Linear Programming

Applications

Agriculture. Diet problem.
Computer science. Compiler register allocation, data mining.
Electrical engineering. VLSI design, optimal clocking.
Energy. Blending petroleum products.
Economics. Equilibrium theory, two-person zero-sum games.
Environment. Water quality management.
Finance. Portfolio optimization.
Logistics. Supply-chain management.
Management. Hotel yield management.
Marketing. Direct mail advertising.
Manufacturing. Production line balancing, cutting stock.
Operations research. Airline crew assignment, vehicle routing.
Physics. Ground states of 3-D Ising spin glasses.
Plasma physics. Optimal stellarator design.
Telecommunication. Network design, Internet routing.
Sports. Scheduling ACC basketball, handicapping horse races.

Brewery Problem: A Toy LP Example

Small brewery produces ale and beer:
- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

<table>
<thead>
<tr>
<th>Beverage</th>
<th>Corn (pounds)</th>
<th>Hops (ounces)</th>
<th>Malt (pounds)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ale (barrel)</td>
<td>5</td>
<td>4</td>
<td>35</td>
<td>13</td>
</tr>
<tr>
<td>Beer (barrel)</td>
<td>15</td>
<td>4</td>
<td>20</td>
<td>23</td>
</tr>
</tbody>
</table>

Limit 480 160 1190

How can brewer maximize profits?
- Devote all resources to ale: 34 barrels of ale ⇒ $442.
- Devote all resources to beer: 32 barrels of beer ⇒ $736.
- 7.5 barrels of ale, 29.5 barrels of beer ⇒ $776.
- 12 barrels of ale, 28 barrels of beer ⇒ $800.

Linear Programming

What is it?
- Quintessential tool for optimal allocation of scarce resources, among a number of competing activities.
- Powerful and general problem-solving method that encompasses:
  - shortest path, network flow, MST, matching
  - Ax = b, 2-person zero sum games

Why significant?
- Widely applicable.
- Fast commercial solvers: CPLEX, OSL.
- Powerful modeling languages: AMPL, GAMS.
- Ranked among most important scientific advances of 20th century.
- Dominates world of industry.

Ex: Delta claims saving $100 million per year using LP
Brewery Problem

\[ \text{max } 13A + 23B \]
\[ \text{s.t. } 5A + 15B \leq 480 \]
\[ 4A + 4B \leq 160 \]
\[ 35A + 20B \leq 1190 \]
\[ A, B \geq 0 \]

Brewery Problem: Objective Function

Brewery Problem: Geometry

Brewery problem observation. Regardless of objective function coefficients, an optimal solution occurs at an extreme point.
"Standard form" LP.
- Input: real numbers \(a_{ij}, c_j, b_i\).
- Output: real numbers \(x_j\).
- \(n\) = # nonnegative variables, \(m\) = # constraints.
- Maximize linear objective function subject to linear inequalities.

\[
\begin{align*}
\text{(P)} \quad & \text{max} \sum_{j=1}^{n} c_j x_j \\
\text{s. t.} & \sum_{j=1}^{n} a_{ij} x_j = b_i \quad 1 \leq i \leq m \\
& x_j \geq 0 \quad 1 \leq j \leq n
\end{align*}
\]

Linear. No \(x^2\), \(xy\), \(\arccos(x)\), etc.

Programming. Planning (term predates computer programming).

Original input.

\[
\begin{align*}
\text{max} \quad & 13A + 23B \\
\text{s. t.} & 5A + 15B \leq 480 \\
& 4A + 4B \leq 160 \\
& 35A + 20B \leq 1190 \\
& A, B \geq 0
\end{align*}
\]

Standard form.
- Add slack variable for each inequality.
- Now a 5-dimensional problem.

\[
\begin{align*}
\text{(P)} \quad & \text{max} c^T x \\
\text{s. t.} & Ax = b \\
& x \geq 0
\end{align*}
\]

Geometry.
- Inequality: halfplane (2D), hyperplane (kD).
- Bounded feasible region: convex polygon (2D), convex polytope (kD).

Convex set. If two points \(a\) and \(b\) are in the set, then so is \(\frac{1}{2}(a + b)\).

Extreme point. A point in the set that can’t be written as \(\frac{1}{2}(a + b)\), where \(a\) and \(b\) are two distinct points in the set.

Extreme point property. If there exists an optimal solution to (P), then there exists one that is an extreme point.

Consequence. Only need to consider finitely many possible solutions.

Challenge. Number of extreme points can be exponential!

Greedy property. Extreme point is optimal iff no neighboring extreme point is better.

\(n\)-dimensional hypercube

Local optima are global optima.
Simplex Algorithm

Simplex algorithm. [George Dantzig, 1947]
- Developed shortly after WWII in response to logistical problems, including Berlin airlift.
- One of greatest and most successful algorithms of all time.

Generic algorithm.
- Start at some extreme point.
- Pivot from one extreme point to a neighboring one.
- Repeat until optimal.

How to implement? Linear algebra.

Simplex Algorithm: Initialization

\[
\begin{align*}
\text{max } Z & \text{ subject to } \\
13A + 23B & = 0 \\
5A + 15B + S_C & = 480 \\
4A + 4B + S_H & = 160 \\
35A + 20B + S_M & = 1190 \\
A, B, S_C, S_H, S_M & \geq 0 \\
\end{align*}
\]

Basis = \{S_C, S_H, S_M\}

Simplex Algorithm: Pivot 1

\[
\begin{align*}
\text{max } Z & \text{ subject to } \\
13A + 23B & = 0 \\
5A + 15B + S_C & = 480 \\
4A + 4B + S_H & = 160 \\
35A + 20B + S_M & = 1190 \\
A, B, S_C, S_H, S_M & \geq 0 \\
\end{align*}
\]

Basis = \{S_C, S_H, S_M\}

Substitute: \( B = 1/15 \ (480 - 5A - S_C) \)
Simplex Algorithm: Pivot 1

Why pivot on column 2?
- Each unit increase in B increases objective value by $23.
- Pivoting on column 1 also OK.

Why pivot on row 2?
- Preserves feasibility by ensuring RHS ≥ 0.
- Minimum ratio rule: min {480/15, 160/4, 1190/20}.

Simplex Algorithm: Optimality

Q. When to stop pivoting?
A. When all coefficients in top row are non-positive.

Q. Why is resulting solution optimal?
A. Any feasible solution satisfies system of equations in tableaux.
- In particular: Z = 800 − S_C − 2S_H
- Thus, optimal objective value Z* ≤ 800 since S_C, S_H ≥ 0.
- Current BFS has value 800 ⇒ optimal.

Simplex Algorithm: Pivot 2

max Z subject to
13A + 23B − Z = 0
5A + 15B + S_C = 480
4A + 4B + S_H = 160
35A + 20B + S_M = 1190
A, B, S_C, S_H, S_M ≥ 0
Basis = {S_C, S_H, S_M}
A = B = 0
Z = 0
S_C = 480
S_H = 160
S_M = 1190

max Z subject to
14 A − 22 S_C − Z = −736
3 A + B + 1/13 S_C = 32
3 A + B + 1/13 S_H = 32
A, B, S_C, S_H, S_M ≥ 0
Basis = {B, S_H, S_M}
A = S_C = 0
Z = 736
B = 32
S_H = 32
S_M = 550


max Z subject to
− S_C − 2 S_H − Z = −800
B + 1/13 S_C + 1/13 S_H = 28
A − 1/13 S_C + 1/13 S_H = 12
A, B, S_C, S_H, S_M ≥ 0
Basis = {A, B, S_M}
S_C = S_H = 0
Z = 800
B = 28
A = 12
S_M = 110

Simplex Algorithm: Bare Bones Implementation

Construct the simplex tableaux.

```
public class Simplex {
    private double[][] a; // simplex tableaux
    private int m, n;

    public Simplex(double[][] A, double[] b, double[] c) {
        M = b.length;
        N = c.length;
        a = new double[M+1][N+1];
        for (int i = 0; i < M; i++)
            for (int j = 0; j < N; j++)
                a[i+1][j] = A[i][j];
        for (int i = 0; i < M; i++)
            a[i][M+N] = b[i];
    }
```
Simplex Algorithm: Degeneracy

Degeneracy. New basis, same extreme point.

“stalling” is common in practice.

Cycling. Get stuck by cycling through different bases that all correspond to same extreme point.

- Doesn’t occur in the wild.
- Bland’s least index rule guarantees finite # of pivots.

Simplex Algorithm: Running Time

Remarkable property. In practice, simplex algorithm typically terminates after at most $2(m+n)$ pivots.

- No polynomial pivot rule known.
- Most pivot rules known to be exponential (or worse) in worst-case.

Pivoting rules. Carefully balance the cost of finding an entering variable with the number of pivots needed.
Simplex Algorithm: Implementation Issues

Implementation issues.
- Avoid stalling.
- Choosing the pivot.
- Numerical stability.
- Maintaining sparsity.
- Detecting infeasibility
- Detecting unboundedness.
- Preprocessing to reduce problem size.

Commercial solvers routinely solve LPs with millions of variables and tens of thousands of constraints.

LP Duality: Economic Interpretation

Brewer’s problem. Find optimal mix of beer and ale to maximize profits.

\[
\begin{align*}
\text{(P)} & \quad \text{max} & & 13A + 23B \\
& \text{s.t.} & & 5A + 15B \leq 480 \\
& & & 4A + 4B \leq 160 \\
& & & 35A + 20B \leq 1190 \\
A, B & \geq 0
\end{align*}
\]

\[
A^* = 12 \\
B^* = 28 \\
\text{OPT} = 800
\]

Entrepreneur’s problem. Buy resources from brewer at min cost.
- \(C, H, M\) = unit price for corn, hops, malt.
- Brewer won’t agree to sell resources if \(5C + 4H + 35M < 13\).

\[
\begin{align*}
\text{(D)} & \quad \text{min} & & 480C + 160H + 1190M \\
& \text{s.t.} & & 5C + 4H + 35M \geq 13 \\
& & & 15C + 4H + 20M \geq 23 \\
C, H, M & \geq 0
\end{align*}
\]

\[
C^* = 1 \\
H^* = 2 \\
M^* = 0 \\
\text{OPT} = 800
\]


CPLEX solver. Industrial strength solver.

\[
\begin{align*}
\text{beer.mod} \\
\text{set INGR := corn hops malt;} \\
\text{set PROD := beer ale;} \\
\text{param: profit :=} \\
\text{ale } & 13 \\
\text{beer } & 23 \\
\text{param: supply :=} \\
\text{corn } & 480 \\
\text{hops } & 160 \\
\text{malt } & 1190; \\
\text{param amt: ale beer :=} \\
\text{corn } & 5 \\
\text{hops } & 4 \\
\text{malt } & 35 \\
\text{solve;} \\
\text{CPLEX 7.1.0: optimal solution; objective 800} \\
\text{total_profit :=} \\
\text{sum \{j in PROD\} amt[j] * x[j] <= supply[j];} \\
\text{subject to constraints \{i in INGR\}:} \\
\text{sum \{j in PROD\} x[j] * profit[j];} \\
\text{var x; x[*] :=} \\
\text{ale 12 beer 28;} \\
\text{separate data from model}
\end{align*}
\]

LP Solvers

Primal and dual LPs. Given real numbers \(a_{ij}, b_i, c_j\), find real numbers \(x_j, y_i\) that optimize \((P)\) and \((D)\).

\[
\begin{align*}
\text{(P)} & \quad \text{max} & & \sum_{j=1}^{n} c_j x_j \\
& \text{s.t.} & & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, \ldots, m \\
& & & x_j \geq 0 \quad \text{for } j = 1, \ldots, n \\
C^* = 1 \\
H^* = 2 \\
M^* = 0 \\
\text{OPT} = 800
\end{align*}
\]

\[
\begin{align*}
\text{(D)} & \quad \text{min} & & \sum_{i=1}^{n} b_i y_i \\
& \text{s.t.} & & \sum_{i=1}^{n} a_{ij} y_i \geq c_j \quad \text{for } j = 1, \ldots, n \\
& & & y_i \geq 0 \quad \text{for } i = 1, \ldots, m
\end{align*}
\]

Duality Theorem. [Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947] If \((P)\) and \((D)\) have feasible solutions, then \(\text{max} = \text{min}\).
LP Duality: Sensitivity Analysis

Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?
A. Corn $1, hops $2, malt $0.

Q. How do I compute marginal prices (dual variables)?
A. Simplex solves primal and dual simultaneously.

Q. New product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?
A. Breakeven: 2 ($1) + 5 ($2) + 24 ($0) = $12 / barrel.

Simplex vs. Interior Point Methods

Simplex vs. Interior Point Methods

interior point faster when polyhedron smooth like disco ball
simplex faster when polyhedron spiky like quartz crystal

Ultimate Problem Solving Model

Ultimate problem-solving model?
- Shortest path.
- Maximum flow.
- Assignment problem.
- Min cost flow.
- Multicommodity flow.
- Linear programming.
- Semidefinite programming.
- Nash equilibrium.
- . . .
- TSP (or any NP-complete problem)

\[ \text{intractable (conjectured)} \]

Does P = NP? No universal problem-solving model exists unless P = NP.

History

1939. Production, planning. [Kantorovich]
1947. Simplex algorithm. [Dantzig]
1950. Applications in many fields.
1975. Nobel prize in Economics. [Kantorovich and Koopmans]
1979. Ellipsoid algorithm. [Khachian]
1984. Projective scaling algorithm. [Karmarkar]
1990. Interior point methods.
200x. Approximation algorithms, large scale optimization.
Assignment Problem

**Assignment problem.** Assign n jobs to n machines to minimize total cost, where $c_{ij}$ = cost of assignment job j to machine i.

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cost $= 3 + 10 + 11 + 20 + 9 = 53$

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<td>1</td>
<td>7</td>
<td>5</td>
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</table>

cost $= 8 + 7 + 20 + 8 + 11 = 44$

**Assignment Problem: LP Formulation**

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \in \{1, \ldots, n\} \\
& \quad \sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \in \{1, \ldots, n\} \\
& \quad x_{ij} \geq 0 \quad \forall i, j \in \{1, \ldots, n\}
\end{align*}
\]

**Theorem.** [Birkhoff 1946, von Neumann 1953] All extreme points of the above polyhedron are (0-1)-valued.

**Corollary.** Can solve assignment problem by solving LP since LP algorithms return an optimal solution that is an extreme point.

**Natural applications.**
- Match jobs to machines.
- Match personnel to tasks.
- Match PU students to writing seminars.

**Non-obvious applications.**
- Vehicle routing.
- Signal processing.
- Virtual output queueing.
- Multiple object tracking.
- Approximate string matching.
- Enhance accuracy of solving linear systems of equations.

**Perspective**

LP is near the deep waters of NP-completeness.
- Solvable in polynomial time.
- Known for $\approx 25$ years.

**Integer linear programming.**
- LP with integrality requirement.
- NP-hard.