Proving Liveness of Parameterized Programs

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```
global t : int              // ticket counter
global s : int              // service counter
local m : int               // my ticket
init s = t

do forever {
    m := t++                // acquire ticket
    do {
                            // busy wait
    } until (m <= s)
    // critical section
    s++                     // bump service counter
}
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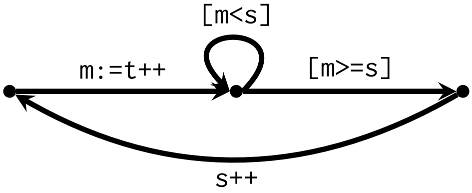
Goal: Prove that no thread starves

- no matter how many threads there are
- automatically

A parameterized concurrent program, P:

- *thread template* = finite directed graph with edges labeled by instructions (in some programming language). Call the set of instructions Σ.
- For any $N \in \mathbb{N}$, $P(N)$ denotes the program with N identical threads, all of which execute P.



Thread identifiers

A **trace** is a sequence $\tau = \langle \sigma_1 : i_1 \rangle \langle \sigma_2 : i_2 \rangle ... \in (\Sigma \times \mathbb{N})^\omega$

Program instructions

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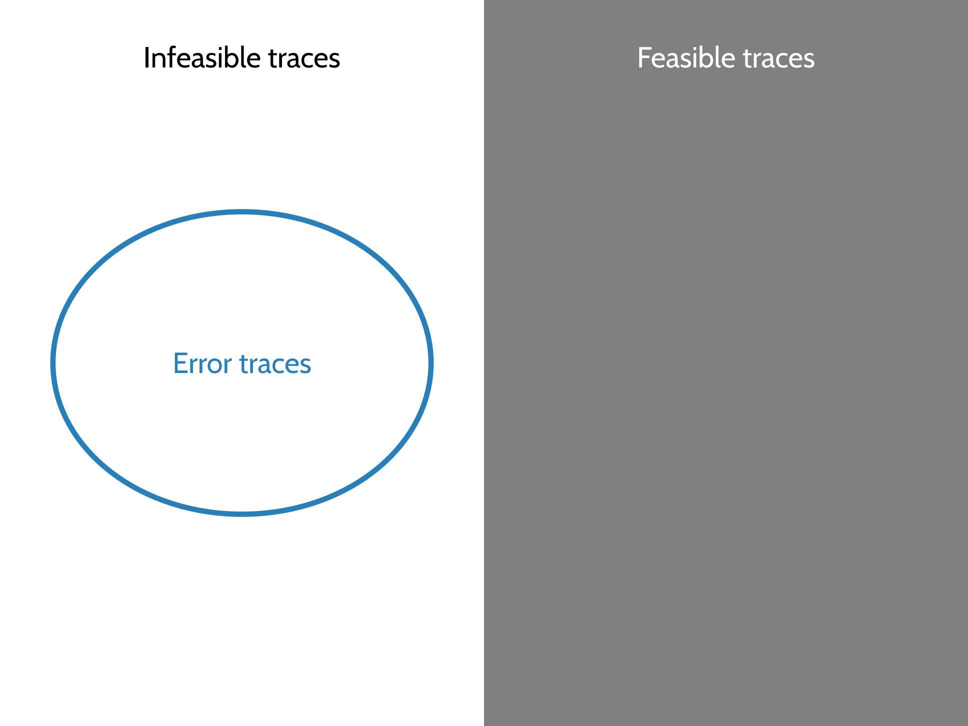
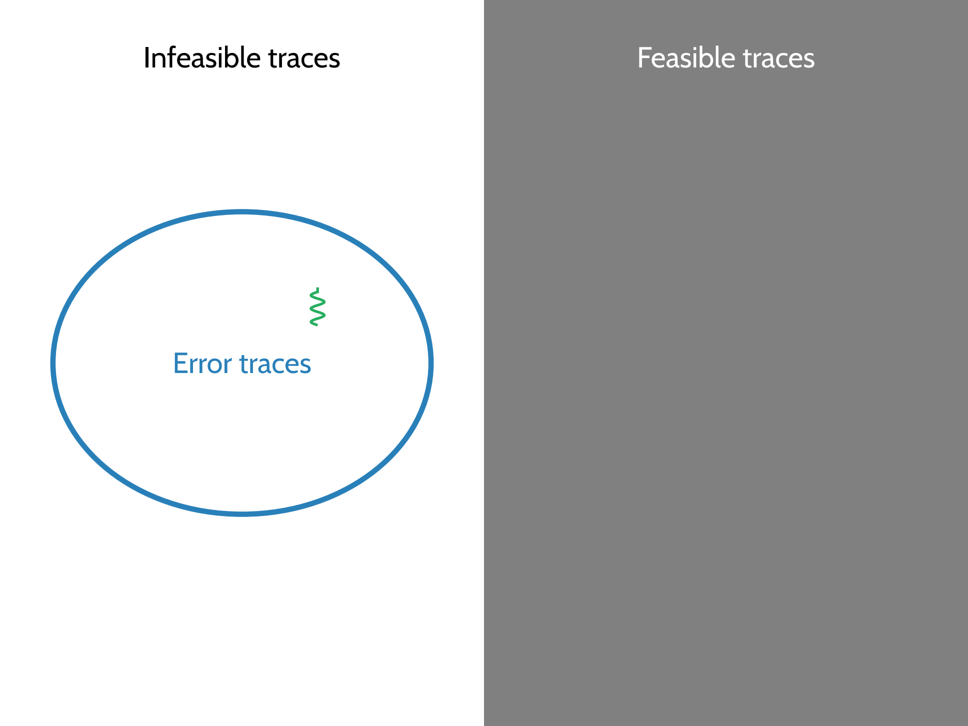
- Program traces $\mathcal{L}(P) = \bigcup_N \mathcal{L}(P(N))$
- P correct $\iff$ every **error trace** in $\mathcal{L}(P) \setminus \mathcal{L}(\Phi)$ is infeasible.

Infeasible traces	Feasible traces
No corresponding executions	At least one corresponding execution


Infeasible traces
Feasible traces
Error traces


Infeasible traces
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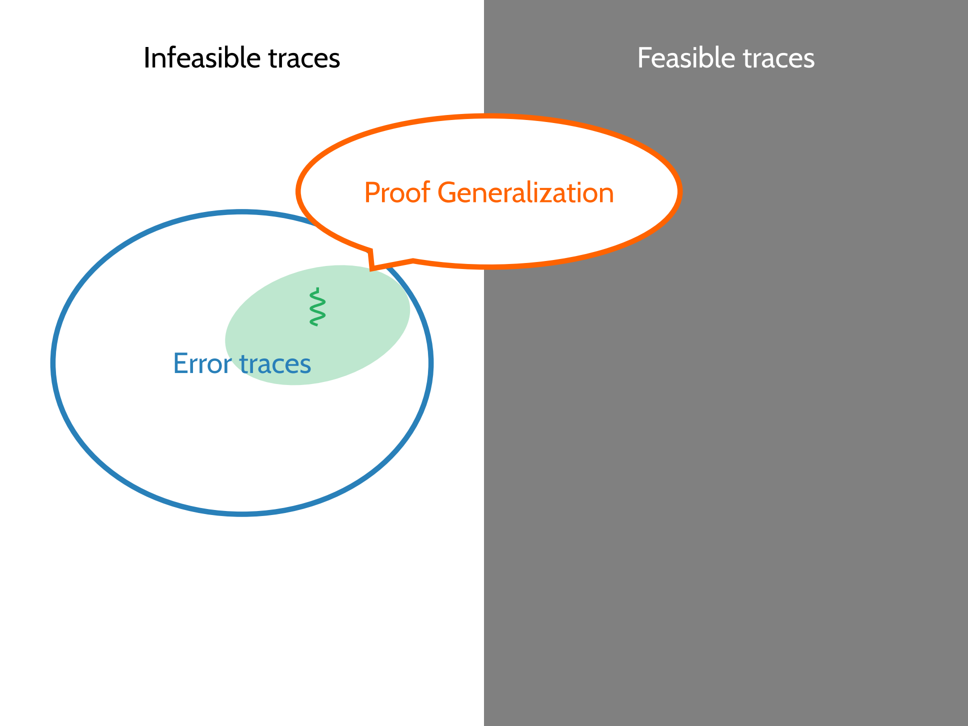
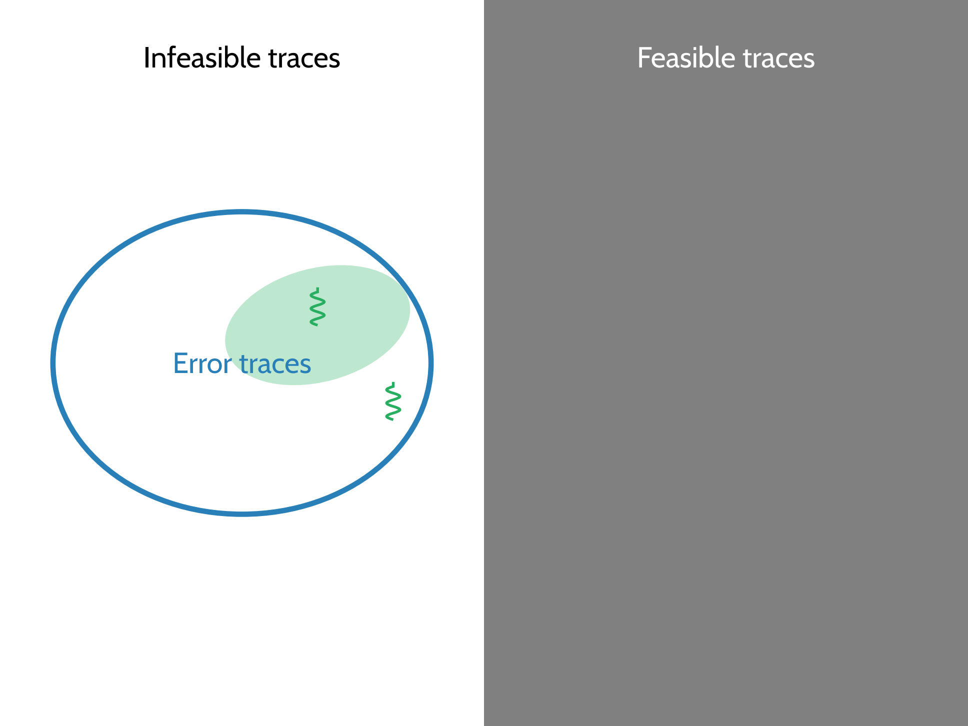


Infeasible traces
Feasible traces
Proof Generalization
Error traces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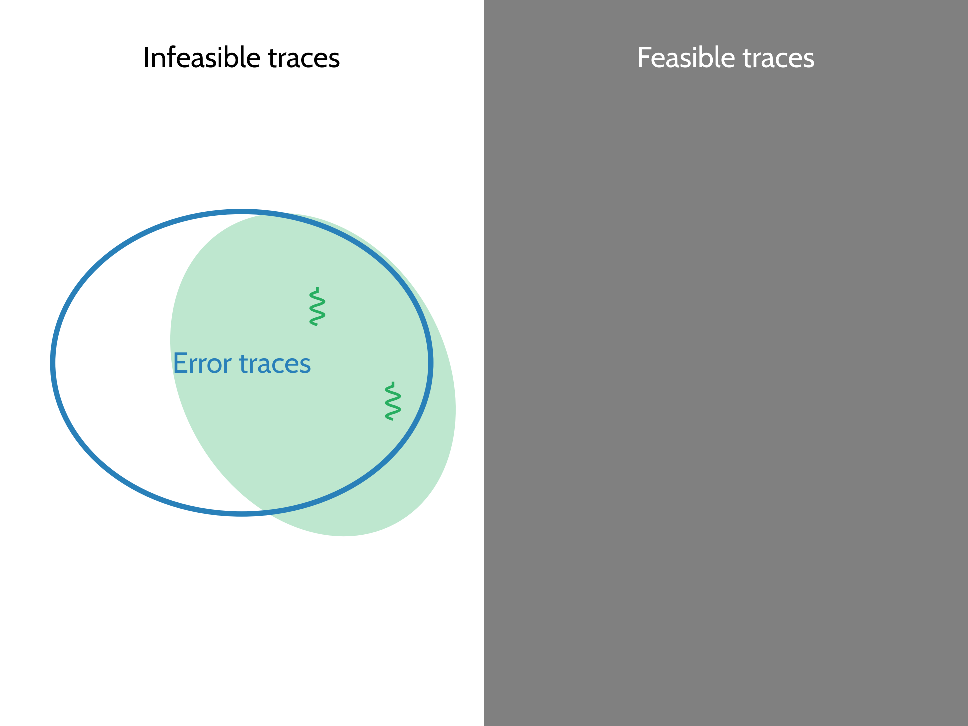

Infeasible traces
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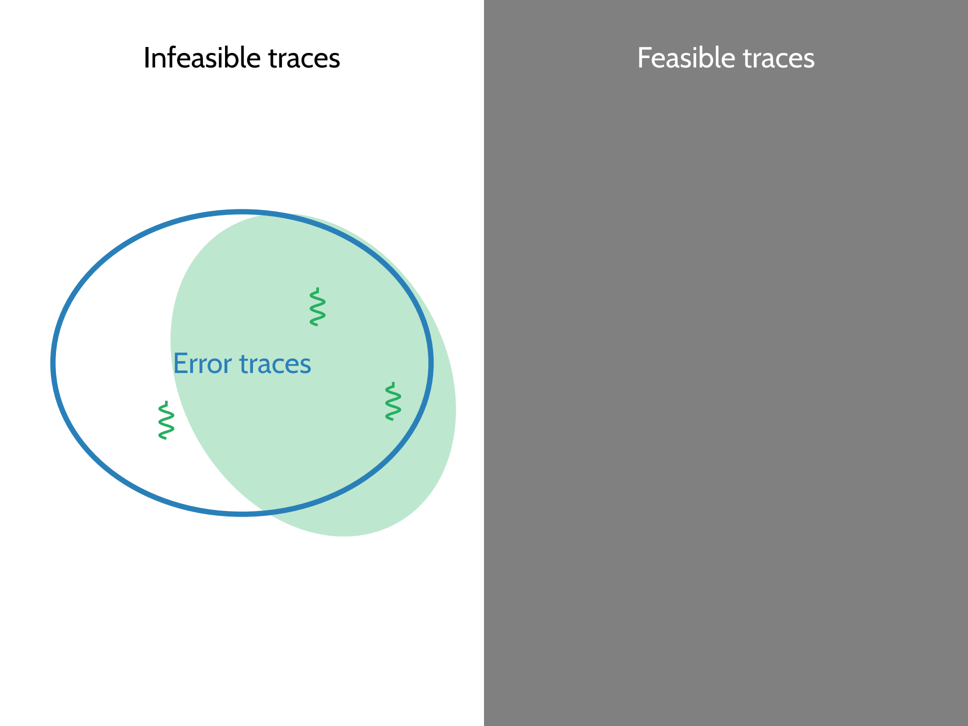
Error traces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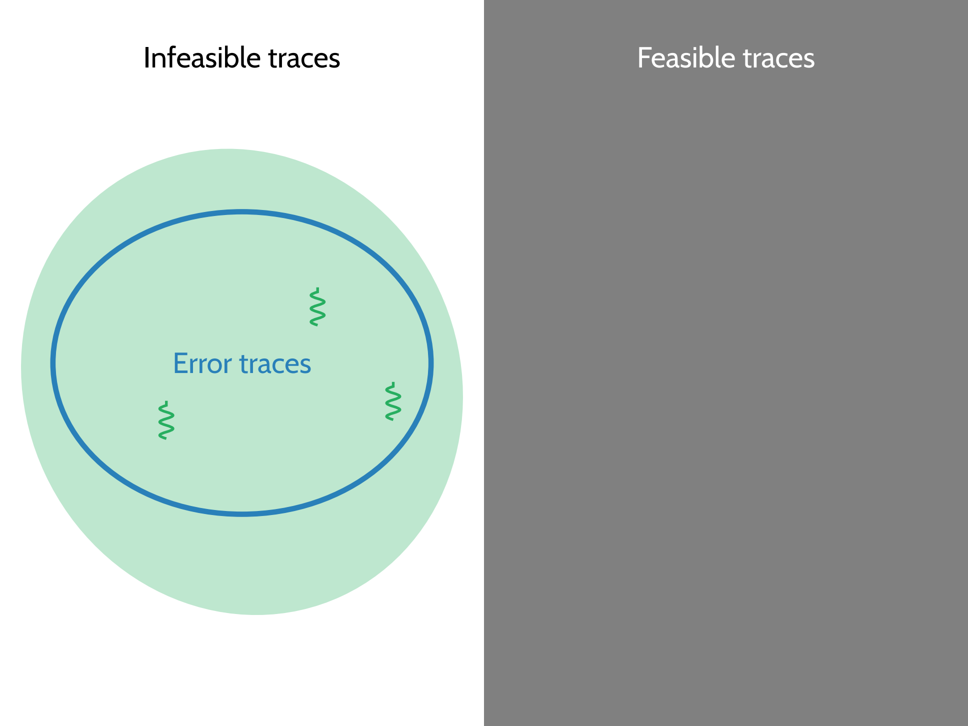
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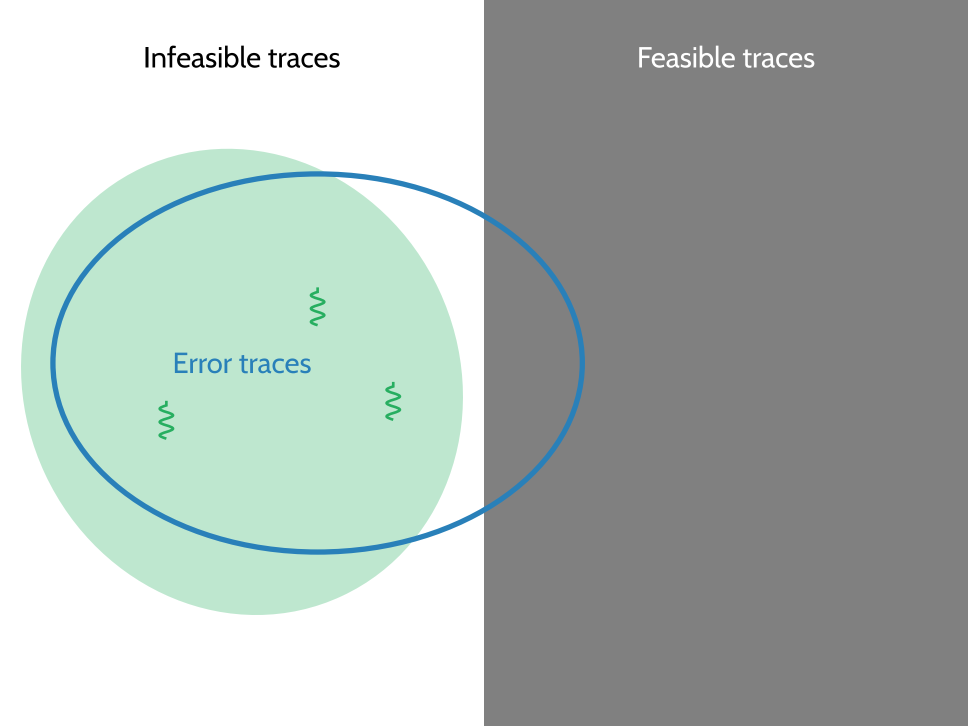
Infeasible traces
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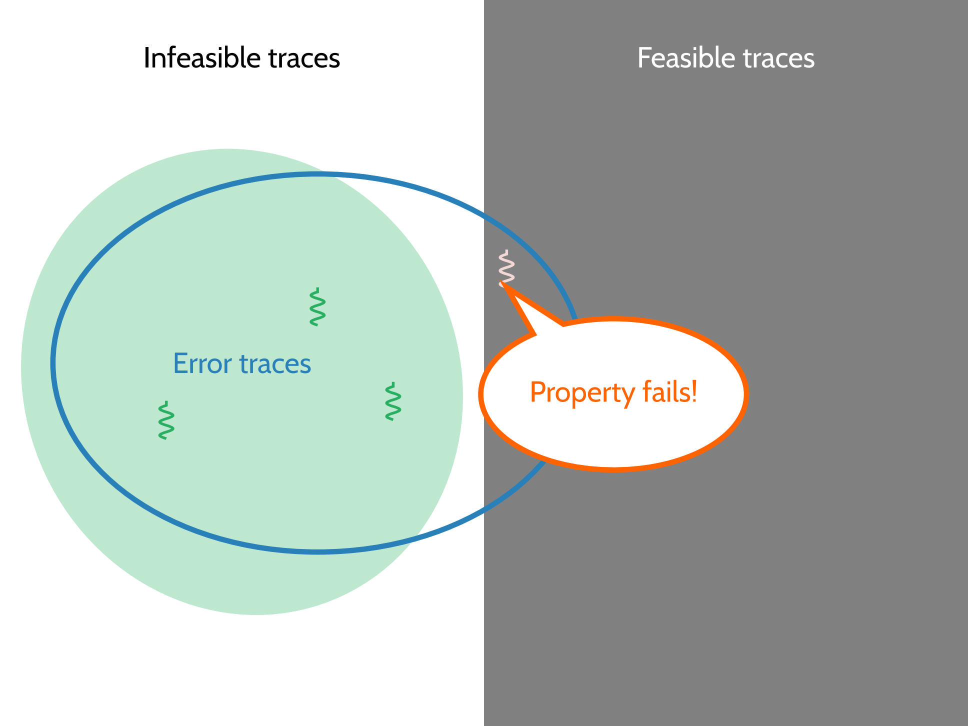
Infeasible traces
Feasible traces
Error traces
Property fails!

Two key problems:

1. How do we generalize proofs?

2. How do we check that a proof is complete?

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   - Parameterization: Same proof applies to many instantiations.
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$$\underbrace{\langle \texttt{m:=t++} : 1\rangle\langle \texttt{m:=t++} : 2\rangle}_{\textsf{Stem}}\ (\underbrace{\langle \texttt{[m>s]} : 2\rangle\langle \texttt{[m<=s]} : 1\rangle\langle \texttt{s++} : 1\rangle\langle \texttt{m:=t++} : 1\rangle}_{\textsf{Loop}})^{\omega}$$

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$$\begin{gathered}
\{\textit{old}(\mathtt{s}) = \mathtt{s}\} \\
\langle \texttt{[m>s]} : 2\rangle \\
\{\textit{old}(\mathtt{s}) = \mathtt{s} \wedge \mathtt{m}(2) \geq \textit{old}(\mathtt{s})\} \\
\langle \texttt{[m<=s]} : 1\rangle \\
\{\textit{old}(\mathtt{s}) = \mathtt{s} \wedge \mathtt{m}(2) \geq \textit{old}(\mathtt{s})\} \\
\langle \texttt{s++} : 1\rangle \\
\{\textit{old}(\mathtt{s}) < \mathtt{s} \wedge \mathtt{m}(2) \geq \textit{old}(\mathtt{s})\} \\
\langle \texttt{m:=t++} : 1\rangle \\
\{\ \textit{old}(\mathtt{s}) < \mathtt{s} \wedge \mathtt{m}(2) \geq \textit{old}(\mathtt{s})\ \}
\end{gathered}$$

Ranking formula

Variance proof

$$\underbrace{\langle \texttt{m:=t++} : 1\rangle\langle \texttt{m:=t++} : 2\rangle}_{\text{Stem}} \; (\underbrace{\langle \texttt{[m>s]} : 2\rangle\langle \texttt{[m<=s]} : 1\rangle\langle \texttt{s++} : 1\rangle\langle \texttt{m:=t++} : 1\rangle}_{\text{Loop}})^{\omega}$$

Variance proof

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$$\langle \texttt{m:=t++} : 1\rangle$$
$$\{\; \mathit{old}(\texttt{s}) < \texttt{s} \wedge \texttt{m}(2) \geq \mathit{old}(\texttt{s}) \;\}$$

Invariance proof

$$\{s = t\}$$
$$\langle \texttt{m:=t++} : 1\rangle$$
$$\{\mathit{true}\}$$
$$\langle \texttt{m:=t++} : 2\rangle$$
$$\{\mathit{true}\}$$
$$\langle \texttt{[m>s]} : 2\rangle$$
$$\{\mathit{true}\}$$
$$\langle \texttt{[m<=s]} : 1\rangle$$
$$\{\mathit{true}\}$$
$$\langle \texttt{s++} : 1\rangle$$
$$\{\mathit{true}\}$$
$$\langle \texttt{m:=t++} : 1\rangle$$
$$\{\mathit{true}\}$$

$$
\begin{array}{ccc}
\{\textit{\textbf{old}}(\mathsf{s}) = \mathsf{s}\} & \{\textit{\textbf{old}}(\mathsf{s}) = \mathsf{s}\} & \{\varphi\} \\
\langle \texttt{[m>s]} : 2 \rangle & \langle \mathsf{s{+}{+}} : 1 \rangle & \langle \sigma : i \rangle \\
\{\texttt{m}(2) \geq \textit{\textbf{old}}(\mathsf{s})\} & \{\textit{\textbf{old}}(\mathsf{s}) < \mathsf{s}\} & \{\varphi\}
\end{array}
$$

Sequencing

$$\{s \leq t\}$$
$$\texttt{m := t++} : \mathbf{1}$$
$$\{m(1) < t\}$$

$$\{m(1) < t\}$$
$$\texttt{m := t++} : \mathbf{2}$$
$$\{m(1) < m(2)\}$$

Sequencing

$\{\mathsf{s} \leq \mathsf{t}\}$
`m := t++ : 1`
$\{\mathtt{m}(1) < \mathsf{t}\}$

$\{\mathtt{m}(1) < \mathsf{t}\}$
`m := t++ : 2`
$\{\mathtt{m}(1) < \mathtt{m}(2)\}$

$\{\mathsf{s} \leq \mathsf{t}\}$
`m := t++ : 1;`
`m := t++ : 2`
$\{\mathtt{m}(1) < \mathtt{m}(2)\}$

Symmetry

$$P(N) = \underbrace{P \parallel P \parallel \cdots \parallel P}_{N \text{ times}}$$

$$\{\mathsf{s} \leq \mathtt{m}(1) \wedge \mathtt{m}(1) < \mathtt{m}(2)\}$$

`[m <= s]`: **2**

$\{$*false*$\}$

Symmetry

$$P(N) = \underbrace{P \parallel P \parallel \cdots \parallel P}_{N \text{ times}}$$

$$\{\mathsf{s} \le \mathtt{m}(1) \wedge \mathtt{m}(1) < \mathtt{m}(2)\} \quad \xrightarrow[{[2 \mapsto 1]}]{[1 \mapsto 2]} \quad \{\mathsf{s} \le \mathtt{m}(2) \wedge \mathtt{m}(2) < \mathtt{m}(1)\}$$

`[m <= s]` : 2 → `[m <= s]` : 1

{*false*} → {*false*}

Symmetry

$$P(N) = \underbrace{P \parallel P \parallel \cdots \parallel P}_{N \text{ times}}$$

$$\{\mathsf{s} \leq \mathtt{m}(1) \wedge \mathtt{m}(1) < \mathtt{m}(2)\}$$
`[m <= s]` : **2**
$\{\textit{false}\}$

$$\xrightarrow[{[2 \mapsto 3]}]{[1 \mapsto 2]}$$

$$\{\mathsf{s} \leq \mathtt{m}(2) \wedge \mathtt{m}(2) < \mathtt{m}(3)\}$$
`[m <= s]` : **3**
$\{\textit{false}\}$

Conjunction

$\{\mathtt{m}(1) < t\}$
`m := t++:3`
$\{\mathtt{m}(1) < \mathtt{m}(3)\}$

$\{\mathtt{m}(2) < t\}$
`m := t++:3`
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Conjunction

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$\{\mathtt{m}(1) < t \wedge \mathtt{m}(2) < t\}$
`m := t++` : 3
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A *Well-founded proof space* (WFPS) $\langle H, R \rangle$ is a set of valid Hoare triples H which is closed under sequencing, symmetry, and conjunction, along with a set of ranking formulas R which is closed under symmetry.

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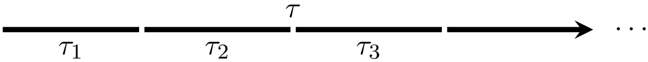
H is a set of theorems about *finite* traces. How do we prove infeasibility of *infinite* traces?

A WFPS $\langle H, R\rangle$ proves a trace τ infeasible if there is some ranking formula $r \in R$, some decomposition of τ:



and some sequence of "intermediate formulas" $\varphi_1, \varphi_2, \ldots$ such that

$$\{\textsf{pre}\}\tau_1\{\varphi_1\} \qquad \{\varphi_1 \land \textit{old}(\mathbf{x}) = \mathbf{x}\}\tau_2\{r\}$$
$$\{\textsf{pre}\}\tau_1\tau_2\{\varphi_2\} \qquad \{\varphi_2 \land \textit{old}(\mathbf{x}) = \mathbf{x}\}\tau_3\{r\}$$
$$\vdots$$
$$\{\textsf{pre}\}\tau_1\tau_2\ldots\tau_i\{\varphi_i\} \qquad \{\varphi_i \land \textit{old}(\mathbf{x}) = \mathbf{x}\}\tau_{i+1}\{r\}$$

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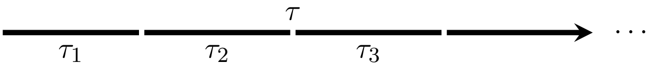

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all belong to H.
The set of traces $\langle H, R \rangle$ proves infeasible is denoted $\omega(H, R)$.

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   - $\mathcal{L}(P) \setminus \mathcal{L}(\Phi) \subseteq \omega(H, R)$: inclusion between infinite sets of infinite words over an infinite alphabet

Infinite traces $\to$ finite traces

An **ultimately periodic trace** is a trace of the form $\pi\rho\rho\rho\cdots$
Every ultimately periodic trace can be written (*not uniquely*) as a **lasso** $\pi\$\rho$.
Given a language $L \subseteq \Sigma^\omega$, define its *lasso language* $\$(L)$ as:

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- For any $N \in \mathbb{N}$, $\mathcal{L}(P) \cap (\Sigma \times \{1, ..., N\})^\omega$ is ω-regular. Same for $\mathcal{L}(\Phi)$ and $\omega(H, R)$.
- Fact: If L_1 and L_2 are ω-regular, then $UP(L_1) \subseteq L_2$ implies $L_1 \subseteq L_2$.

Infinite language $\to$ automaton

Quantified Predicate Automata (QPA): a class of infinite-state automata that recognize words over an infinite alphabet.

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- There is a QPA that recognizes $\$(\mathcal{L}(P))$.
- There is a QPA that recognizes $\$(\mathcal{L}(\Phi))$.
- There is *not* a QPA that recognizes $\$(\omega(H, R))$.

But...

There *is* a QPA that recognizes all lassos $\pi\$\rho$ such that there exists some intermediate assertion φ and some ranking formula $r \in R$ such that

$$\{\text{pre}\}\pi\{\varphi\} \qquad \text{and} \qquad \{\varphi \wedge \textit{old}(\mathbf{x}) = \mathbf{x}\}\rho\{r\}$$

belong to H. Call this language $\$(H, R)$.

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Theorem

If $\$(\mathcal{L}(P)) \setminus \$(\mathcal{L}(\Phi)) \subseteq \$(H, R)$, *then* $\mathcal{L}(P) \setminus \mathcal{L}(\Phi) \subseteq \omega(H, R)$.

- $\pi\rho^\omega \in \mathcal{L}(P) \setminus \mathcal{L}(\Phi) \Rightarrow \pi\rho^n\$\rho^k \in \$(\mathcal{L}(P)) \setminus \$(\mathcal{L}(\Phi))$ for all $n \geq 0, k \geq 1$.
- H contains $\{\text{pre}\}\pi\rho^n\{\varphi_{n,k}\}$ and $\{\varphi_{n,k} \wedge \textit{old}(\mathbf{x}) = \mathbf{x}\}\rho^k\{r_{n,k}\}$. Ramsey!

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QPA language containment can be used to check proofs

Summary

Two key problems:

1. How do we generalize proofs?
   - **Well-founded proof spaces**
2. How do we check that a proof is complete?
   - **Lassos** + **Quantified Predicate Automata**

Thanks!