\[\{3, 6, 7, 9\} = S \text{ or } \{3, 6, 7\} = S \text{ or both.}\]

subject of vertices \( S \subseteq V \) such that \(|S| \leq k\) and for each edge \((u, v)\) either \( u \in S \) or \( v \in S \) or both.

**Vertex Cover**: Given a graph \( G = (V, E) \) and an integer \( k \), is there a vertex cover
constant, then it's also practical.

Remark. If $K$ is a constant, algorithm is poly-time if $K$ is a small

Better. $2^K n = 10^7$ feasible.
Brute. $k^{n+1} = 10^{34}$ infeasible.

Ex. $n = 1,000, K = 10$.

Goal. Limit exponential dependency on $K, e.g., to O(2^K n)$.

Finding Small Vertex Covers

- Takes $O(K \cdot n)$ time to check whether a subset is a vertex cover.
- Try all $O(K \cdot n^K)$ subsets of size $K$.
- Brute force $O(K \cdot n^{K+1})$.
- What if $K$ is small?
Claim. If $G$ has a vertex cover of size $k$, it has $\leq (k-1)$ edges.

Claim. Each vertex covers at most $n-1$ edges.

- Then $S \cup \{u\} \cup \{v\}$ is a vertex cover of $G$.
- Suppose $S$ is a vertex cover of $G - \{u\}$ of size $\leq k - 1$.

$\Rightarrow \quad \text{Pt.}$

$S - \{u\}$ is a vertex cover of $G - \{u\}$.
- $S$ contains either $u$ or $v$ (or both). Assume it contains $u$.

$\Rightarrow \quad \text{Pt.}$

Suppose $S$ is a vertex cover of $S - \{u\}$ of size $\leq k$.

$\Leftarrow \quad \text{Pt.}$

Delete $v$ and all incident edges.

Claim. Let $G - \{u\}$ be an edge of $G$. $G$ has a vertex cover of size $k - 1$.

Finding Small Vertex Covers
• Time: $O(kn)$
• There are $2^{k+1}$ nodes in the recursion tree; each invocation takes $O(1)$ time.
• Correctness follows previous two claims.

```java
{ 
  return a or b
  b = vertex-cover(G - \{\{n\}, \{k-1\}\})
  a = vertex-cover(G - \{\{n\}, \{k-1\}\})
  let (u, v) be any edge of G
  if (G contains u and v) return false
  if (G contains no edges) return true
  boolean vertex-cover(G, k)
}
```

Claim. The following algorithm determines if $G$ has a vertex cover of size $\leq k$ in $O(2^k kn)$ time.

Finding Small Vertex Covers: Algorithm
Finding Small Vertex Covers: Recursion Tree

\[
\begin{align*}
T(m, k) &= 2^c k^m \\
&= \begin{cases} 
2T(m, k-1) + \text{chm} & \text{if } k > 1 \\
0 & \text{if } k = 1
\end{cases}
\end{align*}
\]
Planar Graphs, Four-Coloring, and Hamiltonian Cycles

Duality: \textit{Vertices} \leftrightarrow \textit{Faces}
Planar Graphs, Four-Coloring, and Hamiltonian Cycles

Duality: Vertices $\leftrightarrow$ Faces

Vertex-coloring $\leftrightarrow$ Face-coloring
Reduce to degree 3:

\[
\begin{array}{c}
\text{Hamiltonian cycle } \Rightarrow 4 \text{ coloring} \\
\text{Color inside with 2 colors,} \\
\text{outside with 2 colors:} \\
\text{each side has no cycles}
\end{array}
\]
Planar Graphs, Four-coloring, and Hamiltonian Cycles

Duality: Vertices $\leftrightarrow$ Faces

Vertex-coloring $\leftrightarrow$ Face-coloring
Planar Graphs, Four-Coloring, and Hamiltonian Cycles

Duality: Vertices ↔ Faces

Vertex-coloring ↔ Face-coloring
NP-completeness of Planar Hamilton Cycle