**Administrative Issues**

- **Textbook update:**
  - Bookstore will get one additional copy of textbook.
  - Copies will be available at Triangle, 150 Nassau Street (Tel: 609 924 4630) for $35 beginning today. They are open M-F 8am-6pm.

- **Updated collaboration policy**

- Tutoring ?

- Readings for this week: Matousek and Nesetril, Chapter 10

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**Inclusion-Exclusion principle**

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

\[ = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i \leq k \leq n} |A_i \cap A_k| + \sum_{1 \leq i \leq k \leq l \leq n} |A_i \cap A_k \cap A_l| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n| \]
Hatcheck lady problem

n gentlemen arrive at a party and leave their hats in the cloak room. On their departure, the hatcheck lady absent-mindedly hands back a hat to each man at random.

What is the probability that none of the men receives their own hat?

• n! ways of assigning hats back to men
• What fraction of these assignments are such that no man receives his own hat?

Enter inclusion-exclusion

• $S_n$: set of all permutations
• $A_i = \{\pi \in S_n : \pi(i) = i\}$
• $\bigcup_i A_i$: bad permutations

$|A_i| = (n-1)!$

$|A_i \cap A_j| = (n-1)!$

$|A_i \cap A_j \cap \cdots \cap A_k| = (n-k)!$

Hatcheck lady

• Number hats and men 1,2,...,n
• $\pi(i)$: number of hat received by $i$th man
• $\pi$ is a permutation
• Index $i$ with $\pi(i) = i$ is a fixed point of $\pi$
• $D(n)$: number of permutations with no fixed point

Enter inclusion-exclusion

$|A_i \cap A_j \cap \cdots \cap A_k| = (n-k)!$

$|A_i \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!$

$= \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!}$

$D(n) = n! - |A_i \cup A_2 \cup \cdots \cup A_n|$

$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!}$

$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$
Finishing up

\[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\] converges to \(e^{-1}\)

\[D(n) \approx \frac{n!}{e}\]

Probability that nobody gets their hat back converges to the constant \(e^{-1} = 0.36787\) independent of the number of men!

Algebraic derivation of identities on binomial coefficients

**Binomial Theorem:**

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

\[= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n\]

Equality of two polynomials implies equality of corresponding coefficients

\[\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}\]

\[(1 + x)^n (1 + x)^n = (1 + x)^{2n}\]

Coefficient of \(x^n\) on LHS

\[\binom{n}{0} \binom{n}{n} + \binom{n}{1} \binom{n}{n-1} + \binom{n}{2} \binom{n}{n-2} + \cdots + \binom{n}{n} \binom{n}{0}\]

\[= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}\]

Coefficient of \(x^n\) on RHS

\[= \sum_{k=0}^{n} \binom{2n}{n}\]
Algebraic derivation of identities on binomial coefficients

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} = ? \]

\[ (1-x)^n(1+x)^n = (1-x^2)^n \]

Coefficient of \( x^m \) on LHS = \[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n}{n-k} \]

Coefficient of \( x^m \) on RHS = \[ \begin{cases} 0 & \text{when n odd} \\ (-1)^{n/2} \binom{n}{n/2} & \text{n even} \end{cases} \]

Algebraic derivation of identities on binomial coefficients

\[ \sum_{k=0}^{n} (-1)^k k \binom{n}{k} = ? \]

\[ -n(1-x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k \cdot x^{k-1} \]

Substitute \( x = 1 \)

\[ 0 = \sum_{k=0}^{n} (-1)^k k \binom{n}{k} \]

Power series

Infinite series of the form \( a_0 + a_1 x + a_2 x^2 + \cdots \)

\[ \frac{1}{1-x} = 1 + x + x^2 + \cdots \]

Series converges for \( x \) in the interval \((-1, 1)\)

Function contains all the information about series

Differentiate \( k \) times and substitute \( x = 0 \), we get \( k! \) times coefficient of \( x^k \)

Taylor series of the function \( \frac{1}{1-x} \) at \( x = 0 \)
Power series

\((a_0, a_1, a_2, \ldots)\): sequence of real numbers

\(|a_n| \leq K^n\)

For any number \(x \in (-\frac{1}{K}, \frac{1}{K})\), the series

\[ a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i \]

converges

Values of \(a(x)\) in arbitrarily small neighborhood of 0 uniquely determine \((a_0, a_1, a_2, \ldots)\)

\[ a_n = \frac{a^{(n)}(0)}{n!} \]

Generating functions

\((a_0, a_1, a_2, \ldots)\): sequence of real numbers

Generating function of this sequence is the power series

\[ a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i \]

Generating function basics

What is the generating function of the sequence 

\((1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)\)?

\[ 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots \]

\[ = \frac{\ln(1-x)}{x} \]

\[ x + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots = -\ln(1-x) \]

\[ 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = e^x \]

Generating function toolkit: Generalized binomial theorem

\[ \binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!} \]

\((1+x)^r\) is the generating function

for the sequence \(\binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \binom{r}{3}, \ldots\)

The power series

\[ \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots \]

always converges for all \(|x| < 1\)
Negative binomial coefficients?

\[ \binom{r}{k} = (-1)^k \binom{-r + k - 1}{k} = (-1)^k \binom{-r + k - 1}{-r - 1} \]

\[ \frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1} x + \binom{n+1}{n-1} x^2 + \cdots + \binom{n+k-1}{n-1} x^k + \cdots \]

\[ \frac{1}{1-x} = 1 + x + x^2 + \cdots \]

Operations on power series

- Addition
  \((a_0 + b_0, a_1 + b_1, \ldots)\) has generating function \(a(x) + b(x)\)

- Multiplication by fixed real number
  \((\alpha a_0, \alpha a_1, \ldots)\) has generating function \(\alpha a(x)\)

- Shifting the sequence
  \((0, \ldots, 0, a_0, a_1, \ldots)\) has generating function \(x^n a(x)\)

- Shifting to the left

- Substituting \(\alpha x\) for \(x\)
  \((a_0, \alpha a_1, \alpha^2 a_2, \ldots)\) has generating function \(a(\alpha x)\)

- Substitute \(x^n\) for \(x\)
  \((1, 1, 2, 4, 8, \ldots)\) has generating function \(1 + x + x^2 + x^3 + \cdots\)

- Integration and differentiation
  \((a_0, 2a_1, 3a_2, \ldots)\) has generating function \(\int a(x) dx\)

- Multiplication of generating functions

\[ \frac{1}{1-2x^2} \]

\[ \frac{x}{1-2x^2} \]