Introduction to

Algebraic Program Analysis

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Program analysis

Design algorithms to answer questions about the dynamic behavior of software

- **Correctness**
  - *Is a program correct w.r.t. some specification?*
- **Security**
  - *Can a program over-read a buffer?*
- **Performance**
  - *How much memory will a program consume?*

Algebraic program analysis

A framework for designing program analyses based on **algebra**.

Semantic algebra = space of program properties + composition operators

- Sequencing: $\otimes$
- Choice: $\oplus$
- Iteration: $*$

Why algebraic program analysis?

- *Compositional*
  - Incremental analysis
  - Easy to parallelize
- Opens the door for new ways to compute loop invariants
  - Abstractions of loops are computed from abstractions of loop bodies

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Why *not* algebraic program analysis?

- Loss of contextual information

Outline

- Background
  - Iterative program analysis
  - Abstract interpretation
- Intraprocedural analysis
  - Overview
  - Path expressions
  - Compositional Recurrence Analysis
  - Proving soundness
- Interprocedural analysis
  - Functional approach
  - Newtonian program analysis
  - Newtonian program analysis via tensor product
  - Newtonian program analysis and Gauss-Jordan elimination

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   y = rand() mod x
   x = x + 2
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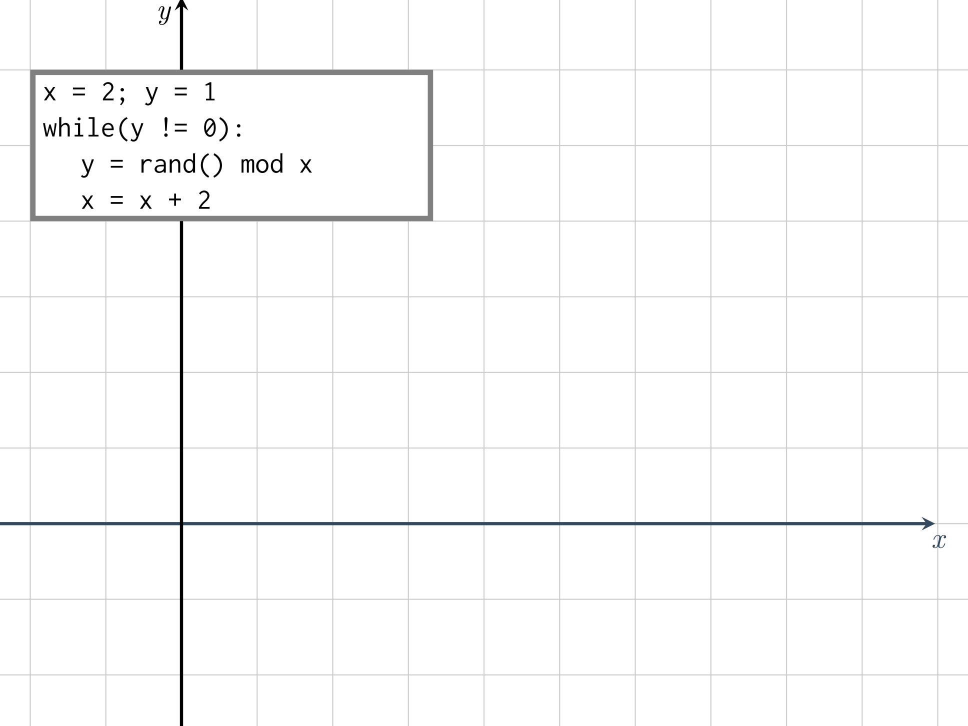
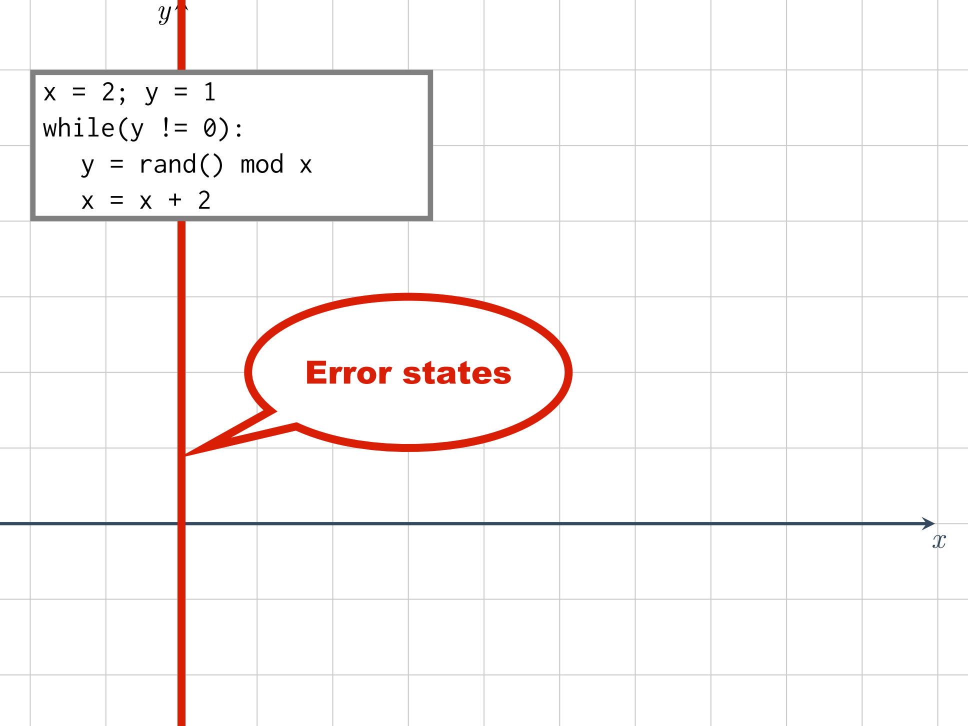
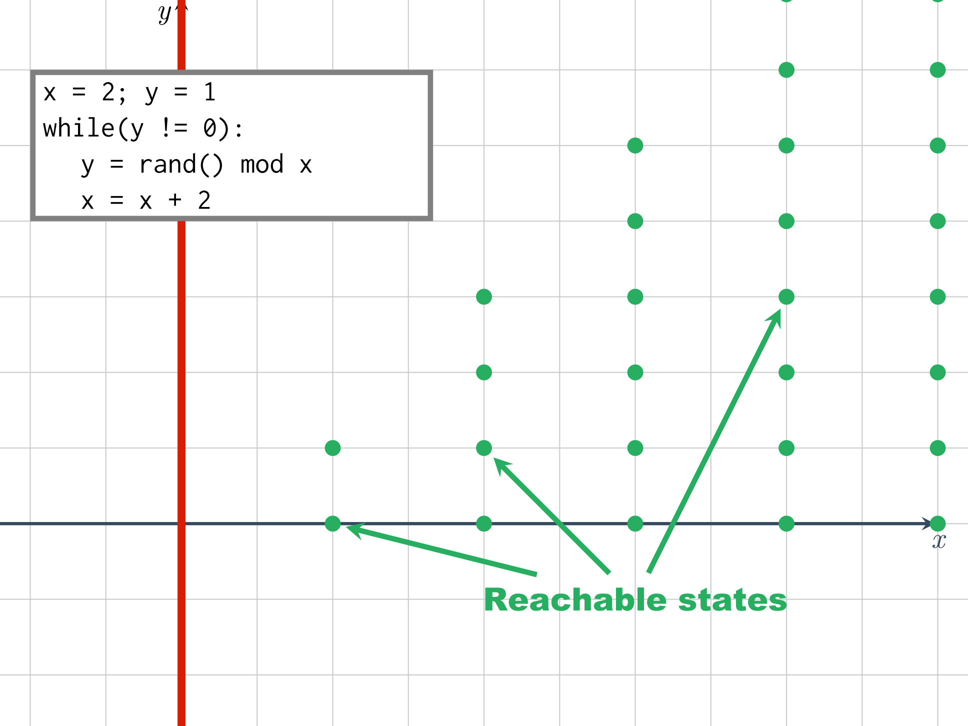




y
x = 2; y = 1
while(y != 0):
y = rand() mod x
x = x + 2
Error states
x



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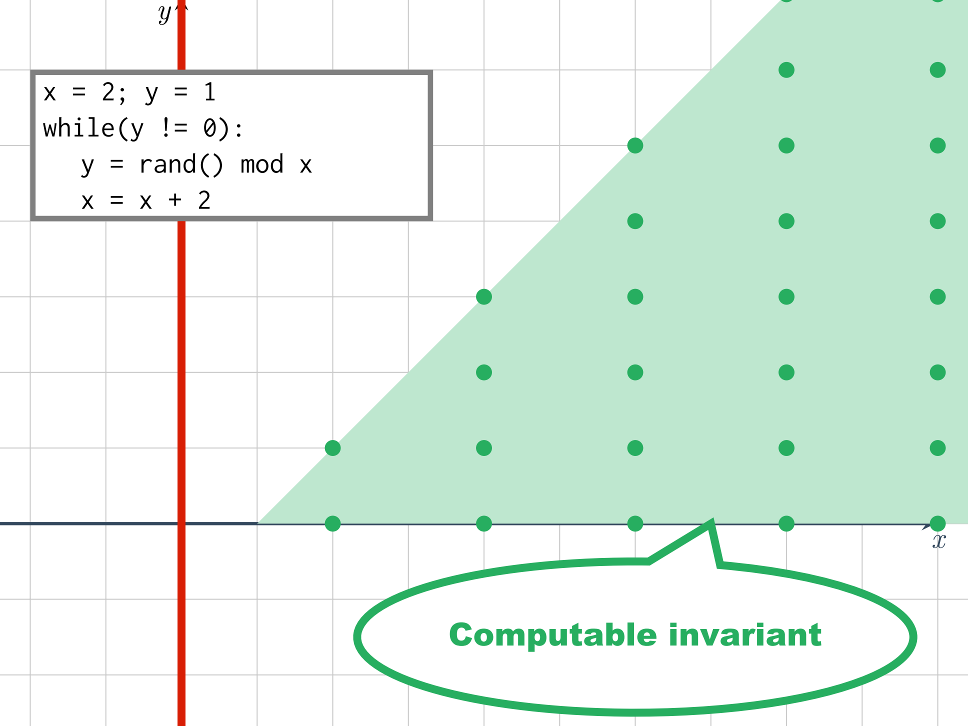
x
Reachable states


y
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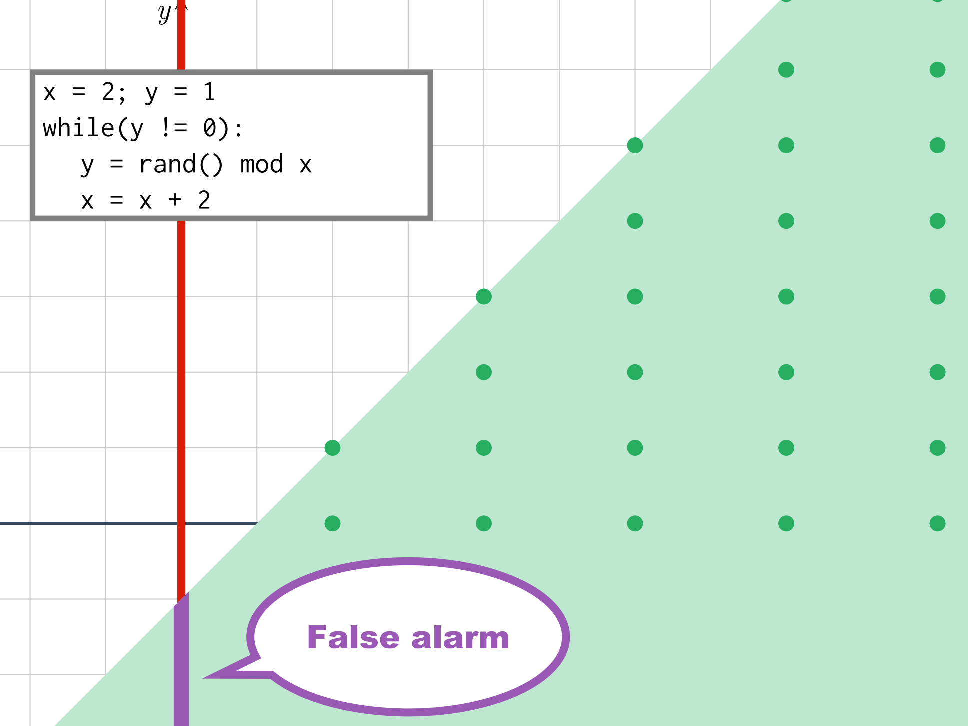
x
Computable invariant


y
x = 2; y = 1
while(y != 0):
y = rand() mod x
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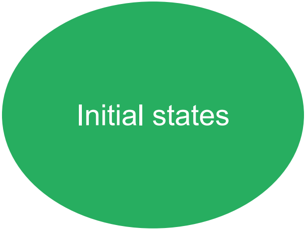
False alarm

Iterative program analysis

Repeatedly evaluate the program under an abstract semantics until convergence upon a property that over-approximates all reachable states.

- SLAM, Astrée, ...



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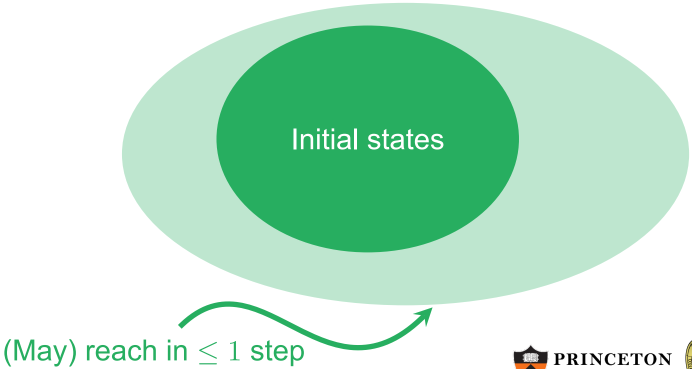

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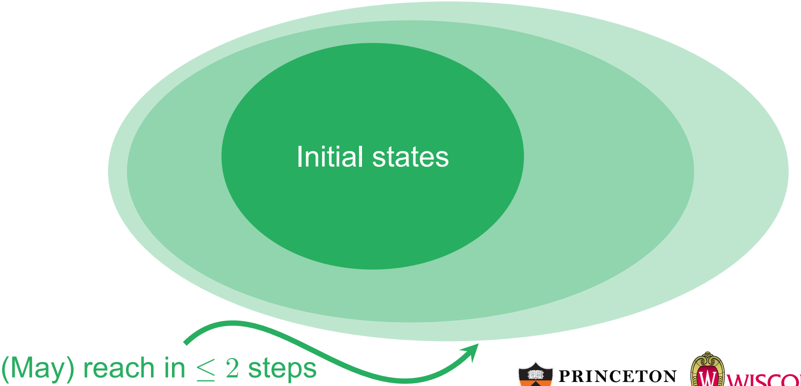

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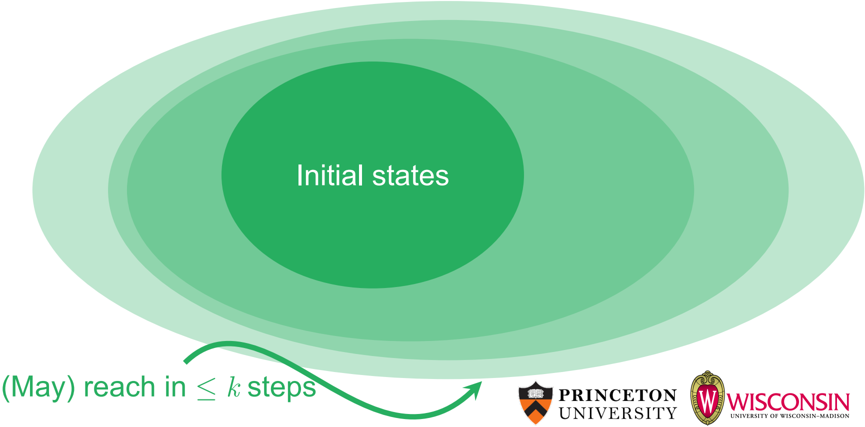

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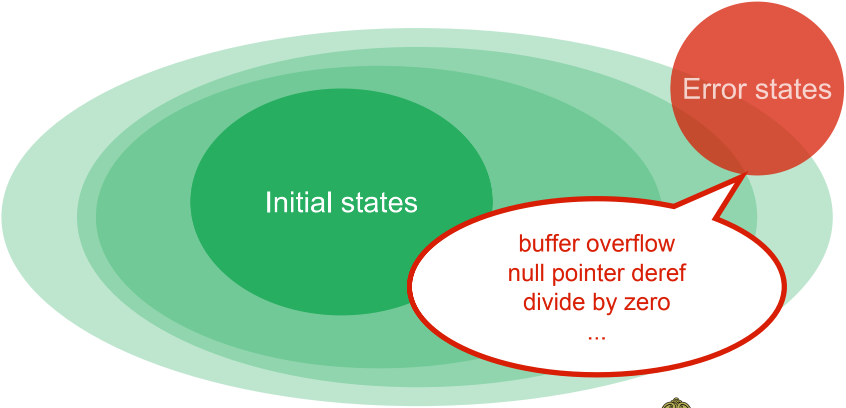

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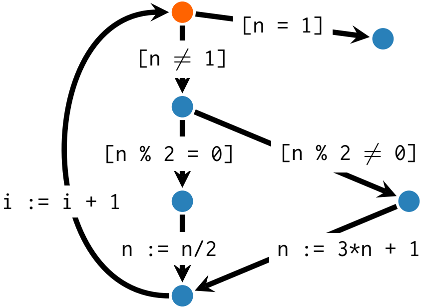
Program model

Control flow graph $G = \langle \mathit{Loc}, \mathit{Edge}, \mathit{root} \rangle$

- *Loc*: set of control locations
- *Edge*: set of instruction-labeled edges
- *root*: root (entry location)

```
while(n ≠ 1){
  if(n % 2 == 0)
    n := n/2;
  else
    n := 3*n+1;
  i := i+1;
}
```



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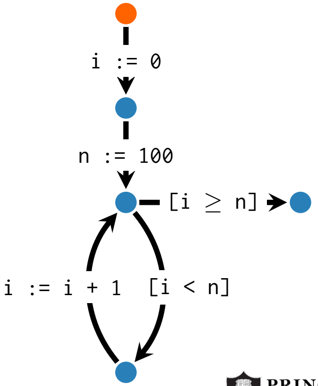
Example: interval analysis

$$\text{Property} \triangleq \underbrace{\text{Var} \to \overbrace{(\mathbb{Z}\cup\{-\infty\})\times(\mathbb{Z}\cup\{\infty\})}^{\text{Interval}}}_{\text{Interval store}}$$



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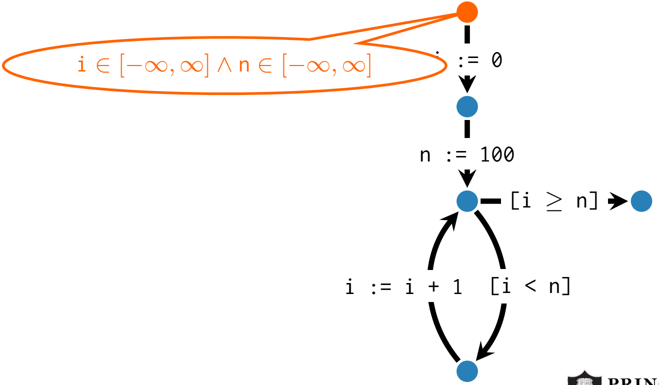

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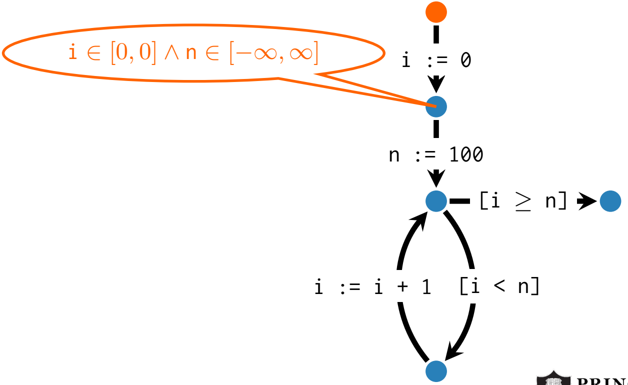

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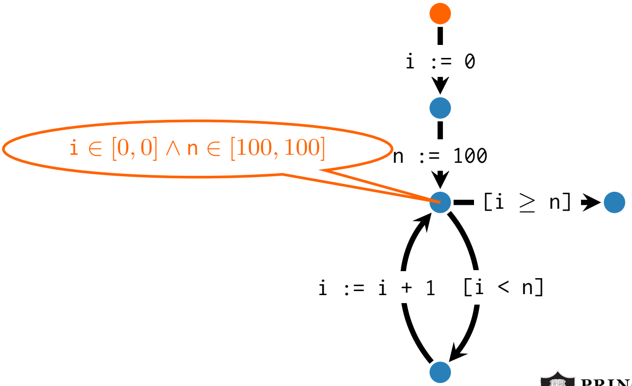

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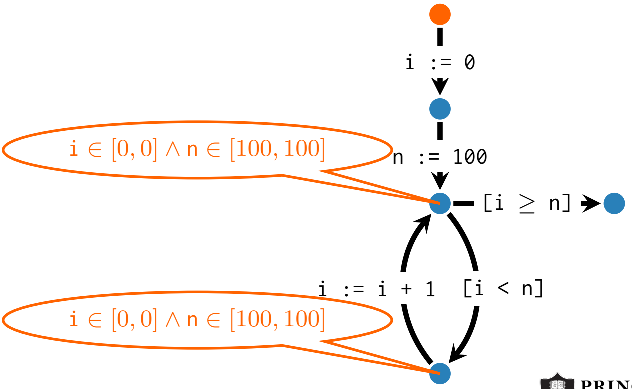

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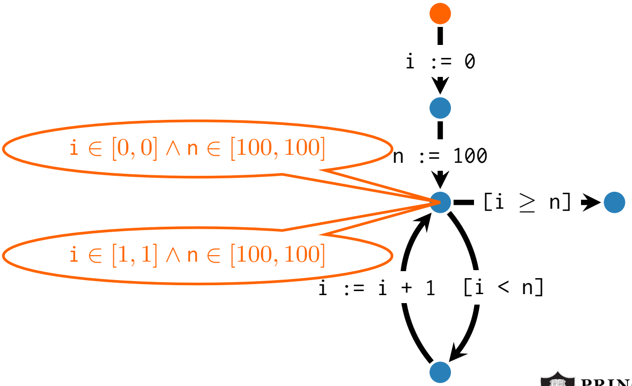

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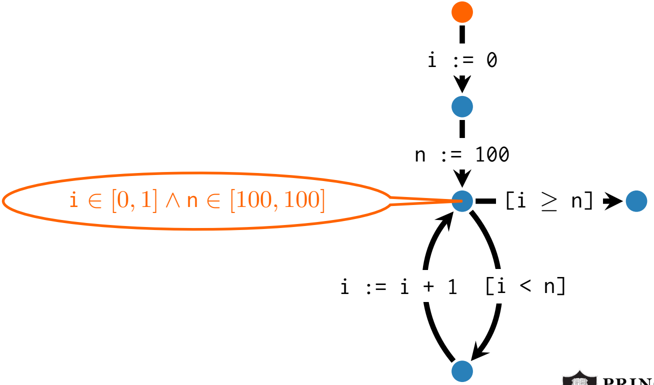

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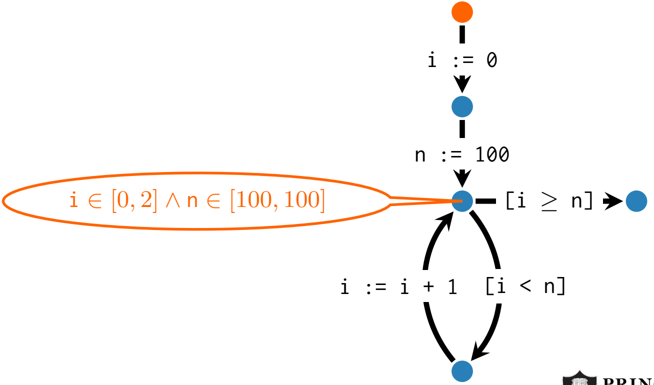

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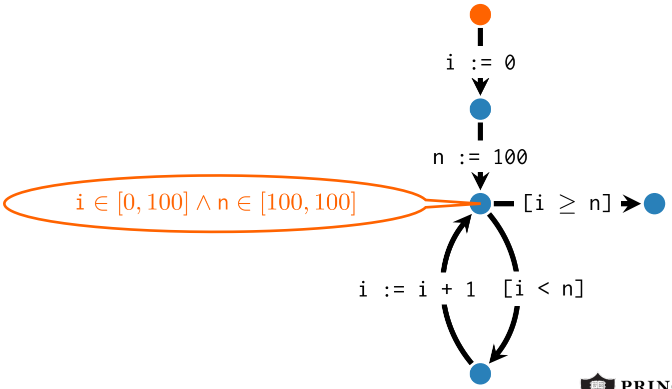

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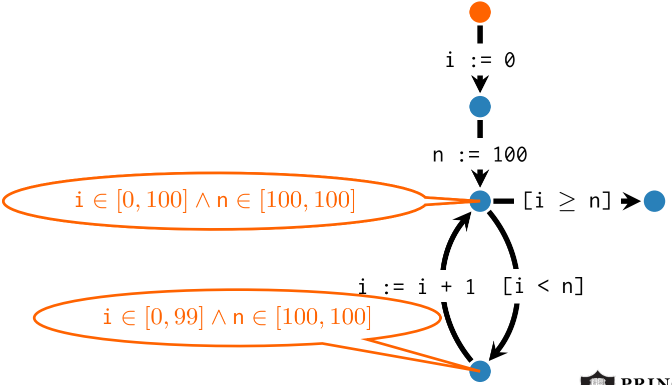

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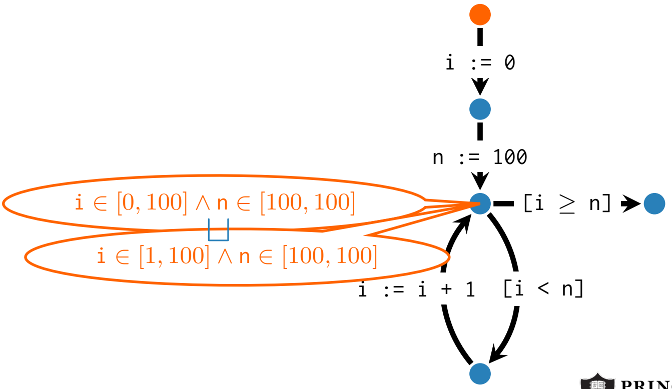

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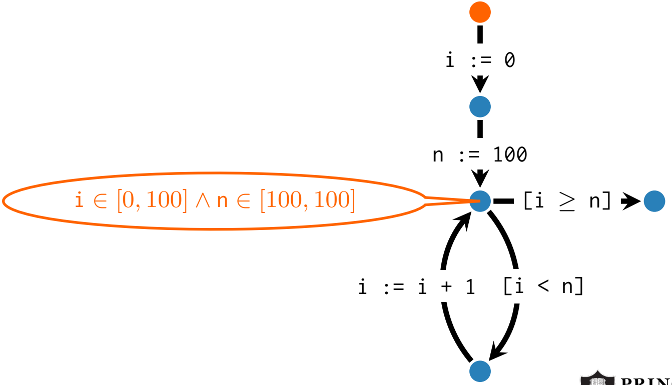

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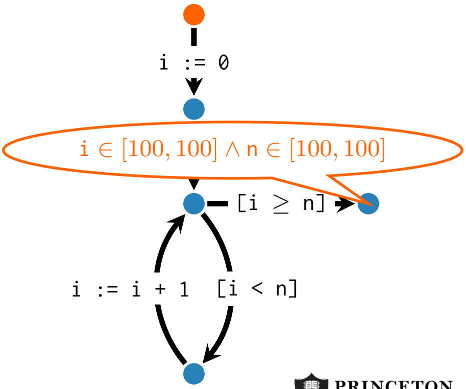

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Approximating a loop

$$\overbrace{p_1 \sqsubseteq p_2 \sqsubseteq p_2 \sqsubseteq \ldots}^{\text{Ascending sequence of properties}}$$

Approximate limit w/ a widening operator

$$\hat{p}_1 = p_1$$
$$\hat{p}_{i+1} = p_i \nabla p_{i+1}$$

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Designing an iterative analysis

1. **Define**:
   - Abstract domain $\mathcal{L} = \langle L, \sqsubseteq, \sqcup, \bot, \nabla \rangle$
     - L: space of program properties
     - $\sqsubseteq \subseteq L \times L$: approximation order
     - $\sqcup : L \times L \to L$: join (least upper bound) operator
     - $\nabla : L \times L \to L$ widening (extrapolation) operator
   - Property transformer: $\mathcal{L}[\![\cdot]\!]$: *Edge* $\to (L \to L)$ maps each command to a monotone function on L

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2. **Apply**: *chaotic iteration algorithm*
   - Computes a map $\mathit{inv} : \mathit{Loc} \rightarrow L$ that is closed under the abstract semantics:

$$\forall (u, v) \in \mathit{Edge}. \mathcal{L}[\![(u, v)]\!](\mathit{inv}(u)) \sqsubseteq \mathit{inv}(v)$$

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Proving soundness [Cousot & Cousot '77]

1. **Define**:
   - Concrete semantics
     - $\mathcal{C} \triangleq \langle 2^{\textit{Store}}, \subseteq, \cup, \emptyset, \cup \rangle$
     - $\mathcal{C}[\![e]\!](S) \triangleq \{s' : \exists s \in S. s \xrightarrow{e} s'\}$
   - *Concretization function* $\gamma : L \to 2^{\textit{Store}}$
     maps properties to set of stores that satisfy it

$$\gamma([x \mapsto [0,1]; y \mapsto [2,3]]) = \left\{ \begin{array}{ll} [x \mapsto 0; y \mapsto 2], & [x \mapsto 0; y \mapsto 3], \\ [x \mapsto 1; y \mapsto 2], & [x \mapsto 1; y \mapsto 3] \end{array} \right\}$$

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3. **Apply** *fixpoint transfer*: Chaotic iteration algorithm computes a map *inv* : *Loc* $\to L$ such that

$$\text{Stores reachable at } v \subseteq \gamma(\textit{inv}(v))$$