COS402- Artificial Intelligence
Fall 2015

Lecture 15: Decision Theory: Utility and Expected values
Outline

• Brief review on approximate inference in Bayes Nets
• Brief review on HMM
• Brief review on Kalman filters
• Brief review on particle filtering
• Decision Theory: Utility and expected value
Approximate inference in BN

• Direct sampling
  – Prior sample algorithm: for joint probability
  – Rejection sampling: for conditional probability
  – Likelihood sampling: for conditional probability

• How to sample the variables?

  • \( P(J=t, M=t) = ? \)
  • \( P(J=t, M=t | B=t) = ? \)
  • \( P(J=t | E=t) = ? \)
Approximate inference in BN

• MCMC
  – A state in MCMC specifies a value for every variable in the BN.
  – Initialize the state with random values for all the non-evidence variable, and copy the evidence for the evidence variables
  – Repeat N times (long enough to assume convergence: stationary distribution.)
    • Randomly choose a non-evidence variable $z$, set the value of $z$ by sampling from $P(z|mb(z))$
  – Estimate $P(X|e)$
Review questions: true/false

1. Approximate inference methods are usually used in large multiply connected networks because exact inference is intractable in these networks.

2. When using direct sampling methods in a Bayesian network, variables can be sampled in any order.

3. The weakness of rejection sampling is that it could take a long time if the evidence rarely happens.

4. Likelihood weighting samples all variables, but does not reject samples that are not consistent with the evidence.
5. Unlike direct sampling methods which generate each sample from scratch, MCMC generates each sample by making a random change to the preceding sample.

6. Gibbs Samplings is a particular form of MCMC. It starts with an arbitrary state by setting a random value for each of the variables in the Bayesian network, and then generates the next state by randomly sampling a value for one of the variables.
Hidden Markov Models

- $X_t$: 
- $E_t$: 

\[ X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \ldots \]

\[ E_1 \rightarrow E_2 \rightarrow \ldots \]
Visible Variables

- \( X_t \): random variable
  - State at time \( t \)
  - Discrete, finite number of values
  - Single variable representing a single state, can be decomposed into several variables
  - Hidden, invisible

- \( E_t \): random variable, evidence at time \( t \)
Hidden Markov Models (parameters)

- $P(X_0)$: the initial state model
- $P(X_t | X_{t-1})$: Transition model (usually assume stationary, same for all $t$)
- $P(E_t | X_t)$: sensor/observation model (usually assume stationary.)
Hidden Markov Models (2 Markov assumptions)

- $P(X_{t+1} | X_{0:t}) = P(X_{t+1} | X_t)$
  - The future is independent of the past given the present.

- $P(E_t | X_{0:t}, E_{1:t-1}) = P(E_t | X_t)$
  - Current evidence only depends on current state.

Note:

- given the 2 Markov assumptions, you can draw the Bayes Net; Given the Bayes net, the 2 Markov assumptions are implied.

- HMMs are special cases of BNs, what is the full joint probability? $P(X_{0:t}, E_{1:t}) = ?$
Hidden Markov Models (4 basic tasks)

- Filtering: Where am I now?
  - \( P(X_{t+1} \mid e_{1:t+1}) = ? \)

- Prediction: Where will I be in k steps?
  - \( P(X_{t+k} \mid e_{1:t}) = ? \)

- Smoothing: Where was I in the past?
  - \( P(X_k \mid e_{1:t}) = ? \) \((k < t)\)

- Finding the most likely sequence
  - Max \( P(X_{0:t} \mid e_{1:t}) = ? \)
Hidden Markov Models (4 basic tasks)

- **Filtering:** $P(X_{t+1} | e_{1:t+1}) = ?$
- **Prediction:** $P(X_{t+k} | e_{1:t}) = ?$
- **Smoothing:** $P(X_k | e_{1:t}) = ?$ (k < t)
- **Finding the most likely sequence:** $\text{Max } P(X_{0:t} | e_{1:t}) = ?$

**Question?**
- Time complexity for the 4 basic tasks? $O(t \cdot (#\text{states})^2)$
- Can we do other inference in HMM? $P(E_2 | X_1, X_3) = ?$, time complexity?
Kth order Hidden Markov Models

- First order HMM
  - \( P(X_{t+1} | X_{0:t}) = P(X_{t+1} | X_t) \)

- Second order HMM
  - \( P(X_{t+1} | X_{0:t}) = P(X_{t+1} | X_t, X_{t-1}) \)

- Kth order HMM?
  - The future is dependent on the last k states.
Kalman Filters

- $P(X_0)$: Gaussian distribution
- $P(X_{t+1} | X_t)$: Linear Gaussian distribution
  - The next state $X_{t+1}$ is a linear function of the current state $X_t$, plus some Gaussian noise.
- $P(E_t | X_t)$: Linear Gaussian distribution
- Filtering: $P(X_{t+1} | e_{1:t+1})$ is also a Gaussian distribution.
Particle Filtering—When to use it?

- In DBNs where state variables are continuous, but both the initial state distribution and transitional model are not Gaussian.
- In DBNs where state variables are discrete, but the state space is huge.
- HMMs with huge state space.
Particle Filtering—How does it work?

- First, a population of $N$ samples is created by sampling from the prior distribution $P(X_0)$.

- Repeat the update cycle for $t = 0, 1, ...$
  
  1. Each sample is propagated forward by sampling the next state value $X_{t+1}$ based on the transitional model $P(X_{t+1} | x_t)$.
  
  2. Each sample is weighted by the likelihood it assigns to the new evidence. $P(e_{t+1} | x_{t+1})$
  
  3. Resample to generate a new population of $N$ samples: The probability that a sample is selected is proportional to its weight. The new samples are un-weighted.
Particle Filtering—Example

\[ P(X_0) = (0.4, 0.2, 0.4) \] , \( e = \{T,F\} \) , \( x = \{A,B,C\} \) , \( N = 10 \)

- \( t = 0 \), \( P(X_0) \)
$P(X_0) = (0.4, 0.2, 0.4)$, $e = \{T,F\}$, $x = \{A,B,C\}$, $N = 10$

- $t=0$, $P(X_0)$

- $t=1$,
  - $P(X_1 | x_0)$
  - $e_1 = T$
  - $P(e_1 | x_1)$
Particle Filtering—Example

\[ P(X_0) = (0.4, 0.2, 0.4), e=\{T,F\}, x=\{A,B,C\}, N=10 \]

- \[ t = 0 \]
  - \[ P(X_0) \]

- \[ t = 1 \]
  - \[ P(X_1 | x_0) \]
  - \[ e_1 = T \]
  - \[ P(e_1 | x_1) \]
  - \[ P(X_1 | e_1) = (0.4, 0.6) \]

- \[ t = 2 \ldots \]
Particle Filtering—Demo?

- [http://robots.stanford.edu/movies/sca80a0.avi](http://robots.stanford.edu/movies/sca80a0.avi)

- A robot is wandering around in some cluster of rooms

- Modeled as HMM

  - States: locations
  
  - Observations: sonar readings
  
  - Task: Determining current state

  - Particle filtering: the green dot is the robot’s actual location; the little red dots are the particles(samples.)
1. HMMS and Kalman Filters are special cases of Dynamic Bayesian Networks (DBNs).

2. HMMS, Kalman Filters, Particle filtering and Dynamic Bayesian Networks (DBNs) are all temporal models to model a world which changes over time.

3. Filtering, prediction, smoothing, and finding the most likely sequence are the four basic tasks in HMMS and they all can be done in time of $O(t \cdot (#\text{states})^2)$.

4. The state variables in Kalman Filters are visible and continuous.
5. In particle filtering, if more samples are used, the estimation of the belief state will be more accurate.

6. Particle filtering can be used in any Bayesian nets.

7. At resampling of particle filtering, the probability that a sample is selected is proportional to its weight.

8. As using more samples will increase the accuracy of the estimation, the number of samples of the new population will increase at each time step in particle filtering.
Decision theory: Utility and expected value

- Expected value (expectation) of a discrete random variable
  - Weighted average of all possible values
    - $E[X] = \sum_x P(X = x) \times x$

- Expected value of a discrete random variable
  - Replace the sum with an integral and the probabilities with probability densities.

- Conditional expectation
  - $E[X|y=a] = \sum_x P(X = x|y = a) \times x$

- Expectation of a real-valued function
  - $E[f(x)] = \sum_x P(X = x) \times f(x)$
Decision theory: Utility and expected value

• Linearity of expectations
  – (1) $E[X + c] = E[X] + c$
  – (2) $E[c \cdot X] = c \cdot E[X]$
  – (3) $E[X + Y] = E[X] + E[Y]$
  – Note: X and Y do not need to be independent.

• Examples:
  – $E[X] = ?$ If X is the result of throwing a die.
  – $E[X] = ?$ If X is the number of heads when throw two fair coins.
Decision theory: MDP

• General principle:
  – Assign utilities to states
  – Take action that yields highest expected utility
  – Rational decision vs. human decision

• Simple decision vs complex decision
  – Simple decision: make a single decision, achieve short-term goal.
  – Complex decision: make a sequence of decisions, achieve long-term goal.
    • We will look at problems of making complex decisions
    • Markov assumption: The next state only depends on current state and action.
• W2 and Quiz 1 will be returned after class today

• W4 has been released and is due on Tuesday Nov. 24th

--- Turn in hard copies in class