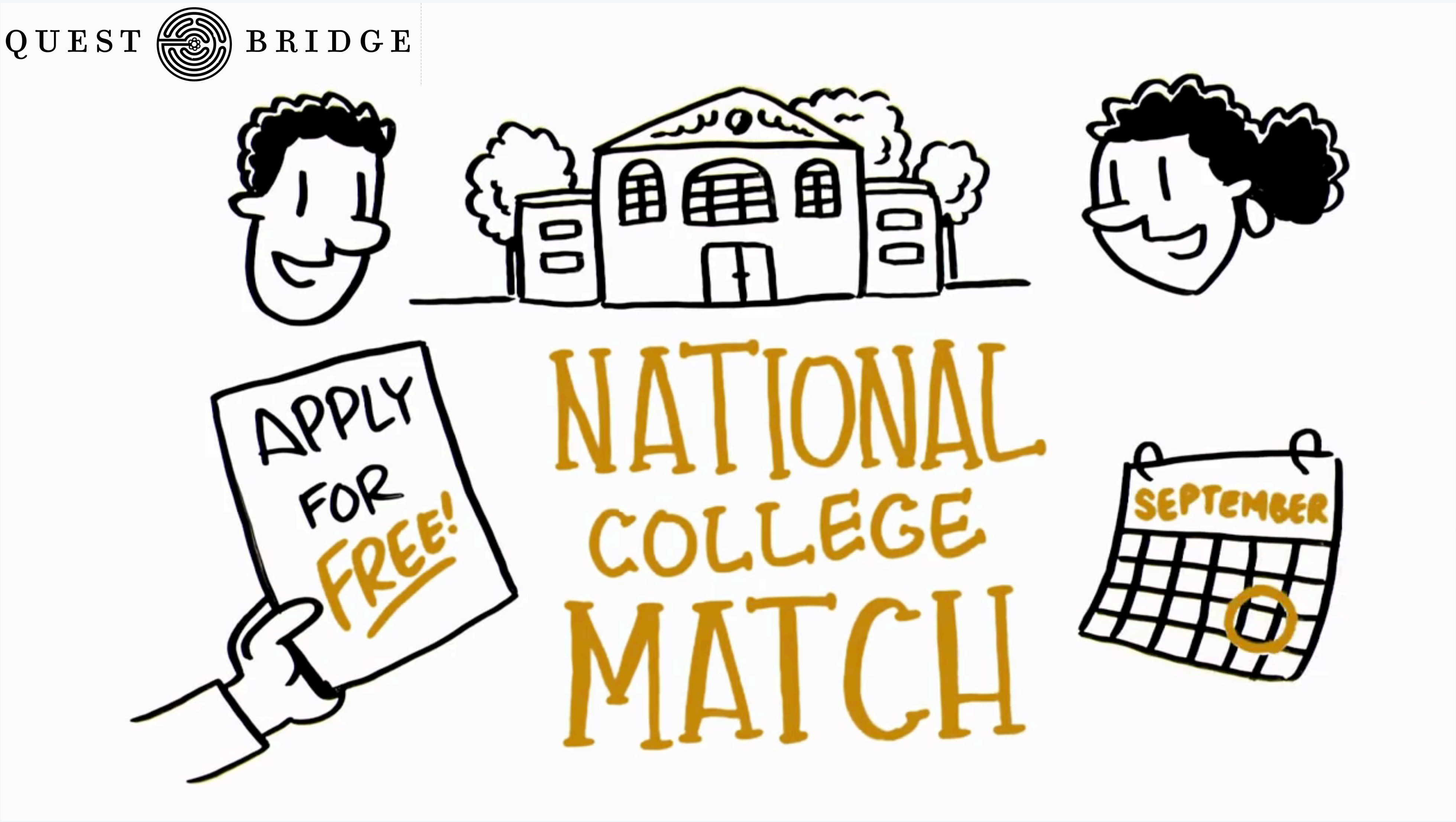
Tuesday was the deadline for eighth graders in New York City to submit applications to secure a spot at one of 426 public high schools. After months of school tours and tests, auditions and interviews, 75,000 students have entrusted their choices to a computer program that will arrange their school assignments for the coming year. The weeks of research and deliberation will be reduced to a fraction of a second of mathematical calculation: In just a couple of hours, all the sorting for the Class of 2019 will be finished.

Questbridge national college match

Low-income student. Ranks colleges.

College. Ranks students willing to admit (and limit).

Goal. Match students to colleges.



A modern application

Content delivery networks. Distribute much of world's content on web.

User. Preferences based on latency and packet loss.

Web server. Preferences based on costs of bandwidth and co-location.

Goal. Assign billions of users to servers, every 10 seconds.

Algorithmic Nuggets in Content Delivery

Bruce M. Maggs
Duke and Akamai
bmm@cs.duke.edu

Ramesh K. Sitaraman
UMass, Amherst and Akamai
ramesh@cs.umass.edu

This article is an editorial note submitted to CCR. It has NOT been peer reviewed.
The authors take full responsibility for this article's technical content. Comments can be posted through CCR Online.

ABSTRACT

This paper "peeks under the covers" at the subsystems that provide the basic functionality of a leading content delivery network. Based on our experiences in building one of the largest distributed systems in the world, we illustrate how sophisticated algorithmic research has been adapted to balance the load between and within server clusters, manage the caches on servers, select paths through an overlay routing network, and elect leaders in various contexts. In each instance, we first explain the theory underlying the algorithms, then introduce practical considerations not captured by the theoretical models, and finally describe what is implemented in practice. Through these examples, we highlight the role of algorithmic research in the design of complex networked systems. The paper also illustrates the close synergy that exists between research and industry where research ideas cross over into products and product requirements drive future research.