Subtyping
COS 441
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Inclusive vs. Coercive Relationships

- Inclusive
  - Every cat is a feline
  - Every dog is a canine
  - Every feline is a mammal
- Coercive/isomorphism
  - Integers can be converted into floating point numbers
  - Booleans can be converted into integers
  - Mammals with a tail can be converted to a mammals without a tail (ouch!)

Subtype Relation

Read $\tau_1 <: \tau_2$ as $\tau_1$ is a subtype of $\tau_2$ or $\tau_2$ is a supertype of $\tau_1$

Subtype relation is reflexive and transitive

\[ \tau_1 <: \tau_2 \quad \text{refl} \quad \tau_1 <: \tau_2, \tau_2 <: \tau_3 \quad \text{trans} \]

We say ($\tau_1 = \tau_2$) iff ($\tau_1 <: \tau_2$) and ($\tau_2 <: \tau_1$)

Implicit vs Explicit

Typing rules for subtyping can be rendered in either implicit or explicit form

\[
\begin{align*}
\Gamma \vdash e : \sigma & \quad \sigma <: \tau \\
\Gamma \vdash e : \tau \\
\Gamma \vdash (\tau) e : \tau
\end{align*}
\]

Simplification for Type-Safety

- Inclusive/Coercive distinction independent of Implicit/Explicit distinction
- Harper associates inclusive with implicit typing and coercive with explicit typing because it simplifies the type safety proof
  - You can have an inclusive semantics with explicit type casts
  - You can have a coercive semantics with implicit typing

Dynamic Semantics

- For inclusive system primitives must operate equally well for all subtypes of a give type for which the primitive is defined
- For coercive systems dynamic semantics simply must cast/convert the value appropriately
Varieties of Systems

- Implicit, Inclusive – Described by Harper
- Explicit, Coercive – Described by Harper
- Implicit, Coercive – Non-deterministic insertion of coercions
- Explicit, Inclusive – Type casts are no-ops in the operational semantics

Subtype Relation (cont.)

Given

\[ \text{bool} \triangleleft \text{int} \quad \text{int} \triangleleft \text{float} \]

via transitivity we can conclude

\[ \text{bool} \triangleleft \text{float} \]

Subtyping of Functions

- weight: \( \text{mammal} \rightarrow \text{float} \)
- numOfTeeth: \( \text{mammal} \rightarrow \text{int} \)
- numOfWiskers: \( \text{feline} \rightarrow \text{int} \)
- pitchOfMeow: \( \text{feline} \rightarrow \text{float} \)
- printInfo: \( (\text{mammal} \times \text{mammal}) \rightarrow \text{unit} \)
- printFelineInfo: \( (\text{feline} \times \text{feline}) \rightarrow \text{unit} \)

Subtyping Quiz

- \( \text{mammal} \rightarrow \text{int} \triangleleft: \text{mammal} \rightarrow \text{float} \)
- \( \text{feline} \rightarrow \text{int} \triangleleft: \text{feline} \rightarrow \text{float} \)
- \( \text{mammal} \rightarrow \text{float} \triangleleft: \text{feline} \rightarrow \text{float} \)
- \( \text{mammal} \rightarrow \text{int} \triangleleft: \text{feline} \rightarrow \text{int} \)
- \( \text{mammal} \rightarrow \text{int} \triangleleft: \text{feline} \rightarrow \text{float} \)

Co/Contra Variance

- \( \tau_1 \triangleleft: \tau_1 \rightarrow \tau_2 \triangleleft: \tau_1 \rightarrow \tau_2' \)
- Both are covariant

Width vs Depth Subtyping

Consider the n-tuple \( (\tau_1 \times \ldots \times \tau_n) \)

Width Subtyping

\[ (\text{int} \times \text{int} \times \text{float}) \triangleleft: (\text{int} \times \text{int}) \]

\[ m > n \quad \text{width} \]

\[ (\tau_1 \times \ldots \times \tau_m) \triangleleft: (\tau_1 \times \ldots \times \tau_n) \]

Depth Subtyping

\[ (\text{int} \times \text{int}) \triangleleft: (\text{float} \times \text{float}) \]

\[ \tau_1 \triangleleft: \tau_1 \rightarrow \tau_2 \triangleleft: \tau_2 \rightarrow \tau_3 \]

\[ (\tau_1 \times \tau_2 \times \tau_3) \triangleleft: (\tau_1 \times \tau_2 \times \tau_3) \]

\[ m > n \quad \text{depth} \]
Width and Depth for Records

Similar rule for records \( \{ l_1 : \tau_1, \ldots, l_n : \tau_n \} \)
Width considers any subset of labels since order of labels doesn’t matter.
Implementing this efficiently can be tricky but doable

Subtyping and Mutability

Mutability destroys the ability to subtype
\( \tau \text{ ref} = (\text{get: unit } \rightarrow \tau, \text{set: } \tau \rightarrow \text{unit}) \)
\( \tau' \text{ ref} = (\text{get: unit } \rightarrow \tau', \text{set: } \tau' \rightarrow \text{unit}) \)
Assume \( \tau <: \tau' \) from that we conclude
\( \text{unit } \rightarrow \tau <: \tau' \rightarrow \text{unit} \) and
\( \tau' \rightarrow \text{unit} <: \tau \rightarrow \text{unit} \)

Subtyping Defines Preorder/DAG

Subtyping relation can form any DAG

Typechecking With Subtyping

With explicit typing every expression has a unique type so we can use type synthesis to compute the type of an expression
Under implicit typing an expression may have many different types, which one should we choose?
e.g. CalicoCat : mammal, CalicoCat : feline, and CalicoCat : cat

Which Type to Use?

Consider weight: mammal \( \rightarrow \) float
   countWiskers: feline \( \rightarrow \) int

let val c = CalicoCat
in (weight c, countWiskers c)
end

What type should we use for c?

Which Type to Use?

Consider weight: mammal \( \rightarrow \) float
   countWiskers: feline \( \rightarrow \) int

let val c: mammal = CalicoCat
in (weight c, countWiskers c)
end

What type should we use for c?
Which Type to Use?
Consider weight: mammal \rightarrow \text{float}
\quad \text{countWiskers: feline} \rightarrow \text{int}

let val c: \text{feline} = \text{CalicoCat}
in (weight c, \text{countWiskers} c)
end

How do we know this is the “best” solution?

Which Type to Use?
Consider weight: mammal \rightarrow \text{float}
\quad \text{countWiskers: feline} \rightarrow \text{int}

let val c: \text{cat} = \text{CalicoCat}
in (weight c, \text{countWiskers} c)
end

Choose the most specific type.

Principal Types
Principal type is the “most specific” type. It is the least type in a given pre-order defined by the subtype relation
Lack of principal types makes type synthesis impossible with *implicit* subtyping unless programmer annotates code
Not at as big a problem for *explicit* subtyping rules

Subtyping Defines Preorder/DAG
Q: What is the least element “principal type” for “mammal”?

Subtyping Defines Preorder/DAG
A: “mammal” has no principal type in the subtyping relation defined below

Implementing Subtyping
For inclusive based semantics maybe hard to implement or impose restrictions on representations of values
Coercive based semantics give more freedom on choosing appropriate representation of values
Can use “type-directed translation” to convert inclusive system to coercive system
Subtyping with Coercions

Define a new relation $\tau_1 :<:\tau_2 \rightsquigarrow v$
Where $v$ is a function of type ($\tau_1 \rightarrow \tau_2$)

$\tau_1 :<:\tau_2 \rightsquigarrow \rho :<:\sigma \rightsquigarrow \nu$
$\rho :<:\tau \rightarrow \nu' \Rightarrow \nu' :<:\tau \rightarrow \nu$

$\text{int} :<:\text{float} \rightsquigarrow \text{float}$

Subtyping with Coercions (cont)

1. Primitive conversion: $\text{to.float}$.
2. Identity: $\text{id}_x = \text{fn}.x : \text{x}$.
3. Composition: $v;v' = \text{fn}.x : \text{x} \rightarrow \nu'(v(x))$.
4. Functions: $v_1 ; v_2 = \text{fn}.f : \text{c} \rightarrow \text{c} \in \text{fn}.x : \text{c} \in \nu_1 \rightarrow \nu_2(f(v_1(x)))$.

Implementing Record Subtyping

Implementing subtyping on tuples is easy since address index “does the right thing”

$((1, 2, 3) : (\text{int} \times \text{int} \times \text{int})).2$

Selecting the field label with records is more challenging

$\{a=1,b=2,c=3\} : \{a : \text{int}, b : \text{int}, c : \text{int}\}.c$

Approaches to Record Subtyping

Represent record as a “hash-table” keyed by label name
Convert record to tuple when coercing create new tuple that represents different record with appropriate fields
Two level approach represent record as “view” and value. Dynamically coerce views.
(Java interfaces are implemented this way, but you can statically compute all the views in Java)

By Name vs Structural Subtyping

Harper adopts a structural view of subtyping.
Things are subtypes if they are somehow isomorphic.
Java adopts a “by name” view. Things are subtypes if they are structurally compatible and the user declared them as subtypes.
Java approach leads to simpler type-checking and implementation but is arguably less modular than a pure structural approach

Summary

Coercive vs Inclusive
Operational view of what subtyping means
Implicit vs Explicit
How type system represents subtyping
Systems can support all possible combinations
Need to think things through to avoid bugs
Tuples/records have both width and depth subtyping
Functions are contravariant in argument type
References are invariant