Solutions to Final Exam

Problem 1

\[ \sum_{n \geq 2} b_n x^n = \sum_{n \geq 2} b_{n-1} x^n + \sum_{n \geq 2} x^n \sum_{1 \leq k \leq n-1} b_k b_{n-1-k} \]

\[ = x \sum_{n \geq 2} b_{n-1} x^{n-1} + x \sum_{m \geq 1} x^m \sum_{1 \leq k \leq m} b_k b_{m-k}. \]

This implies

\[ B(x) - b_1 - b_1 x = x(B(x) - b_0) + x(b_1 x + b_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots), \]

ie,

\[ B(x) - 1 - 2x = x(B(x) - 1) + x(B(x) - 1)B(x). \]

This leads to \( xB(x)^2 - B(x) + (1 + x) = 0 \), and hence using \( B(0) = b_0 = 1 \) we have

\[ B(x) = \frac{1 - \sqrt{1 - 4x(1 + x)}}{2x}. \]

Problem 2

(a) For \( n = 1 \), \( G_n \) consists of two isolated vertices and is thus by definition Eulerian. For \( n > 1 \), \( G_n \) is Eulerian since (A) it is connected (vertex 1 is connected to vertex \( n + 1 \) through \( 1 - 2 - (n + 1) \), and vertex 1 has an edge to each of the remaining vertices) and (B) every vertex has even degree (in fact \( 2n - 2 \)).

(b) For \( n = 1 \), \( G_n \) consists of two isolated vertices and has no Hamiltonian circuit. For \( n > 1 \), \( G_n \) has the following Hamiltonian circuit \( 1, 2, 3, \ldots, n - 1, n + 1, n + 2, \ldots, 2n, 1 \).

(c) The answer is \( \omega(G_n) = n \). Note that \( \omega(G_n) \geq n \) since \( \{1, 2, \ldots, n\} \) is a clique; \( \omega(G_n) < n + 1 \) since any clique can contain at most one of the vertices \( i, n + i \) for each \( 1 \leq i \leq n \).

(d) The answer is \( \chi(G_n) = n \). Note that \( \chi(G_n) \geq n \) since \( \{1, 2, \ldots, n\} \) is a clique and thus each vertex in it has to be painted with a different color; \( \chi(G_n) \leq n \) since we can just paint both vertices \( i, n + i \) with color \( i \), for each \( 1 \leq i \leq n \).

Problem 3 Let \( E_0 = \{\{4n + 1, n\}, \{4n + 1, 2n\}, \{4n + 1, 3n\}, \{4n + 1, 4n\}\} \), and

\[ E_1 = \{\{4n, 1\}, \{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}\}, \]
\[ E_2 = \{(n, n + 1), (n + 1, n + 2), (n + 2, n + 3), \ldots, (2n - 1, 2n)\}, \]
\[ E_3 = \{(2n, 2n + 1), (2n + 1, 2n + 2), (2n + 2, 2n + 3), \ldots, (3n - 1, 3n)\}, \]
\[ E_4 = \{(3n, 3n + 1), (3n + 1, 3n + 2), (3n + 2, 3n + 3), \ldots, (4n - 1, 4n)\}. \]

Then \( E = \cup_{0 \leq i \leq 4} E_i \).

A spanning tree of \( H_n \) has \( 4n \) edges, and can be specified by the 4 edges missing from \( E \). For \( \alpha \in \{0, 1, 2, 3, 4\} \), let \( s_{n,\alpha} \) be the number of spanning trees of \( H_n \) for which \( \alpha \) of the missing edges are from \( E_0 \). Then
\[
s_n = \sum_{0 \leq \alpha \leq 4} s_{n,\alpha}.
\]

Clearly, \( s_{n,4} = 0 \) since at least one edge from \( E_0 \) is needed to keep vertex \( 4n + 1 \) from being isolated.

To calculate \( s_{n,3} \), we count first how many spanning trees there are that contain \( \{4n + 1, n\} \) but no other edge from \( E_0 \). A spanning tree is now specified by the one missing edge from \( \cup_{1 \leq i \leq 4} E_i \), so that number is \( |\cup_{1 \leq i \leq 4} E_i| = 4n \). We can prove the same result if we count the number of spanning trees that contain any one specific edge but no other edges in \( E_0 \). Thus,
\[ s_{n,3} = 4 \cdot 4n = 16n. \]

To calculate \( s_{n,2} \), let \( a_n \) be the number of spanning trees containing \( \{4n + 1, n\}, \{4n + 1, 2n\} \) but no other edges in \( E_0 \); let \( b_n \) be the number of spanning trees containing \( \{4n + 1, n\}, \{4n + 1, 3n\} \) but no other edges in \( E_0 \). Clearly,
\[ s_{n,2} = 4a_n + 2b_n. \]

We compute \( a_n \). A spanning tree of this type is specified by a missing edge chosen from \( E_2 \), and a missing edge from \( E_1 \cup E_3 \cup E_4 \). Thus,
\[ a_n = |E_2| \cdot |E_1 \cup E_3 \cup E_4| = 3n^2. \]

Similarly,
\[ b_n = |E_2 \cup E_3| \cdot |E_1 \cup E_4| = 4n^2. \]

This leads to
\[ s_{n,2} = 4 \cdot 3n^2 + 2 \cdot 4n^2 = 20n^2. \]

To calculate \( s_{n,1} \), let \( c_n \) be the number of spanning trees containing \( \{4n + 1, n\}, \{4n + 1, 2n\}, \{4n + 1, 3n\} \) but no other edges in \( E_0 \). Then \( s_{n,1} = 4c_n \). To compute \( c_n \), note that

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such a spanning tree is specified by a missing edge from each of the sets $E_2, E_3, E_4 \cup E_1$. Hence,

$$s_{n,1} = 4 \cdot 2n^3 = 8n^3.$$  

To calculate $s_{n,0}$, note that such a spanning tree is specified by a missing edge from each of the sets $E_1, E_2, E_3, E_4$. Thus,

$$s_{n,0} = |E_1| \cdot |E_2| \cdot |E_3| \cdot |E_4| = n^4.$$  

Putting everything together, we have

$$s_n = \sum_{0 \leq \alpha \leq 4} s_{n,\alpha} = n^4 + 8n^3 + 20n^2 + 16n.$$