COS 341   Discrete Mathematics

Counting
Administrative Issues

• Bookstore has run out of copies of textbook.
• Readings for this week: Matousek and Nesetril, Chapter 2
  next week: Chapter 10

• Homework policy:
• All problems on a homework carry the same weight, unless
  stated otherwise
• All homeworks will be equally weighted
• Homeworks due in class on Wednesday
• Late homeworks submitted by 5pm Friday will be
  penalized 50%
• No late homeworks accepted after 5pm Friday.
How many ways are there to write a nonnegative integer $m$ as a sum of $r$ nonnegative integers (order is important) ?

e.g. $m = 3$, $r = 2$

$3 = 0 + 3$
$= 1 + 2$
$= 2 + 1$
$= 3 + 0$

How many ordered $r$-tuples $(i_1, i_2, \ldots, i_r)$ of nonnegative integers satisfy the equation

$$i_1 + i_2 + \cdots + i_r = m$$
\[ i_1 + i_2 + \cdots + i_r = m \]

\( m \) indistinguishable balls and \( r \) boxes

How many ways of placing \( m \) balls in \( r \) boxes?

Each placement gives a solution of the equation

\[ 0 + 1 + 0 + 3 + 1 + 2 = 7 \]
Placing balls in boxes

Every arrangement of \( m \) balls and \( r-1 \) walls gives a unique placement of \( m \) balls into \( r \) boxes
\( m+r-1 \) objects arranged in a row,

\( m \) balls, \( r-1 \) walls

Number of choices of \( r-1 \) positions amongst \( m+r-1 \)

\[
\binom{m + r - 1}{r - 1}
\]

Number of solutions of the equation

\[ i_1 + i_2 + \cdots + i_r = m \]
Properties of Binomial coefficients

\[
\binom{n}{k} = \binom{n}{n-k}
\]

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}
\]
A messy proof

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(n-k)!(k-1)!} + \frac{(n-1)!}{(n-k-1)!k!}
\]

\[
= \frac{(n-1)!}{(n-k-1)!(k-1)!} \left(\frac{1}{n-k} + \frac{1}{k}\right)
\]

\[
= \frac{(n-1)!}{(n-k-1)!(k-1)!} \left(\frac{n}{(n-k)k}\right)
\]

\[
= \frac{n!}{(n-k)!k!} = \binom{n}{k}
\]
A more elegant proof

Fix an element \( a \) of an \( n \) element set

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}
\]

Number of k-element subsets that include a

Number of k-element subsets that do not include a

Total number of k-element subsets
Pascal’s triangle

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}
\]

\[
\begin{array}{cccccc}
& & & & 1 & \\
& & & 1 & 1 & \\
& & 1 & 2 & 1 & \\
& 1 & 3 & 3 & 1 & \\
1 & 4 & 6 & 4 & 1 & \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

\[
\binom{4}{1} + \binom{4}{2} = \binom{5}{2}
\]
What is the sum of numbers in the $n$th row of Pascal’s triangle?

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Number of $k$-element subsets of $n$-element set

Total number of subsets of $n$-element set
Binomial Theorem

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

\[= \binom{n}{0} y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \ldots + \binom{n}{n-1} x^{n-1} y + \binom{n}{n} x^n\]
Proof of Binomial theorem

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

\[(x + y)^n = (x + y) \cdot (x + y) \cdot \ldots \cdot (x + y) \quad \text{\(n\) times}\]

Coefficient of \(x^k y^{n-k}\)

\[= \text{number of ways of picking } k \text{ } x's \text{ from } n \text{ terms}\]

\[= \binom{n}{k}\]
\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

\[= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n\]
Proofs using the binomial theorem

Substitute $x = 1$ in

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$
Proofs using the binomial theorem

Substitute \( x = -1 \) in

\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k
\]

\[
0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots = \sum_{k=0}^{n} (-1)^k \binom{n}{k}
\]

\[
2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots
\]

\[
2^n = 2 \left[ \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots \right]
\]
Further identities

\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}
\]
How many distinct words can you obtain by permuting the letters of MISSISSIPPI?

11 letters: 1 M, 4 I, 4 S, 2 P

M₁ I₁ S₁ S₂ I₂ S₃ S₄ I₃ P₁ P₂ I₄

11! permutations

How many indexed words give you a particular unindexed word?

e.g. SIPIS MSSIPIS

4! ways to place indices of I
4! ways to place indices of S
2! ways to place indices of P
1! way for indices of M

number of distinct words

= \frac{11!}{4!4!2!1!}
Generalization

$n$ objects of $m$ different kinds

$k_i$ indistinguishable objects of $i$th kind

$k_1 + k_2 + \ldots + k_m = n$

Then the total number of distinct arrangements is

\[
\frac{n!}{k_1!k_2!\ldots k_m!}
\]

\[
\binom{n}{k_1, k_2, \ldots, k_m}
\]

Multinomial coefficient
\[ m = 2 \]
\[
\binom{n}{k, n-k} = \binom{n}{k}
\]

**Multinomial theorem**

\[
(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}
\]

\[ k_1, \ldots, k_m \geq 0 \]
Inclusion-Exclusion principle

Given \(|A|, |B|, |A \cap B|\), what is \(|A \cup B|\)?

\[ |A \cup B| = |A| + |B| - |A \cap B| \]
Inclusion-Exclusion principle

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]
Inclusion-Exclusion principle

\[ | A_1 \cup A_2 \cup \cdots \cup A_n | \]

\[ = \sum_{i=1}^{n} | A_i | - \sum_{1 \leq i_1 \leq i_2 \leq n} | A_{i_1} \cap A_{i_2} | \]

\[ + \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq n} | A_{i_1} \cap A_{i_2} \cap A_{i_3} | \]

\[ - \cdots + (-1)^{n-1} | A_1 \cap A_2 \cap \cdots \cap A_n | \]
Proof of inclusion-exclusion principle

Consider an element \( x \in A_1 \cup A_2 \cup \cdots \cup A_n \)

Contribution to LHS = 1

What is the contribution to RHS?

Suppose \( x \) belongs to \( j \) sets.

Rename sets to be \( A_1, A_2, \ldots, A_j \)

\( x \) appears in intersection of every \( k \)-tuple of sets amongst \( A_1, A_2, \ldots, A_j \)
Proof of inclusion-exclusion principle

$x$ appears in intersection of every $k$-tuple of sets amongst $A_1, A_2, \ldots, A_j$

correction of $x$ to RHS

$$= \sum_{i=1}^{k} (-1)^{k-1} \binom{j}{k}$$

$$= j - \binom{j}{2} + \binom{j}{3} - \cdots + (-1)^{j-1} \binom{j}{j}$$

$$= 1$$