High Dimensional Expanders

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By

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Abstract

This work studies the relation between spectral and combinatorial expansion in simplicial complexes. More precisely, we study the spectrum of the simplicial Hodge Laplacian defined by Eckmann, generalizing well known theorems from graph theory: the Cheeger inequalities concerning a graph's isoperimetric constant; the "Expander Mixing Lemma"; properties of random walks and return probabilities; the theorems of Alon-Boppana and of Kesten regarding infinite trees; the study of random complexes, and of "Ramanujan complexes", and Gromov's geometric overlap property.

Contents

1 Introduction **5**
- 1.1 Isoperimetric constant . . . 6
- 1.2 Expander Mixing Lemmas . . . 7
- 1.3 Examples and applications . . . 9
- 1.4 High dimensional random walk . . . 10
- 1.5 Infinite complexes . . . 13
- 1.6 Isospectrality . . . 14

2 Preliminaries **16**
- 2.1 Simplicial Hodge Laplacians . . . 18
- 2.2 The spectrum of complexes . . . 19
- 2.3 Complexes with a complete skeleton . . . 20

3 Isoperimetric constant **24**
- 3.1 A Cheeger-type inequality . . . 24
- 3.2 Towards a lower Cheeger inequality . . . 26

4 Mixing and pseudo-randomness **30**
- 4.1 The complete skeleton case . . . 30
- 4.2 The general case . . . 32

5 Examples and Applications **37**
- 5.1 Gromov's geometric overlap . . . 37
- 5.2 Chromatic bounds . . . 39
- 5.3 Ideal expanders . . . 40
- 5.4 Linial-Meshulam complexes . . . 40
- 5.5 Ramanujan triangle complexes . . . 42

6 High dimensional random walk **50**
- 6.1 The $(d-1)$-walk and expectation process . . . 50
- 6.2 Normalized Laplacians . . . 51
- 6.3 Walk and spectrum . . . 55

7 Infinite complexes **59**
- 7.1 Infinite graphs . . . 59
- 7.2 Infinite complexes of general dimension . . . 59
- 7.3 Example - arboreal complexes . . . 60
- 7.4 Continuity of the spectral measure . . . 65
- 7.5 Alon-Boppana type theorems . . . 67
- 7.6 Analysis of balls in T_2^2 . . . 70
- 7.7 Spectral radius and random walk . . . 74
- 7.8 Amenability, transience and recurrence . . . 75

8 Isospectrality **78**
- 8.1 G-sets . . . 78
- 8.2 Action and spectrum . . . 81
- 8.3 Construction of unbalanced pairs . . . 84
- 8.4 Computation . . . 90

9 Generalizations and open questions **92**

1 Introduction

Most of my PhD research was devoted to the ongoing quest of understanding "high-dimensional expanders". Expanders are sparse graphs which are highly connected. This is manifested in their geometry (isoperimetric constant), combinatorics (pseudo-randomness), and dynamics (behavior of random walks), but it turns out that the notion of *spectral expansion* - boundedness of the Laplace spectrum of the graph - is often the most useful for mathematical analysis. In my study I sought to generalize these notions of expansion to simplicial complexes of higher dimension, and to connect them to the spectrum of the high-dimensional Laplacian defined in the 1940's by Eckmann. My study, several parts of which were conducted in collaboration with other PhD students in our department, concerns several notions of expansion and related questions:

- Isoperimetric constant: this is a generalization of the Cheeger constant of graphs. It was studied, in joint work with Ron Rosenthal and Ran Tessler in [PRT14], and the results appear here in §3.
- pseudo-randomness (mixing): this is a natural notion of combinatorial expansion, which is close in spirit to the isoperimetric constant. This study is described in [PRT14, Par13a], and here in §4.
- Dynamical expansion: together with Ron Rosenthal I have defined and studied a stochastic process which generalizes random walks on graphs, and gives a notion of dynamic expansion which relates to the homology of a complex. This is described in [PR12, §2] and here in §6.
- Geometric overlap: this is a notion of expansion which is due to Gromov. Its relation to high-dimensional spectral expansion is explained in §5.1.
- Random complexes: random graphs are excellent expanders. Together with Ron Rosenthal I have studied the expansion properties of Random Linial-Meshulam complexes, and the results are described in [PRT14, §4.5] and here in §5.4
- Ramanujan complexes: these complexes are high-dimensional analogues of the Ramanujan graphs constructed in [LPS88]. Together with Konstantin Golubev I have studied the spectral and combinatorial properties of triangle Ramanujan complexes. Some of our results appear in §5.5.
- Asymptotic questions: in [PR12, §3] Ron Rosenthal and I have studied the spectrum of infinite complexes and questions concerning sequences of complexes, generalizing results of Kesten and of Alon-Boppana. This is described in §7.

Apart from spectral expansion, I have studied *isospectrality*: the challenge of constructing geometric objects (graphs, complexes, manifolds and orbifolds) which have the same Laplace spectrum. The results appear in §8, which is based on [Par13b]. Another research I was involved in during my PhD studies is that of words in free groups. The results are not described here, and appear in [PS13, PP14a, PP14b]. The rest of this section gives a summary of the work and its main results.

1.1 Isoperimetric constant

The *Cheeger constant* of a finite graph $G = (V, E)$ on n vertices is usually taken to be

$$\varphi(G) = \min_{\substack{A \subseteq V \\ 0 < |A| \le \frac{n}{2}}} \frac{|E(A, V \backslash A)|}{|A|}$$

where $E(A, B)$ is the set of edges with one vertex in A and the other in B. In this work, however, we use the following version:

$$h(G) = \min_{0 < |A| < n} \frac{n |E(A, V \backslash A)|}{|A| |V \backslash A|}. \tag{1.1}$$

Since $\varphi(G) \le h(G) \le 2\varphi(G)$, defining expanders by φ or by h is equivalent. The *spectral gap* of G, denoted $\lambda(G)$, is the second smallest eigenvalue of the *Laplacian* $\Delta^+ : \mathbb{R}^V \to \mathbb{R}^V$, which is defined by

$$\left(\Delta^+ f\right)(v) = \deg(v) f(v) - \sum_{w \sim v} f(w). \tag{1.2}$$

The discrete Cheeger inequalities [Tan84, Dod84, AM85, Alo86] relate the Cheeger constant and the spectral gap:

$$\frac{h^2(G)}{8k} \le \lambda(G) \le h(G), \tag{1.3}$$

where k is the maximal degree of a vertex in G.(†) In particular, the bound $\lambda \le h$ shows that spectral expanders are combinatorial expanders. This proved to be of immense importance since the spectral gap is approachable by many mathematical tools (coming from linear algebra, spectral methods, representation theory and even number theory - see [HLW06, Lub10, Lub12a] and the references within). In contrast, the Cheeger constant is usually hard to analyze directly, and even to compute it for a given graph is NP-hard [BKV+81, MS90].

Moving on to higher dimension, let X be an (abstract) simplicial complex with vertex set V. This means that X is a collection of subsets of V, called *cells* (and also *simplexes*, *faces*, or *hyperedges*), which is closed under taking subsets, i.e., if $\sigma \in X$ and $\tau \subseteq \sigma$, then $\tau \in X$. The *dimension* of a cell σ is $\dim \sigma = |\sigma| - 1$, and X^j denotes the set of cells of dimension j. The dimension of X is the maximal dimension of a cell in it. The *degree* of a j-cell (a cell of dimension j) is the number of $(j+1)$-cells which contain it. Throughout this work we denote by d the dimension of the complex at hand, and by n the number of vertices in it (which will be finite except for Section 7). We shall occasionally add the assumption that the complex has a *complete skeleton*, by which we mean that every possible j-cell with $j < d$ belongs to X.

We define the following generalization of the Cheeger constant:

Definition 1.1. For a finite d-complex X with n vertices V,

$$h(X) = \min_{V = \coprod_{i=0}^{d} A_i} \frac{n \cdot |F(A_0, A_1, \ldots, A_d)|}{|A_0| \cdot |A_1| \cdot \ldots \cdot |A_d|},$$

(†) For φ they are given by $\frac{\varphi^2(G)}{2k} \le \lambda(G) \le 2\varphi(G)$.

where the minimum is taken over all partitions of V into nonempty sets $A_0, \ldots, A_d$, and $F(A_0, \ldots, A_d)$ denotes the set of d-dimensional cells with one vertex in each A_i.

For $d = 1$, this coincides with the Cheeger constant of a graph (1.1). To formulate an analogue of the Cheeger inequalities, we need a high-dimensional analogue of the spectral gap. Such an analogue is provided by the work of Eckmann on discrete Hodge theory [Eck44]. In order to give the definition we shall need more terminology, and we defer this to §2.2[(†)]. The basic idea, however, is the same as for graphs, namely, the spectral gap $\lambda(X)$ is the smallest nontrivial eigenvalue of a suitable Laplace operator. The following theorem, whose proof appears in §3.1, generalizes the upper Cheeger inequality to higher dimensions:

Theorem 1.2 (Cheeger Inequality, [PRT14])**.** *For a finite complex X with a complete skeleton, $\lambda(X) \le h(X)$.*

Remarks. (1) If the skeleton of X is not complete, then $h(X) = 0$, since there exist some $\{v_0, \ldots, v_{d-1}\} \notin X^{d-1}$, and then $F(\{v_0\}, \{v_1\}, \ldots, \{v_{d-1}\}, V \setminus \{v_0, \ldots, v_{d-1}\}) = 0$. This suggests that a different definition of h is called for. We give such a definition in §5.5 (see (5.10)), and a corresponding Cheeger inequality is proved in Theorem 5.9. A different generalization of h appears in the open questions section §9.

(2) The existence of a lower Cheeger inequality is still an open question, and some progress in this direction is described in §3.2.

In [LM06] Linial and Meshulam introduced the following model for random simplicial complexes: for a given $p = p(n) \in (0,1)$, $X(d,n,p)$ is a d-dimensional simplicial complex on n vertices, with a complete skeleton, and with every d-cell being included independently with probability p. Using Theorem 1.2 we show in §5.4 the following:

Proposition 1.3. *Let $X = X\left(d, n, \frac{C \log n}{n}\right)$.*

(1) For large enough C, a.a.s. $h(X) \ge \left(C - O\left(\sqrt{C}\right)\right) \log n$.

(2) For $C < 1$, a.a.s. $h(X) = 0$.

The proof appears as part of Corollary 5.7.

1.2 Expander Mixing Lemmas

The Cheeger inequalities (1.1) bound the expansion along the partitions of a graph, in terms of its spectral gap. Nevertheless, a large spectral gap does not suffice to control the number of edges between *any* two sets of vertices. For example, the bipartite Ramanujan graphs constructed in [LPS88, MSS13] are regular graphs with very large spectral gaps, which are bipartite. This means that they contain disjoint sets $A, B \subseteq V$ of size $\frac{n}{4}$, with $E(A,B) = \varnothing$. The *Expander Mixing Lemma* [FP87, AC88, BMS93] (see also [HLW06]) remedies this inconvenience, using not only the spectral gap but also the maximal eigenvalue of the Laplacian:

(†) The spectral gap appears in Definition 2.2, and is given alternative characterizations in Proposition 2.4.

Theorem (Expander Mixing Lemma, [FP87, AC88, BMS93])**.** *Let $G = (V, E)$ be a graph on n vertices. If the nontrivial spectrum of its Laplacian is contained within $[k - \rho, k + \rho]$, then for any two sets of vertices A, B one has*

$$\left| |E(A,B)| - \frac{k|A||B|}{n} \right| \le \rho \cdot \sqrt{|A||B|}. \tag{1.4}$$

If k is the average degree of a vertex in G, then $\frac{k|A||B|}{n}$ is about the expected size of $|E(A,B)|$ (the exact value is $\frac{k}{n-1}|A||B|$). Thus, the Lemma means that a concentrated spectrum indicates a pseudo-random behavior. The deviation of $|E(A,B)|$ from its expected value $p|A||B|$, where $p = \frac{k}{n} \approx |E|/\binom{n}{2}$ is the edge density, is called the *discrepancy* of A and B. In a similar fashion, if k is the average degree of a $(d-1)$-cell in a d-complex X with a complete skeleton, we call the deviation

$$\left| |F(A_0, \ldots, A_d)| - \frac{|X^d|}{\binom{n}{d+1}} \cdot |A_0| \cdot \ldots \cdot |A_d| \right| \approx \left| |F(A_0, \ldots, A_d)| - \frac{k|A_0| \cdot \ldots \cdot |A_d|}{n} \right|$$

the discrepancy of $A_0, \ldots, A_d$ (the question of using $\frac{|X^d|}{\binom{n}{d+1}}$ or $\frac{k}{n}$ is addressed in Remark 4.1). The following theorem generalizes the Expander Mixing Lemma to these settings:

Theorem 1.4 (Mixing Lemma, [PRT14])**.** *If X is a d-dimensional complex with a complete skeleton, and the nontrivial spectrum of its Laplacian is contained within $[k - \rho, k + \rho]$, then for any disjoint sets of vertices $A_0, \ldots, A_d$ one has*

$$\left| |F(A_0, \ldots, A_d)| - \frac{k \cdot |A_0| \cdot \ldots \cdot |A_d|}{n} \right| \le \rho \cdot (|A_0| \cdot \ldots \cdot |A_d|)^{\frac{d}{d+1}}.$$

The Laplacian of X is defined in §2.1, and the proof of Theorem 1.4 appears in §4.1.

What happens when the skeleton of X is not complete? A d-dimensional complex has, in fact, d Laplace operators, with the j-th one acting on the cells of dimension j $(0 \le j < d)$. It turns out that the assumption of a complete skeleton can be replaced by the assumption that of all these operators have concentrated spectra: Let us say that X a (j, k, ε)*-expander* if $\varepsilon < 1$, and the nontrivial spectrum of the j-th Laplacian is contained within $[k(1-\varepsilon), k(1+\varepsilon)]$ (†). We then have (this is a special case of Proposition 4.2):

Theorem 1.5 ([Par13a])**.** *If a d-dimensional complex X is a (j, k_j, ε_j)-expander for every $0 \le j < d$, and $A_0, \ldots, A_d$ are disjoint sets of vertices in X then*

$$\left| |F(A_0, \ldots, A_d)| - \frac{k_0 \ldots k_{d-1}}{n^d} |A_0| \cdot \ldots \cdot |A_d| \right| \le c_d k_0 \ldots k_{d-1} (\varepsilon_0 + \ldots + \varepsilon_{d-1}) \max |A_i|,$$

where c_d depends only on d.

The understanding of $F(A_0, \ldots, A_d)$ in the case of general complexes is achieved by studying a wider counting problem:

(†) With this definition, Theorem 1.4 applies to a d-complex with a complete skeleton which is a $\left(d-1, k, \frac{\rho}{k}\right)$-expander.

Definition 1.6. Given disjoint sets $A_0, \ldots, A_\ell \subseteq V$, and $j \leq \ell$, a *j-gallery in* $A_0, \ldots, A_\ell$ is a sequence of j-cells $\sigma_0, \ldots, \sigma_{\ell-j} \in X^j$, such that σ_i is in $F(A_i, \ldots, A_{i+j})$, and σ_i and σ_{i+1} intersect in a $(j-1)$-cell (which must lie in $F(A_{i+1}, \ldots, A_{i+j})$). We denote the set of j-galleries in $A_0, \ldots, A_\ell$ by $F^j(A_0, \ldots, A_\ell)$.

Example.

(1) An ℓ-gallery in $A_0, \ldots, A_\ell$ is just a single ℓ-cell, so that $F^\ell(A_0, \ldots, A_\ell) = F(A_0, \ldots, A_\ell)$.

(2) A 0-gallery is any sequence of vertices, so that $F^0(A_0, \ldots, A_\ell) = A_0 \times \ldots \times A_\ell$.

(3) $F^2(A, B, C, D, E)$ is the number of triplets of triangles $t_1 \in F(A, B, C)$, $t_2 \in F(B, C, D)$, $t_3 \in F(C, D, E)$ such that the boundaries of t_1 and t_2 share a common edge (necessarily in $F(B, C)$), and likewise for t_2 and t_3.

The heart of our analysis is the following lemma, which estimates the size of $F^{j+1}(A_0, \ldots, A_\ell)$ in terms of that of $F^j(A_0, \ldots, A_\ell)$. Repeatedly applying this lemma allows us to estimate $|F(A_0, \ldots, A_d)| = \left|F^d(A_0, \ldots, A_d)\right|$ in terms of $\left|F^0(A_0, \ldots, A_d)\right| = |A_0| \cdot \ldots \cdot |A_d|$.

Lemma 1.7 (Descent Lemma, [Par13a])**.** *Let $A_0, \ldots, A_\ell$ be disjoint sets of vertices in X. If X is an (i, k_i, ε_i)-expander for $i = j-1, i = j$, then*

$$\left|\left|F^{j+1}(A_0, \ldots, A_\ell)\right| - \left(\frac{k_j}{k_{j-1}}\right)^{\ell-j} \left|F^j(A_0, \ldots, A_\ell)\right|\right|$$
$$\leq (\ell - j)\, k_j^{\ell-j} (\varepsilon_j + \varepsilon_{j-1}) \sqrt{|F(A_0, \ldots, A_j)|\, |F(A_{\ell-j}, \ldots, A_\ell)|}.$$

The proofs of this lemma and of the mixing lemma it implies (Theorem 1.5) appear in §4.2.

1.3 Examples and applications

If a graph $G = (V, E)$ has a large Cheeger constant, then given a mapping $\varphi : V \to \mathbb{R}$, there exists a point $x \in \mathbb{R}$ which is covered by many edges in the linear extension of φ to E (namely, $x = \text{median}(\{\varphi(v) \mid v \in V\})$. This observation led Gromov to define the *geometric overlap* of a complex ([Gro10], see also [FGL+12, MW11]):

Definition 1.8. Let X be a d-dimensional simplicial complex. The overlap of X is defined by

$$\text{overlap}(X) = \min_{\varphi : V \to \mathbb{R}^d} \max_{x \in \mathbb{R}^d} \frac{\#\left\{\sigma \in X^d \,\middle|\, x \in \text{conv}\{\varphi(v) \mid v \in \sigma\}\right\}}{|X^d|}.$$

In other words, X has overlap $\geq \varepsilon$ if for every simplicial mapping of X into $\mathbb{R}^d$ (a mapping induced linearly by the images of the vertices), some point in $\mathbb{R}^d$ is covered by at least an ε-fraction of the d-cells of X.

A theorem of Pach [Pac98], together with our mixing Lemmas yield a connection between the spectrum of the Laplacian and the overlap property:

Proposition. *There exist positive constants C_d and C'_d with the following property: if a d-complex X is a (j, k_j, ε_j)-expander for $0 \leq j < d$ then*

$$\operatorname{overlap} X > C_d - C'_d \left(\varepsilon_0 + \ldots + \varepsilon_{d-1}\right).$$

Corollary 5.2 proves this for complexes with a complete skeleton, and Proposition 5.4 for the general case. As an application, we show that Linial-Meshulam complexes have geometric overlap for suitable parameters:

Corollary. *There exist $\vartheta > 0$ such that for large enough C a.a.s.* $\operatorname{overlap}\left(X\left(d, n, \frac{C \cdot \log n}{n}\right)\right) > \vartheta$.

This is a part of Corollary 5.7, which is proved in §5.4. Another application of the expander mixing lemma is bounding the chromatic number of a complex, defined in §5.2:

Proposition 1.9. *There exists a constant C_d with the following property: if a d-complex X is a (j, k_j, ε_j)-expander for $0 \leq j < d$ then*

$$\chi(X) \geq \frac{C_d}{\sqrt[d]{\varepsilon_0 + \ldots + \varepsilon_{d-1}}},$$

where $\chi(X)$ is the chromatic number of X.

The Ramanujan graphs constructed in [LPS88, Mar88] form a celebrated example of excellent expanders. Their construction and study was generalized to *Ramanujan complexes* in [CSŻ03, Li04, LSV05a, LSV05b], but as of now little is known on their combinatorial expansion. In §5.5, which is based on [GP13], we study the Hodge spectrum of Ramanujan triangle complexes, i.e. complexes of dimension two. We obtain the following isoperimetric bound (for the definitions see §5.5):

Theorem 1.10. *If X is a non-3-colorable Ramanujan triangle complex with n vertices, vertex degree $k_0 = 2\left(q^2+q+1\right)$ and edge degree $k_1 = q+1$, then*

$$\frac{|F(A,B,C)|}{|A|\,|B|\,|C|} \geq \frac{1}{n^2}\left(q+1-2\sqrt{q}\right)\left(2q^2+2q+2-6q\left(1+\frac{10}{9\,|A|\,|B|\,|C|}\right)\right)$$

holds for any partition $V(X) = A \coprod B \coprod C$.

This can be stated in terms of an appropriate Cheeger constant (see (5.10) and (5.11)). Furthermore, we show that the major part of the spectrum of X is concentrated, giving hope of establishing pseudo-randomness in the future.

1.4 High dimensional random walk

There are well known connections between dynamical, topological and spectral properties of graphs: The random walk on a graph reflects both its topological and algebraic connectivity, which are reflected by the 0^{th}-homology and the spectral gap, respectively. In §6 we present a stochastic process which generalizes these connections to higher dimensions. In particular, for a finite d-dimensional complex, the asymptotic behavior of the process reflects the existence of a nontrivial $(d-1)$-homology, and its rate of convergence is dictated by the *normalized* spectral gap (see §6.2). In order to give a flavor of the results without plunging into the most general definitions, we present here, without proofs the special case of regular triangle complexes.

First, let us observe the $\frac{1}{2}$-lazy random walk on a k-regular graph $G = (V, E)$: the walker starts at a vertex v_0, and at each step remains in place with probability $\frac{1}{2}$ or moves to each of its k neighbors with probability $\frac{1}{2k}$. Let $\mathbf{p}_n^{v_0}(v)$ denote the probability of finding the walker at the vertex v at time n. The following observations are classic:

(1) If G is finite, then $\mathbf{p}_\infty^{v_0} = \lim_{n\to\infty} \mathbf{p}_n^{v_0}$ exists, and it is constant if and only if G is connected.

(2) Furthermore, the rate of convergence is given by

$$\left\|\mathbf{p}_n^{v_0} - \mathbf{const}\right\| = O\left(\left(1 - \frac{1}{2}\lambda(G)\right)^n\right),$$

where $\lambda(G)$ is the spectral gap of G (the definition follows below).

(3) When G is infinite and connected, the spectral gap is related to the return probability of the walk by

$$\lim_{n\to\infty} \sqrt[n]{\mathbf{p}_n^{v_0}(v_0)} = 1 - \frac{1}{2}\lambda(G). \tag{1.5}$$

Let us denote in this section by Δ^+ the *normalized* Laplacian of G, which acts on $\mathbb{R}^V$ by

$$\left(\Delta^+ f\right)(v) = f(v) - \frac{1}{k}\sum_{w\sim v} f(w)$$

If G is finite, then its *spectral gap* $\lambda(G)$ is the minimal Laplacian eigenvalue on a function whose sum on V vanishes. When G is infinite, its spectral gap is defined to be $\lambda(G) = \min \operatorname{Spec}\left(\Delta^+\big|_{L^2(V)}\right)$ (for more on this see §7.1).

Moving one dimension higher, let $X = (V, E, T)$ be a *k-regular triangle complex*, namely every edge in $E = X^1$ is contained in exactly k triangles in $T = X^2$. For $\{v, w\} \in E$ we denote the directed edge $\overset{v}{\bullet}\!\!\longrightarrow\!\!\overset{w}{\bullet}$ by $[v, w]$, and the set of all directed edges by $E_\pm$ (so that $|E_\pm| = 2|E|$). For $e \in E_\pm$, $\overline{e}$ denotes the edge with the same vertices and opposite direction, i.e. $\overline{[v, w]} = [w, v]$.

The following definition is the basis of the process which we shall study:

Definition 1.11. Two directed edges $e, e' \in E_\pm$ are called *neighbors* (indicated by $e \sim e'$) if they have the same origin or the same terminus, and the triangle they form is in the complex. Namely, if $e = [v, w]$ and $e' = [v', w']$, then $e \sim e'$ means that either $v = v'$ and $\{v, w, w'\} \in T$, or $w = w'$ and $\{v, v', w'\} \in T$.

We study the following lazy random walk on $E_\pm$: The walk starts at some directed edge $e_0 \in E_\pm$. At every step, the walker stays put with probability $\frac{1}{2}$, or else move to a uniformly chosen neighbor. Figure 1.1 illustrates one step of the process, in two cases (the right one is non-regular, but the walk is defined in the same manner).

As in the random walk on a graph, this process induces a sequence of distributions on $E_\pm$,

$$\mathbf{p}_n(e) = \mathbf{p}_n^{e_0}(e),$$

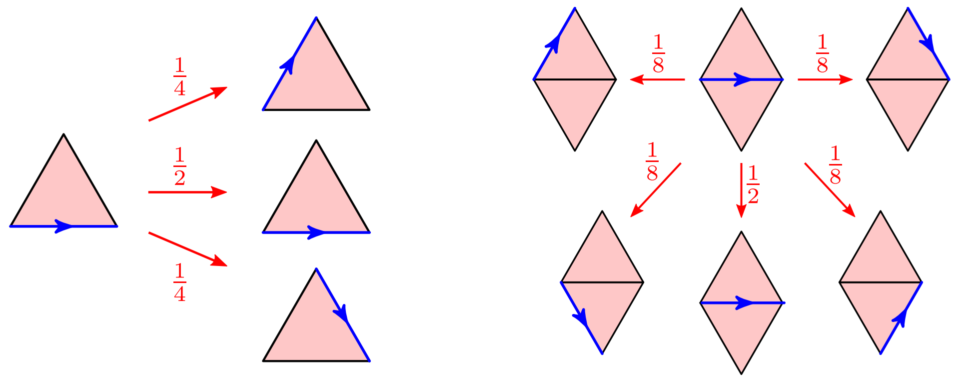


Figure 1.1: One step of the edge walk.

describing the probability of finding the walker at the directed edge e at time n (having started from e_0). However, studying the evolution of $\mathbf{p}_n$ amounts to studying the traditional random walk on the graph with vertices $E_\pm$ and edges defined by $\sim$. This will not take us very far, and in particular will not reveal the presence or absence of first homology in X. Instead, we study the evolution of what we call the "expectation process" on X:

$$\mathcal{E}_n(e) = \mathcal{E}_n^{e_0}(e) = \mathbf{p}_n^{e_0}(e) - \mathbf{p}_n^{e_0}(\overline{e}),$$

i.e. the probability of finding the walker at time n at e, minus the probability of finding it at the opposite edge $\overline{e}$ (for the reasons behind the name see Remark 6.4).

It is tempting to look at $\mathcal{E}_\infty^{e_0} = \lim_{n\to\infty} \mathcal{E}_n^{e_0}$ as is done in graphs, but a moment of reflection will show the reader that $\mathcal{E}_\infty^{e_0} \equiv 0$ for any finite triangle complex, and any starting point e_0. Namely, the probabilities of reaching e and $\overline{e}$ become arbitrarily close, for every e. While this might cause initial worry, it turns out that the rate of decay of $\mathcal{E}_n$ is always the same: for any finite triangle complex one has $\|\mathcal{E}_n^{e_0}\| = \Theta\left(\left(\frac{3}{4}\right)^n\right)$. It is therefore reasonable to turn our attention to the *normalized expectation process,*

$$\widetilde{\mathcal{E}}_n^{e_0}(e) = \left(\tfrac{4}{3}\right)^n \mathcal{E}_n^{e_0}(e) = \left(\tfrac{4}{3}\right)^n \left[\mathbf{p}_n^{e_0}(e) - \mathbf{p}_n^{e_0}(\overline{e})\right],$$

and observe its limit,

$$\widetilde{\mathcal{E}}_\infty^{e_0} = \lim_{n\to\infty} \widetilde{\mathcal{E}}_n^{e_0}.$$

For a finite triangle complex this limit always exists, and is nonzero. This is the object which reveals the first homology of the complex. To see how, we need the following definition: We say that $f : E_\pm \to \mathbb{R}$ is *exact* if its sum along every closed path vanishes; namely, if

$$v_0 \sim v_1 \sim \ldots \sim v_n = v_0 \qquad \Longrightarrow \qquad \sum_{i=0}^{n-1} f([v_i, v_{i+1}]) = 0.$$

This is the one-dimensional analogue of constant functions (for reasons which will become clear in §2), and the following holds:

(1) For a finite X, $\widetilde{\mathcal{E}}_\infty^{e_0}$ is exact for every $e_0 \in E_\pm$ if and only if G has a trivial first homology.

(2) Furthermore, the rate of convergence is given by

$$\left\|\widetilde{\mathcal{E}}_n^{e_0}-\mathbf{exact}\right\|=O\left(\left(1-\frac{1}{3}\lambda\left(X\right)\right)^n\right),$$

where $\lambda\left(X\right)$ is the spectral gap of X (see §6.2).

(3) If X is infinite and every vertex in X is of infinite degree, then its spectral gap (which is defined in §7.2) is revealed by the "return expectation":

$$\sup_{e_0\in E_\pm}\lim_{n\to\infty}\sqrt[n]{\widetilde{\mathcal{E}}_n^{e_0}\left(e_0\right)}=1-\frac{1}{3}\lambda\left(X\right).$$

What if one is interested not only in the existence of a first homology, but also in its dimension? The answer is manifested in the walk as well. In graphs the number of connected components is given by the dimension of Span $\{\mathbf{p}_\infty^{v_0}\,|\,v_0\in V\}$, and an analogue statement holds here (see Theorem 6.9).

Remark. If the non-lazy walk on a finite graph is observed, then apart from disconnectedness there is another obstruction for convergence to the uniform distribution: *bipartiteness.* We shall see that this is a special case of an obstruction in general dimension, which we call *disorientability* (see Definition 6.6). In our example we have avoided this problem by considering the lazy walk, both on graphs and on triangle complexes.

The analogue process for general dimension, and for non-regular complexes, is defined in §6.1. In §6.3 we define the corresponding normalized expectation process $\widetilde{\mathcal{E}}_n^{\sigma_0}$. In §6.3 it is shown that the limit of this process $\widetilde{\mathcal{E}}_\infty^{\sigma_0}=\lim_n\widetilde{\mathcal{E}}_n^{\sigma_0}$ always exists and captures various properties of X, according to the amount of laziness p (this is an abridged version of Theorem 6.9):

Theorem. *When $\frac{d-1}{3d-1}<p<1$, $\widetilde{\mathcal{E}}_\infty^{\sigma_0}$ is exact for every starting point σ_0 if and only if the $(d-1)$-homology of X is trivial. If furthermore $p\geq\frac{1}{2}$ then the rate of convergence is controlled by the spectral gap of X:*

$$\operatorname{dist}\left(\widetilde{\mathcal{E}}_n^{\sigma_0},\widetilde{\mathcal{E}}_\infty^{\sigma_0}\right)=O\left(\left(1-\frac{1-p}{p\left(d-1\right)+1}\lambda\left(X\right)\right)^n\right).$$

When $p=\frac{d-1}{3d-1}$, $\widetilde{\mathcal{E}}_\infty^{\sigma_0}$ is exact for every starting point σ_0 if and only if the $(d-1)$-homology of X is trivial, and in addition X has no disorientable $(d-1)$-components (see Definitions 6.2, 6.6).

1.5 Infinite complexes

In §7 we we turn to infinite complexes, studying the high-dimensional analogues of classic properties and theorems regarding infinite graphs. In this study we encounter new phenomena along the familiar ones, which reveal that graphs present only a degenerated case of a broader theory.

In §7.3 we define a family of simplicial complexes (which we call *arboreal complexes*) generalizing the notion of trees. In Theorem 7.3 we compute their spectra, extending Kesten's classic result on the spectrum of regular trees [Kes59]. The spectra of the regular arboreal complexes of high dimension and low regularity exhibit a surprising new phenomenon - an isolated eigenvalue.

Sections 7.4 and 7.5 are devoted to study the behavior of the spectrum with respect to a limit in the space of complexes. In particular we are interested in the high-dimensional analogue of the

Alon-Boppana theorem, which states that if a sequence of graphs G_n convergences to a graph G, then $\liminf_{n\to\infty}\lambda(G_n)\leq\lambda(G)$. We show that in general this need not hold in higher dimension (Theorem 7.10). This uses the isolated eigenvalue of the 2-regular arboreal complex of dimension two, which is shown in Figure 7.1 on page 61, as well as a study of the spectrum of balls in this complex (shown in Figure 7.2 on page 63).

Even though the Alon-Boppana theorem does not hold in general in high dimension, we show that under a variety of conditions it does hold :

Theorem 1.12 ([PR12])**.** *If $X_n \overset{n\to\infty}{\longrightarrow} X$, and one of the following holds:*

(1) The spectral gap of X is nonzero,

(2) zero is a non-isolated point in the spectrum of X, or

(3) the $(d-1)$-skeletons of the complexes X_n form a family of $(d-1)$-expanders,

then $\liminf_{n\to\infty}\lambda(X_n)\leq\lambda(X)$.

In §7.7 we show that the connection between the spectrum of a graph, and the return probability of the random walk on it (see e.g. [Kes59, Lemma 2.2]), generalizes to the high dimensional random walk defined in §6. In §7.8, the final section on infinite complexes, we address the high-dimensional analogues of the concepts of amenability, recurrence and transience, proving some properties of these (Proposition 7.16), and raising many open questions.

1.6 Isospectrality

Two graphs, complexes, or Riemannian manifolds are said to be *isospectral* if they have the same spectrum of the Laplace operator (see Definition 8.5). The question whether isospectral manifolds are necessarily isometric has gained popularity as "*Can one hear the shape of a drum?*" [Kac66], and it was answered negatively for many classes of manifolds (e.g., [Mil64, Bus86, GWW92, CDS94]). In 1985, Sunada described a general group-theoretic method for constructing isospectral Riemannian manifolds [Sun85], and recently this method was presented as a special case of a more general one [PB10]. In [Par13b] we explore a broader special case of the latter theory, obtaining the following, somewhat surprising, result:

Proposition (Corollary 8.14)**.** *Let G be a finite non-cyclic group which acts faithfully by isometries on a compact connected Riemannian manifold M. Then there exist $r\in\mathbb{N}$ and subgroups $H_1,\ldots,H_r$ and $K_1,\ldots,K_r$ of G such that the disjoint unions $\bigcup_{i=1}^{r} M/H_i$ and $\bigcup_{i=1}^{r} M/K_i$ are isospectral non-isometric manifolds (or orbifolds[†]).*

From this follows:

Theorem (Corollary 8.15)**.** *If M is a compact connected Riemannian manifold (or orbifold) whose fundamental group has a finite non-cyclic quotient, then M has isospectral non-isometric covers.*

[†]If G does not act freely on M (i.e. some $g\in G\setminus\{e\}$ acts on M with fixed points), then $\bigcup M/H_i$ and $\bigcup M/K_i$ are in general orbifolds. A reader not interested in these can assume that we discuss only manifolds, at the cost of limiting the study to free actions.

Let M denote a compact Riemannian manifold, and let G be a finite group which acts on it by isometries. In these settings, Sunada's theorem [Sun85] states that if two subgroups $H, K \leq G$ satisfy

$$\forall g \in G: \quad |[g] \cap H| = |[g] \cap K|, \tag{1.6}$$

where $[g]$ denotes the conjugacy class of g in G, then the quotients ${}^{M}\!/_{H}$ and ${}^{M}\!/_{K}$ are isospectral. In fact, it is not harder to show (see Corollary 8.6) that if two collections $H_1, \ldots, H_r$ and $K_1, \ldots, K_r$ of subgroups of G satisfy

$$\forall g \in G: \quad \sum_{i=1}^{r} \frac{|[g] \cap H_i|}{|H_i|} = \sum_{i=1}^{r} \frac{|[g] \cap K_i|}{|K_i|} \tag{1.7}$$

then $\bigcup {}^{M}\!/_{H_i}$ and $\bigcup {}^{M}\!/_{K_i}$ are isospectral[(†)]. We shall see, however, that in contrast with Sunada pairs (H, K satisfying (1.6)), collections satisfying (1.7) are rather abundant. In fact, we will show that every finite non-cyclic group G has such collections, and furthermore, that some of them (which we denote *unbalanced,* see Definition 8.7) necessarily yield non-isometric quotients.

1.6.1 Example

Let T be the torus ${}^{\mathbb{R}^2}\!/_{\mathbb{Z}^2}$. Let $G = \{e, \sigma, \tau, \sigma\tau\}$ be the non-cyclic group of size four (i.e. $G \cong {}^{\mathbb{Z}}\!/_{2\mathbb{Z}} \times {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$), and let $\sigma, \tau \in G$ act on T by rotations: $\sigma \cdot (x, y) = \left(x, y + \frac{1}{2}\right)$ and $\tau \cdot (x, y) = \left(x + \frac{1}{2}, y\right)$ (Figure 1.2).

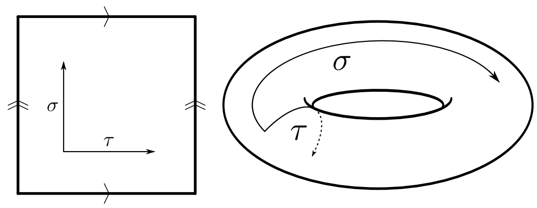


Figure 1.2: Two views of an action of $G = \{e, \sigma, \tau, \sigma\tau\} \cong {}^{\mathbb{Z}}\!/_{2\mathbb{Z}} \times {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$ on the torus T.

The subgroups

$$\begin{array}{ll} H_1 = \{e, \sigma\} & \\ H_2 = \{e, \tau\} & K_1 = \{e\} \\ & K_2 = K_3 = G \\ H_3 = \{e, \sigma\tau\} & \end{array} \tag{1.8}$$

satisfy (1.7): since G is abelian, (1.7) becomes $\forall g \in G: \sum_{i\,:\,g \in H_i} \frac{1}{|H_i|} = \sum_{i\,:\,g \in K_i} \frac{1}{|K_i|}$, which is easy to verify. Thus, the unions of tori $\bigcup {}^{T}\!/_{H_i} = {}^{T}\!/_{\langle\sigma\rangle} \bigcup {}^{T}\!/_{\langle\tau\rangle} \bigcup {}^{T}\!/_{\langle\sigma\tau\rangle}$ and $\bigcup {}^{T}\!/_{K_i} = T \bigcup {}^{T}\!/_{G} \bigcup {}^{T}\!/_{G}$ are isospectral (Figure 1.3).

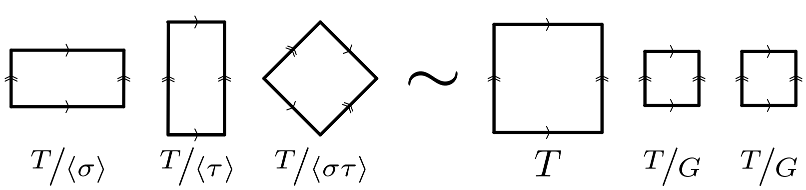


Figure 1.3: An isospectral pair consisting of quotients of the torus T (Figure 1.2) by the subgroups of G described in (1.8).

(†) In what follows $\bigcup$ always stands for disjoint union.

2 Preliminaries

Throughout this work X denotes a finite d-dimensional simplicial complex with vertex set V of size n (with $n < \infty$ until §7), and X^j denotes the set of j-cells of X, where $-1 \le j \le d$. In particular, we have $X^{-1} = \{\varnothing\}$. For $j \ge 1$, every j-cell $\sigma = \{\sigma_0, \dots, \sigma_j\}$ has two possible orientations, corresponding to the possible orderings of its vertices, up to an even permutation (1-cells and the empty cell have only one orientation). We denote an oriented cell by square brackets, and a flip of orientation by an overbar. For example, one orientation of $\sigma = \{x, y, z\}$ is $[x, y, z]$, which is the same as $[y, z, x]$ and $[z, x, y]$. The other orientation of σ is $\overline{[x, y, z]} = [y, x, z] = [x, z, y] = [z, y, x]$. We denote by $X^j_\pm$ the set of oriented j-cells (so that $\left|X^j_\pm\right| = 2\left|X^j\right|$ for $j \ge 1$ and $X^j_\pm = X^j$ for $j = -1, 0$).

We now describe the so-called *simplicial Hodge theory*, due to Eckmann [Eck44]. This is a discrete analogue of Hodge theory in Riemannian geometry, but in contrast, the proofs of the statements are all exercises in finite-dimensional linear algebra. Furthermore, it applies to any complex, and not only to manifolds.

The space of *j-forms on X*, denoted $\Omega^j(X)$, is the vector space of skew-symmetric functions on oriented j-cells:

$$\Omega^j = \Omega^j(X) = \left\{ f : X^j_\pm \to \mathbb{R} \,\middle|\, f(\overline{\sigma}) = -f(\sigma)\ \forall \sigma \in X^j_\pm \right\}.$$

In particular, Ω^0 is the space of functions on V, and $\Omega^{-1} = \mathbb{R}^{\{\varnothing\}}$ can be identified in a natural way with $\mathbb{R}$. With every oriented j-cell $\sigma \in X^j$ we associate the Dirac j-form $\mathbb{1}_\sigma$ defined by

$$\mathbb{1}_\sigma(\sigma') = \begin{cases} 1 & \sigma' = \sigma \\ -1 & \sigma' = \overline{\sigma} \\ 0 & \text{otherwise} \end{cases}$$

(for $j = 0$ this is the standard Dirac function, and $\mathbb{1}_\varnothing$ is the constant 1).

For a cell σ (either oriented or non-oriented) and a vertex v, we write $v \triangleleft \sigma$ if $v \notin \sigma$ and $\{v\} \cup \sigma$ is a cell in X. If $\sigma = [\sigma_0, \dots, \sigma_j]$ is oriented and $v \triangleleft \sigma$, then $v\sigma$ denotes the oriented $(j+1)$-cell $[v, \sigma_0, \dots, \sigma_j]$. An oriented j-cell $[\sigma_0, \dots, \sigma_j]$ induces orientations on its *faces* - the $(j-1)$-cells which form its boundary - as follows: the face $\{\sigma_0, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_j\}$ is oriented as $(-1)^i [\sigma_0, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_j]$, where $(-1)\tau = \overline{\tau}$.[(†)]

The j^{th} *boundary operator* $\partial_j : \Omega^j \to \Omega^{j-1}$ is

$$(\partial_j f)(\sigma) = \sum_{v \triangleleft \sigma} f(v\sigma),$$

and in particular, $\partial_0 : \Omega^0 \to \Omega^{-1}$ is defined by $(\partial_0 f)(\varnothing) = \sum_{v \in X^0} f(v)$. The sequence $(\Omega^\bullet, \partial_\bullet)$ is a

[(†)] An edge $e = [v_0, v_1]$ induces a "negative orientation" on v_0. We do not bother to make this formal, as everything we study is well known and understood in dimension zero.

chain complex, i.e., $\partial_{j-1}\partial_j = 0$ for all j, and one denotes

$$\begin{aligned} Z_j &= \ker \partial_j && j\text{-cycles} \\ B_j &= \operatorname{im} \partial_{j+1} && j\text{-boundaries} \\ H_j &= {}^{Z_j}\!/_{B_j} && \text{the } j^{\text{th}} \text{ homology of } X (\text{over } \mathbb{R}). \end{aligned}$$

The notions presented so far go back to the nineteenth century; Eckmann's innovation was introducing a "simplicial Riemannian structure", by endowing each space Ω^j with an inner product. Until §6, we assume that X is a finite complex and work with inner product

$$\langle f, g \rangle = \sum_{\sigma \in X^j} f(\sigma)\, g(\sigma) \tag{2.1}$$

(note that $f(\sigma)\, g(\sigma)$ is well defined even without choosing an orientation for σ). In §6 we will choose a different inner product (see §6.2), which is better suited to analyze the stochastic process studied there. Section §7 treats infinite complexes, for which the situation is more involved, and most of the statements to follow in this section are false. The proper adjustments are addressed in §7.2.

As Ω^j and Ω^{j-1} are finite dimensional inner product spaces, ∂_j has an adjoint operator. This is the *differential*, or *coboundary operator* $\delta_j = \partial_j^* : \Omega^{j-1} \to \Omega^j$, given by

$$\left(\partial_j^* f\right)(\sigma) = \sum_{\substack{\tau \text{ is a} \\ \text{face of } \sigma}} f(\tau) = \sum_{i=0}^{j} (-1)^i f(\sigma \backslash \sigma_i),$$

where $\sigma \backslash \sigma_i = [\sigma_0, \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots \sigma_j]$. Here the standard terms are

$$\begin{aligned} Z^j &= \ker \partial_{j+1}^* = B_j^{\perp} && \text{closed } j\text{-forms (or cocycles)} \\ B^j &= \operatorname{im} \partial_j^* = Z_j^{\perp} && \text{exact } j\text{-forms (or coboundaries)} \\ H^j &= {}^{Z^j}\!/_{B^j} && \text{the } j^{\text{th}} \text{ cohomology of } X (\text{over } \mathbb{R}). \end{aligned}$$

Example. For $j = 0$, Z^0 consists of the locally constant functions (functions constant on connected components); B^0 consists of the constant functions; Z_0 of the functions whose sum vanishes, and B_0 of the functions whose sum on each connected component vanishes.

For $j = 1$, Z^1 are the forms whose sum along the boundary of every triangle in the complex vanishes; in B^1 lie the forms whose sum along every closed path vanishes; Z_1 are the *Kirchhoff forms*, also known as *flows*, those for which the sum over all edges incident to a vertex, oriented inward, is zero; and B_1 are the forms spanned (over $\mathbb{R}$) by oriented boundaries of triangles in the complex. The chain of simplicial forms in dimensions -1 to 2 is depicted in Figure 2.1.

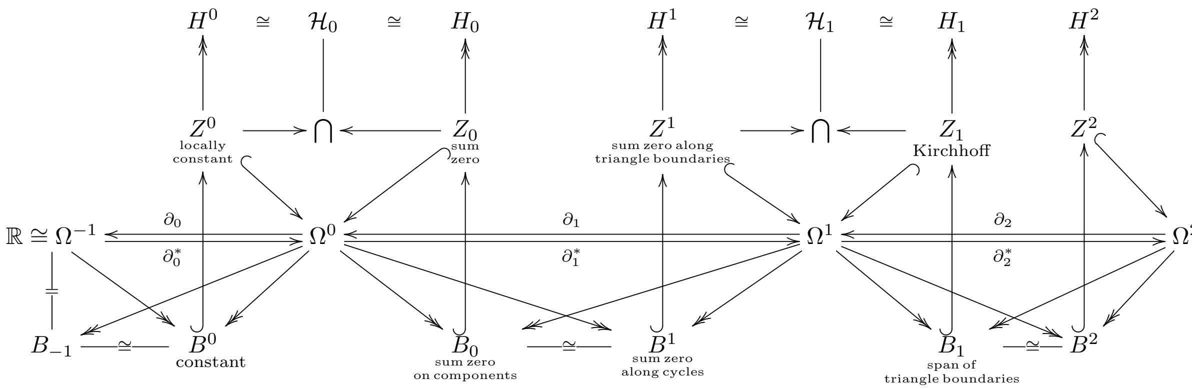


Figure 2.1: The lowermost part of the chain complex of simplicial forms.

2.1 Simplicial Hodge Laplacians

The spectral theory of complexes starts with the definition of the *upper, lower,* and *full Laplacians*:

$$\Delta_j^+ = \partial_{j+1}\delta_{j+1}, \qquad \Delta_j^- = \delta_j\partial_j, \qquad \text{and} \qquad \Delta_j = \Delta_j^+ + \Delta_j^-$$

respectively, all acting on Ω^j. These operators, especially Δ_j^+ and Δ_j, were studied in several prominent works, e.g. [Gar73, Żuk96, Fri98, KRS00, ABM05], sometimes under the name *combinatorial Laplacian.* We will work mainly with the upper Laplacian Δ_j^+. For a d-complex with a complete skeleton the case $j = d-1$ is the most important (e.g., the 0-th Laplacian of a graph), and in this case Δ^+ will stand for Δ_{d-1}^+.

All of the Laplacians are self-adjoint and decompose with respect to the orthogonal decompositions $\Omega^j = B^j \oplus Z_j = B_j \oplus Z^j$. In addition, $\ker \Delta_j^+ = Z^j$ and $\ker \Delta_j^- = Z_j$. The space of *harmonic j-forms* on X is $\mathcal{H}^j = \ker \Delta_j$. If $f \in \mathcal{H}^j$ then

$$0 = \langle \Delta f, f \rangle = \langle \partial_j f, \partial_j f \rangle + \left\langle \partial_{j+1}^* f, \partial_{j+1}^* f \right\rangle$$

which shows that $\mathcal{H}^j = Z^j \cap Z_j = \left(B_j \oplus B^j\right)^\perp$, giving the *discrete Hodge decomposition*

$$\Omega^j = \overbrace{B_j \oplus \underbrace{\mathcal{H}^j}_{}}^{Z_j}\!\!\underbrace{\;\oplus B^j}_{Z^j}. \tag{2.2}$$

In particular, it follows that the space of harmonic forms can be identified with the cohomology of X:

$$H^j = \frac{Z^j}{B^j} = \frac{B_j^\perp}{B^j} = \frac{B^j \oplus \mathcal{H}^j}{B^j} \cong \mathcal{H}^j.$$

The same holds for the homology of X, giving

$$H^j \cong \mathcal{H}^j \cong H_j. \tag{2.3}$$

The dimension of $\ker \Delta_j \cong H_j \cong H^j$ is the j^{th} (reduced) *Betti number* of X, denoted by β_j.

Remark. For comparison, the original Hodge decomposition states that for a Riemannian manifold M and $0 \leq j \leq \dim M$, there is an orthogonal decomposition

$$\Omega^j(M) = d\left(\Omega^{j-1}(M)\right) \oplus \mathcal{H}^j(M) \oplus \delta\left(\Omega^{j+1}(M)\right)$$

where Ω^j are the smooth j-forms on M, d is the exterior derivative, δ its Hodge dual, and $\mathcal{H}^j$ the smooth harmonic j-forms on M. As in the discrete case, this gives an isomorphism between the j^{th} de-Rham cohomology of M and the space of harmonic j-forms on it.

The combinatorial meaning of the Laplacians is better understood via the following adjacency relations on oriented cells:

Definition 2.1. Let σ and σ' be two distinct oriented j-cells in X.

(1) We denote $\sigma \pitchfork \sigma'$ if σ and σ' intersect in a common $(j-1)$-cell and induce the same orientation on it; for edges this means that they have a common origin or a common endpoint, and for vertices $v \pitchfork v'$ holds whenever $v \neq v'$.

(2) We denote $\sigma \sim \sigma'$, and say that σ and σ' are *neighbors*, if $\sigma \pitchfork \sigma'$, and in addition the $(j+1)$-cell $\sigma \cup \sigma'$ is in X. For vertices this is the common relation of neighbors in a graph.

Using these relations, the Laplacians can be expressed as follows (recall that the degree of a j-cell is the number of $(j+1)$-cells in which it is contained):

$$\begin{aligned} \left(\Delta_j^+\varphi\right)(\sigma) &= \deg(\sigma)\,\varphi(\sigma) - \sum_{\sigma'\sim\sigma}\varphi(\sigma') \\ \left(\Delta_j^-\varphi\right)(\sigma) &= (j+1)\,\varphi(\sigma) + \sum_{\sigma'\pitchfork\sigma}\varphi(\sigma') \\ \left(\Delta_j\varphi\right)(\sigma) &= (\deg\sigma + j + 1)\,\varphi(\sigma) + \sum_{\substack{\sigma'\pitchfork\sigma \\ \sigma'\nsim\sigma}}\varphi(\sigma') \end{aligned} \tag{2.4}$$

We also define adjacency operators on Ω^j which correspond to the $\sim$ and $\pitchfork$ relations:

$$\left(\mathcal{A}_j^{\sim}\varphi\right)(\sigma) = \sum_{\sigma'\sim\sigma}\varphi(\sigma'), \qquad \left(\mathcal{A}_j^{\pitchfork}\varphi\right)(\sigma) = \sum_{\sigma'\pitchfork\sigma}\varphi(\sigma'), \tag{2.5}$$

so that $\Delta_j^- = (j+1)\cdot I + \mathcal{A}_j^{\pitchfork}$ and $\Delta_j^+ = D_j - \mathcal{A}_j^{\sim}$, where D_j is the degree operator $(D_j f)(\sigma) = \deg(\sigma) f(\sigma)$.

2.2 The spectrum of complexes

The spectra we are primarily interested in are those of Δ_j^+ for $0 \leq j < d$.[†] For $j = 0$, this is the standard graph Laplacian

$$\left(\Delta_0^+ f\right)(v) = \deg(v) f(v) - \sum_{v'\sim v} f(v').$$

[†] It is sometimes useful to consider the Laplacian Δ_{-1}^+ as well. This operator acts on $\Omega^{-1} \cong \mathbb{R}$ as multiplication by $\deg\varnothing = |V| = n$, so that $\operatorname{Spec}\Delta_{-1}^+ = \{n\}$.

Every graph has a "trivial zero" in the spectrum of its Laplacian, corresponding to the constant functions, i.e. B^0. Similarly, since $(\Omega^\bullet, \delta_\bullet)$ is a co-chain complex, $B^j = \operatorname{im}\delta_j$ is always contained in the kernel of $\Delta_j^+ = \partial_{j+1}\delta_{j+1}$, and the eigenvalues which correspond to forms in B^j are considered to be the *trivial spectrum* of Δ_j^+. As $\left(B^j\right)^\perp = Z_j$, this leads to the following definition:

Definition 2.2. The *nontrivial spectrum* of Δ_j^+ is $\operatorname{Spec}\Delta_j^+\big|_{Z_j}$, and the j-dimensional *spectral gap,* denoted $\lambda_j(X)$, is the minimal nontrivial eigenvalue of Δ_j^+:

$$\lambda(X) = \min \operatorname{Spec}\left(\Delta_j^+\big|_{Z_j}\right)$$

(Note that we also have $\lambda_j(X) = \min\operatorname{Spec}\left(\Delta_j\big|_{Z_j}\right)$ since $\Delta_j\big|_{Z_j} \equiv \Delta_j^+\big|_{Z_j}$.)

Zero is a nontrivial eigenvalue of Δ_j^+ (i.e. $\lambda_j(X) = 0$) precisely when $\mathcal{H}^j = Z_j \cap Z^j \neq 0$, which by (2.3) happens iff X has nontrivial j-th homology. For example, the nontrivial spectrum of Δ_0^+ corresponds to Z_0, which are the functions whose sum on all vertices vanish, and zero is a nontrivial eigenvalue of Δ_0^+ iff the complex is disconnected.

Since $\lambda_j(X) = 0$ indicates a non-trivial j-th homology, a large value of $\lambda_j(X)$ should indicate a "very trivial j-th homology". For example, a graph with a large spectral gap should be "very connected". The Cheeger inequality for graphs gives precise meaning to this intuition, and our ambition is to generalize this to higher dimensions.

2.3 Complexes with a complete skeleton

Complexes with a complete skeleton appear to be particularly well behaved, in comparison with the general case. For these complexes we are mainly interested in $\lambda_{d-1}, \Delta_{d-1}^+, \Delta_{d-1}^-, \Delta_{d-1}, D_{d-1}, \mathcal{A}_{d-1}^\sim, \mathcal{A}_{d-1}^\pitchfork$, and we denote them simply by $\lambda, \Delta^+, \ldots$ (this will be the case until §4.2, and again in §6 and §7).

The following proposition lists some observations regarding these complexes. These will be used in the proofs of the main theorems in §3, §4.1, and also to obtain simpler characterizations of the spectral gap in this case.

Proposition 2.3. *If X has a complete skeleton, then*

(1) If $\overline{X}$ is the complement complex of X, i.e., $\overline{X}^{d-1} = X^{d-1} = \binom{V}{d}$ (†) and $\overline{X}^d = \binom{V}{d+1}\backslash X^d$, then

$$\Delta_{\overline{X}}^+ = n \cdot I - \Delta_X. \tag{2.6}$$

(2) The spectrum of Δ lies in the interval $[0, n]$.

(3) The lower Laplacian of X satisfies

$$\Delta^- = n \cdot \mathbb{P}_{B^{d-1}} \tag{2.7}$$

where $\mathbb{P}_{B^{d-1}}$ is the orthogonal projection onto B^{d-1}.

(†) $\binom{V}{j}$ denotes the set of subsets of V of size j.

Proof. By (2.4) we have $(\Delta_X f)(\sigma) = (\deg\sigma + d) f(\sigma) + \sum_{\substack{\sigma'\pitchfork\sigma \\ \sigma'\nsim\sigma}} f(\sigma')$, and as $\sigma' \sim \sigma$ in $\overline{X}$ iff $\sigma' \pitchfork \sigma$ and $\sigma' \nsim \sigma$ in X,

$$\left(\Delta_{\overline{X}}^{+} f\right)(\sigma) = (n - d - \deg(\sigma)) f(\sigma) - \sum_{\substack{\sigma'\pitchfork\sigma \\ \sigma'\nsim\sigma}} f(\sigma'),$$

hence (*1*) follows. From (*1*) we conclude that $\operatorname{Spec}\Delta_{\overline{X}}^{+} = \left\{n - \gamma \,\middle|\, \gamma \in \operatorname{Spec}\Delta_X\right\}$, and since Δ_X and $\Delta_{\overline{X}}^{+}$ are positive semidefinite, (*2*) follows. To establish (*3*), recall that $\left(B^{d-1}\right)^{\perp} = Z_{d-1} = \ker\Delta^{-}$, and it is left to show that $\Delta^{-} f = nf$ for $f \in B^{d-1}$. Note that $B^{d-1} \subseteq Z^{d-1} = \ker\Delta_{\overline{X}}^{+}$, and in addition, that since B^{d-1} only depends on X's $(d-1)$-skeleton,

$$B^{d-1}(X) = B^{d-1}\left(\overline{X}\right) \subseteq Z^{d-1}\left(\overline{X}\right) = \ker\Delta_{\overline{X}}^{+}.$$

Now from (*1*) it follows that for $f \in B^{d-1}$

$$\Delta_X^{-} f = \Delta_X^{-} f + \Delta_X^{+} f = \Delta_X f = nf - \Delta_{\overline{X}}^{+} f = nf$$

as desired. □

The following proposition gives several alternate characterizations of $\lambda(X)$:

Proposition 2.4. *Let $\lambda(X) = \lambda_{d-1}(X)$ be the $(d-1)$-th spectral gap.*

(1) $\lambda(X)$ is the $(r+1)$-th smallest eigenvalue of Δ_{d-1}^{+}, where $r = \left(\left|X^{d-1}\right| - \beta_{d-1}\right) - \left(\left|X^{d}\right| - \beta_{d}\right)$.

Furthermore, if X has a complete skeleton, then

(2) $\lambda(X)$ is the $\binom{n-1}{d-1} + 1$ smallest eigenvalue of Δ^{+},

(3) and

$$\lambda(X) = \min\operatorname{Spec}\Delta. \tag{2.8}$$

Remarks.

(1) For a graph $G = (V, E)$ we have $\lambda(G) = \lambda_r$, where $r = |V| - |E| - \beta_0 + \beta_1 = 1$ (this follows from Euler's formula), hence $\lambda(G)$ is the second smallest eigenvalue of the graph's Laplacian. Alternatively, (2.8) gives $\lambda(G) = \min\operatorname{Spec}(\Delta^{+} + J)$, where $J = \Delta^{-} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$.

(2) In general (2.8) does not hold: for example, for the triangle complex ⋈, $\lambda = \min\operatorname{Spec}\left(\Delta\big|_{Z_1}\right) = 3$ but $\min\operatorname{Spec}\Delta = 1$.

Proof.

(1) Since Δ^{+} decomposes w.r.t. $\Omega^{d-1} = B^{d-1} \oplus Z_{d-1}$, and $\Delta^{+}\big|_{B^{d-1}} \equiv 0$, the spectrum of Δ^{+} consists of $r = \dim B^{d-1}$ zeros, followed by the spectral gap. To compute r, we observe that

$$\begin{aligned} \dim B^{j-1} &= \dim Z^{j-1} - \dim H^{j-1} = \operatorname{null}\partial_j^{*} - \beta_{j-1} \\ &= \dim\Omega^{j-1} - \operatorname{rank}\partial_j^{*} - \beta_{j-1} = \left|X^{j-1}\right| - \dim B^{j} - \beta_{j-1} \end{aligned}$$

and therefore

$$
\begin{aligned}
r = \dim B^{d-1} &= \left|X^{d-1}\right| - \dim B^{d} - \beta_{d-1} = \left|X^{d-1}\right| - \left(\left|X^{d}\right| - \dim B^{d+1} - \beta_{d}\right) - \beta_{d-1} \\
&= \left(\left|X^{d-1}\right| - \beta_{d-1}\right) - \left(\left|X^{d}\right| - \beta_{d}\right).
\end{aligned}
$$

(2) The Euler characteristic satisfies $\sum_{i=-1}^{d} (-1)^{i} \left|X^{i}\right| = \chi(X) = \sum_{i=-1}^{d} (-1)^{i} \beta_{i}$. Therefore,

$$
\begin{aligned}
r &= \left(\left|X^{d-1}\right| - \beta_{d-1}\right) - \left(\left|X^{d}\right| - \beta_{d}\right) \\
&= \left(\left|X^{d-1}\right| - \beta_{d-1}\right) - \left(\left|X^{d}\right| - \beta_{d}\right) + (-1)^{d} \sum_{i=-1}^{d} (-1)^{i} \left(\left|X^{i}\right| - \beta_{i}\right) \\
&= \sum_{i=-1}^{d-2} (-1)^{d+i} \left(\left|X^{i}\right| - \beta_{i}\right).
\end{aligned}
$$

Since the $(d-1)$-skeleton is complete, $\left|X^{i}\right| = \binom{n}{i+1}$ and $\beta_i = 0$ for $-1 \le i \le d-2$, and so

$$
r = \sum_{i=-1}^{d-2} (-1)^{d+i} \binom{n}{i+1} = \binom{n-1}{d-1}.
$$

(3) First, since Δ decomposes w.r.t. $\Omega^{d-1} = B^{d-1} \oplus Z_{d-1}$ we have

$$
\operatorname{Spec} \Delta = \operatorname{Spec} \Delta\big|_{B^{d-1}} \cup \operatorname{Spec} \Delta\big|_{Z_{d-1}} = \operatorname{Spec} \Delta^{-}\big|_{B^{d-1}} \cup \operatorname{Spec} \Delta^{+}\big|_{Z_{d-1}}.
$$

By Proposition 2.3, $\operatorname{Spec} \Delta^{-}\big|_{B^{d-1}} = \{n\}$ and $\operatorname{Spec} \Delta \subseteq [0,n]$, which implies that $\lambda = \min \operatorname{Spec} \left(\Delta^{+}\big|_{Z_{d-1}}\right) = \min \operatorname{Spec} \Delta$.

□

We finish with a note on the density of d-cells in X, which will come in handy later. A generalization of this to complexes with a non-complete skeleton appears in Lemma 5.5.

Proposition 2.5. *Let X be a d-complex with a complete skeleton. Let $\mathfrak{D}$ denote the d-cell density of X, $\mathfrak{D} = \frac{\left|X^{d}\right|}{\binom{n}{d+1}}$, let k denote the average degree of a $(d-1)$-cell, and let λ_{avg} denote the average nontrivial eigenvalue of $\Delta^{+} = \Delta^{+}_{d-1}$. Then*

$$
\mathfrak{D} = \frac{\lambda_{avg}}{n} = \frac{k}{n-d}.
$$

Proof. On the one hand,

$$
\mathfrak{D} = \frac{\left|X^{d}\right|}{\binom{n}{d+1}} = \frac{\left|X^{d-1}\right| \frac{k}{d+1}}{\binom{n}{d+1}} = \frac{\binom{n}{d} \frac{k}{d+1}}{\binom{n}{d+1}} = \frac{k}{n-d}.
$$

On the other,

$$\binom{n}{d}k = \left|X^{d-1}\right| k = \sum_{\sigma \in X^{d-1}} \deg \sigma = \operatorname{trace} \Delta^+ = \sum_{\lambda \in \operatorname{Spec} \Delta^+} \lambda = \sum_{\lambda \in \operatorname{Spec} \Delta^+|_{Z_{d-1}}} \lambda,$$

and by Proposition 2.4

$$\lambda_{avg} = \frac{1}{\binom{n}{d} - \binom{n-1}{d-1}} \sum_{\lambda \in \operatorname{Spec} \Delta^+|_{Z_{d-1}}} \lambda = \frac{1}{\binom{n-1}{d}} \sum_{\lambda \in \operatorname{Spec} \Delta^+|_{Z_{d-1}}} \lambda = \frac{n}{n-d} \cdot k.$$

$\square$

3 Isoperimetric constant

3.1 A Cheeger-type inequality

This section is devoted to the proof of Theorem 1.2: For a complex with a complete skeleton, the generalized Cheeger constant (Definition 1.1) is bounded from below by the spectral gap (Definition 2.2).

Proof of Theorem 1.2. Recall that we seek to show

$$\min \operatorname{Spec}\left(\Delta^{+}\big|_{Z_{d-1}}\right) = \lambda(X) \le h(X) = \min_{V=\coprod_{i=0}^{d} A_i} \frac{n \cdot |F(A_0, A_1, \ldots, A_d)|}{|A_0| \cdot |A_1| \cdot \ldots \cdot |A_d|}.$$

Let $A_0, \ldots, A_d$ be a partition of V which realizes the minimum in h. We define $f \in \Omega^{d-1}$ by

$$f([\sigma_0\, \sigma_1\, \ldots\, \sigma_{d-1}]) = \begin{cases} \operatorname{sgn}(\pi)\left|A_{\pi(d)}\right| & \exists \pi \in \operatorname{Sym}_{\{0\ldots d\}} \text{ with } \sigma_i \in A_{\pi(i)} \text{ for } 0 \le i \le d-1 \\ 0 & \text{else, i.e. } \exists k, i \ne j \text{ with } \sigma_i, \sigma_j \in A_k. \end{cases} \tag{3.1}$$

Note that $f(\pi'\sigma) = \operatorname{sgn}(\pi') f(\sigma)$ for any $\pi' \in \operatorname{Sym}_{\{0\ldots d-1\}}$ and $\sigma \in X^{d-1}$. Therefore, f is a well-defined skew-symmetric function on oriented $(d-1)$-cells, i.e., $f \in \Omega^{d-1}$. Figure 3.1 illustrates f for $d = 1, 2$.

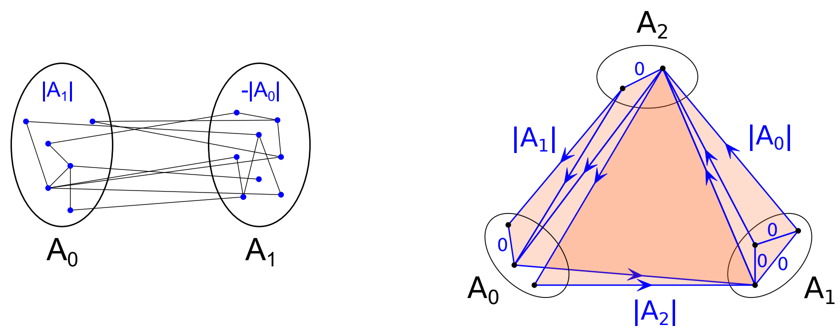


Figure 3.1: The form $f \in \Omega^{d-1}$ defined in (3.1), for complexes of dimensions one and two.

We proceed to show that $f \in Z_{d-1}$. Let $\sigma = [\sigma_0, \sigma_1, \ldots, \sigma_{d-2}] \in X^{d-2}_{\pm}$. As we assumed that X^{d-1} is complete,

$$(\partial_{d-1} f)(\sigma) = \sum_{v \lhd \sigma} f([v, \sigma_0, \sigma_1, \ldots, \sigma_{d-2}]) = \sum_{v \notin \sigma} f([v, \sigma_0, \sigma_1, \ldots, \sigma_{d-2}]).$$

If for some k and $i \ne j$ we have $\sigma_i, \sigma_j \in A_k$, this sum vanishes. On the other hand, if there exists $\pi \in \operatorname{Sym}_{\{0\ldots d\}}$ such that $\sigma_i \in A_{\pi(i)}$ for $0 \le i \le d-2$ then

$$\begin{aligned} (\partial_{d-1} f)(\sigma) &= \sum_{v \in A_{\pi(d-1)}} f([v, \sigma_0, \sigma_1, \ldots, \sigma_{d-2}]) + \sum_{v \in A_{\pi(d)}} f([v, \sigma_0, \sigma_1, \ldots, \sigma_{d-2}]) \\ &= \sum_{v \in A_{\pi(d-1)}} (-1)^{d-1} \operatorname{sgn}\pi \left|A_{\pi(d)}\right| + \sum_{v \in A_d} (-1)^{d} \operatorname{sgn}\pi \left|A_{\pi(d-1)}\right| \\ &= (-1)^{d-1} \operatorname{sgn}\pi \left(\left|A_{\pi(d-1)}\right| \left|A_{\pi(d)}\right| - \left|A_{\pi(d)}\right| \left|A_{\pi(d-1)}\right|\right) = 0 \end{aligned}$$

and in both cases $f \in Z_{d-1}$. Thus, by Rayleigh's principle

$$\lambda(X) = \min \operatorname{Spec}\left(\Delta^+\big|_{Z_{d-1}}\right) \leq \frac{\langle \Delta^+ f, f\rangle}{\langle f, f\rangle} = \frac{\langle \partial_d^* f, \partial_d^* f\rangle}{\langle f, f\rangle}. \tag{3.2}$$

The denominator is

$$\langle f, f\rangle = \sum_{\sigma \in X^{d-1}} f(\sigma)^2,$$

and a $(d-1)$-cell σ contributes to this sum only if its vertices are in different blocks of the partition, i.e., there are no k and $i \neq j$ with $\sigma_i, \sigma_j \in A_k$. In this case, there exists a unique block, A_i, which does not contain a vertex of σ, and σ contributes $|A_i|^2$ to the sum. Since X^{d-1} is complete, there are $|A_0| \cdot \ldots \cdot |A_{i-1}| \cdot |A_{i+1}| \cdot \ldots \cdot |A_d|$ non-oriented $(d-1)$-cells whose vertices are in distinct blocks and which do not intersect A_i, hence

$$\langle f, f\rangle = \sum_{i=0}^{d} \left(\prod_{j \neq i} |A_j|\right) |A_i|^2 = n \prod_{i=0}^{d} |A_i|.$$

To evaluate the numerator in (3.2), we first show that for $\sigma \in X^d$

$$|(\partial_d^* f)(\sigma)| = \begin{cases} n & \sigma \in F(A_0, \ldots, A_d) \\ 0 & \sigma \notin F(A_0, \ldots, A_d). \end{cases} \tag{3.3}$$

First, let $\sigma \notin F(A_0, \ldots, A_d)$. If σ has three vertices from the same A_i, or two pairs of vertices from the same blocks (i.e. $\sigma_i, \sigma_j \in A_k$ and $\sigma_{i'}, \sigma_{j'} \in A_{k'}$), then for every summand in

$$(\partial_d^* f)(\sigma) = \sum_{i=0}^{d} (-1)^i f(\sigma \backslash \sigma_i),$$

the cell $\sigma \backslash \sigma_i$ has two vertices from the same block, and therefore $(\partial_d^* f)(\sigma) = 0$. Next, assume that σ_j and σ_k (with $j < k$) is the only pair of vertices in σ which belong to the same block. The only non-vanishing terms in $(\partial_d^* f)(\sigma) = \sum_{i=0}^{d} (-1)^i f(\sigma \backslash \sigma_i)$ are $i = j$ and $i = k$, i.e.,

$$(\partial_d^* f)(\sigma) = (-1)^j f(\sigma \backslash \sigma_j) + (-1)^k f(\sigma \backslash \sigma_k).$$

Since the value of f on a simplex depends only on the blocks to which its vertices belong,

$$\begin{aligned} f(\sigma \backslash \sigma_j) &= f([\sigma_0\, \sigma_1\, \ldots\, \sigma_{j-1}\, \sigma_{j+1}\, \ldots \sigma_{k-1}\, \sigma_k\, \sigma_{k+1} \ldots\, \sigma_d]) \\ &= f([\sigma_0\, \sigma_1\, \ldots\, \sigma_{j-1}\, \sigma_{j+1}\, \ldots \sigma_{k-1}\, \sigma_j\, \sigma_{k+1} \ldots\, \sigma_d]) \\ &= f\left((-1)^{k-j+1} [\sigma_0\, \sigma_1\, \ldots\, \sigma_{j-1}\, \sigma_j\, \sigma_{j+1}\, \ldots\, \sigma_{k-1}\, \sigma_{k+1} \ldots\, \sigma_d]\right) \\ &= (-1)^{k-j+1} f(\sigma \backslash \sigma_k), \end{aligned}$$

so that

$$(\partial_d^* f)(\sigma) = (-1)^j (-1)^{k-j+1} f(\sigma \backslash \sigma_k) + (-1)^k f(\sigma \backslash \sigma_k) = 0.$$

The remaining case is $\sigma \in F(A_0, \ldots, A_d)$. Here, there exists $\pi \in \mathrm{Sym}_{\{0\ldots d\}}$ with $\sigma_i \in A_{\pi(i)}$ for $0 \leq i \leq d$. Observe that

$$f(\sigma \backslash \sigma_i) = \mathrm{sgn}\left(\pi \cdot (d\ d{-}1\ d{-}2\ \ldots\ i)\right)\left|A_{\pi(i)}\right| = (-1)^{d-i}\,\mathrm{sgn}\,\pi\left|A_{\pi(i)}\right|$$

and therefore

$$(\partial_d^* f)(\sigma) = \sum_{i=0}^{d} (-1)^i f(\sigma \backslash \sigma_i) = (-1)^d\,\mathrm{sgn}\,\pi \sum_{i=0}^{d} \left|A_{\pi(i)}\right| = (-1)^d\,\mathrm{sgn}\,\pi \cdot n.$$

Therefore, $|(\partial_d^* f)(\sigma)| = n$. This establishes (3.3), which implies that

$$\langle \partial_d^* f, \partial_d^* f \rangle = \sum_{\sigma \in X^d} |(\partial_d^* f)(\sigma)|^2 = n^2\,|F(A_0, \ldots, A_d)|$$

and in total

$$\lambda(X) \leq \frac{\langle \partial_d^* f, \partial_d^* f \rangle}{\langle f, f \rangle} = \frac{n\,|F(A_0, \ldots, A_d)|}{\prod_{i=0}^{d} |A_i|} = h(X).$$

$\square$

3.2 Towards a lower Cheeger inequality

The first observation to be made regarding a lower Cheeger inequality, is that no bound of the form $C \cdot h(X)^m \leq \lambda(X)$ can be found. Had such a bound existed, one would have that $\lambda(X) = 0$ implies $h(X) = 0$, but a counterexample to this is provided by the minimal triangulation of the Möbius strip (Figure 3.2).

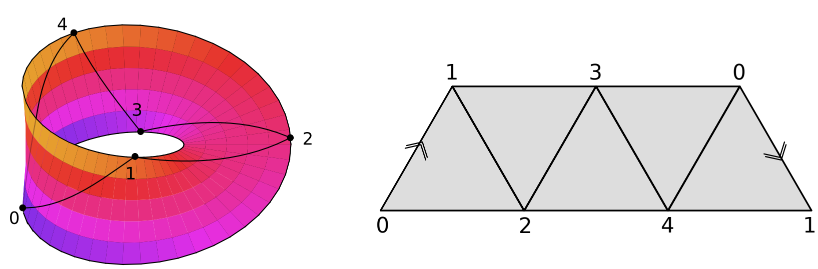


Figure 3.2: A triangulation of the Möbius strip for which $h(X) = 1\frac{1}{4}$ but $\lambda(X) = 0$.

Nevertheless, numerical experiments hint that a bound of the form $C \cdot h(X)^2 - c \leq \lambda(X)$ should hold, where C and c depend on the dimension and the maximal degree of a $(d-1)$-cell in X.

An attempt towards an upper bound for the Cheeger constant can be made by connecting it to "local Cheeger constants", as follows. For every $\tau \in X^{d-2}$ we consider the *link* of τ (see Figure 3.3),

$$\mathrm{lk}\,\tau = \{\sigma \in X \mid \sigma \cap \tau = \varnothing \text{ and } \sigma \cup \tau \in X\}.$$

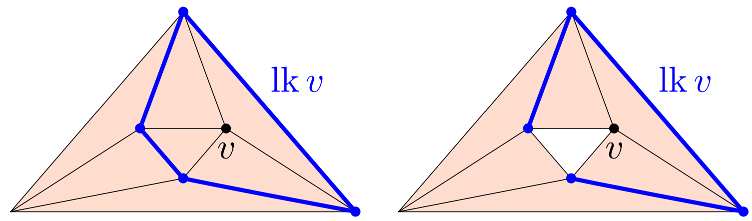


Figure 3.3: Two examples for the link of a vertex in a triangle complex.

Since $\dim \tau = d-2$, $\mathrm{lk}\,\tau$ is a graph, and there is a $1-1$ correspondence between vertices (edges) of $\mathrm{lk}\,\tau$ and $(d-1)$-cells (d-cells) of X which contain τ. We have the following bound for the Cheeger constant of X:

Proposition 3.1. *The bound $h(X) \le \frac{h(\mathrm{lk}\,\tau)}{1-\frac{d-1}{n}}$ holds for any d-complex X and $\tau \in X^{d-2}$.*

Proof. Write $\tau = [\tau_0, \tau_1, \dots, \tau_{d-2}]$ and denote $A_i = \{\tau_i\}$ for $0 \le i \le d-2$. Due to the correspondence between $(\mathrm{lk}\,\tau)^j$ and cells in X^{d-1+j} containing τ,

$$h(\mathrm{lk}\,\tau) \overset{def}{=} \min_{B \coprod C = (\mathrm{lk}\,\tau)^0} \frac{|E_{\mathrm{lk}\,\tau}(B,C)| \cdot \left|(\mathrm{lk}\,\tau)^0\right|}{|B| \cdot |C|} = \min_{B \coprod C = (\mathrm{lk}\,\tau)^0} \frac{|F(A_0, \dots, A_{d-2}, B, C)| \cdot \left|(\mathrm{lk}\,\tau)^0\right|}{|B| \cdot |C|}.$$

Assume that the minimum is attained by $B = B_0$ and $C = C_0$. We define

$$A_{d-1} = B_0, \qquad A_d = V \backslash \left(\bigcup_{i=0}^{d-1} A_i \right).$$

Now $A_0, \dots, A_d$ is a partition of V, and

$$F(A_0, \dots, A_{d-2}, B_0, C_0) = F(A_0, \dots, A_{d-2}, A_{d-1}, A_d)$$

since no d-cell containing τ has a vertex in $A_d \backslash C_0$. In addition,

$$\begin{aligned}
\frac{\left|(\mathrm{lk}\,\tau)^0\right| |A_d|}{n\,|C_0|} &\ge \frac{\left|(\mathrm{lk}\,\tau)^0\right| |A_d| - |A_{d-1}| \left(|A_d| - |C_0|\right)}{n\,|C_0|} \\
&= \frac{\left[n - (d-1) - (|A_d| - |C_0|)\right] |A_d| - |A_{d-1}| \left(|A_d| - |C_0|\right)}{n\,|C_0|} \\
&= \frac{(n - (d-1)) |A_d| - (|A_{d-1}| + |A_d|) \left(|A_d| - |C_0|\right)}{n\,|C_0|} \\
&= \frac{(n - (d-1)) \left[|A_d| - (|A_d| - |C_0|)\right]}{n\,|C_0|} = 1 - \frac{d-1}{n},
\end{aligned}$$

which implies

$$
\begin{aligned}
h\left(\operatorname{lk}\tau\right) &= \frac{F\left(A_0,\dots,A_{d-2},A_{d-1},A_d\right)\left|\left(\operatorname{lk}\tau\right)^0\right|}{\left|B_0\right|\cdot\left|C_0\right|} = \frac{F\left(A_0,\dots,A_{d-2},A_{d-1},A_d\right)n}{\left|A_0\right|\cdot\ldots\cdot\left|A_d\right|}\cdot\frac{\left|\left(\operatorname{lk}\tau\right)^0\right|\left|A_d\right|}{n\left|C_0\right|} \\
&\geq h\left(X\right)\cdot\frac{\left|\left(\operatorname{lk}\tau\right)^0\right|\left|A_d\right|}{n\left|C_0\right|} \geq \left(1-\frac{d-1}{n}\right)h\left(X\right).
\end{aligned}
$$

□

Since $\operatorname{lk}\tau$ is a graph, its Cheeger constant can be bounded using the lower inequality in (1.1). We also note that the degree of a vertex in $\operatorname{lk}\tau$ corresponds to the degree of a $(d-1)$-cell in X, and therefore

$$
\frac{\left(1-\frac{d-1}{n}\right)^2}{8k}h^2\left(X\right) \leq \frac{h\left(\operatorname{lk}\tau\right)^2}{8k} \leq \frac{h\left(\operatorname{lk}\tau\right)^2}{8k_\tau} \leq \lambda\left(\operatorname{lk}\tau\right) \tag{3.4}
$$

where k is the maximal degree of a $(d-1)$-cell in X, and k_τ of a vertex in $\operatorname{lk}\tau$.

We now see that a bound of the spectral gap of links by that of the complex would yield a lower Cheeger inequality. Such a bound was discovered by Garland [Gar73], and was studied further by several authors [Żuk96, ABM05, GW12]. The following lemma appears in [GW12], for a normalized version of the Laplacian. We give here its form for the Laplacian we use.

Lemma 3.2 ([Gar73, GW12])**.** *Let X be a d-dimensional simplicial complex. Given $f\in\Omega^{d-1},\sigma\in X^{d-1},\tau\in X^{d-2}$ define a function $f_\tau:\left(\operatorname{lk}\tau\right)^0\to\mathbb{R}$ by $f_\tau\left(v\right)=f\left(v\tau\right)$, and an operator $\Delta_\tau^+:\Omega^{d-1}\left(X\right)\to\Omega^{d-1}\left(X\right)$ by*

$$
\left(\Delta_\tau^+f\right)\left(\sigma\right)=\begin{cases}\deg_\tau\left(\sigma\right)f\left(\sigma\right)-\sum\limits_{\substack{\sigma'\sim\sigma\\ \tau\subseteq\sigma'}}f\left(\sigma'\right) & \tau\subset\sigma\\ 0 & \tau\nsubseteq\sigma\end{cases}
$$

where $\deg_\tau\left(\sigma\right)=\#\left\{\sigma'\sim\sigma\,|\,\tau\subseteq\sigma'\right\}=\deg_{\operatorname{lk}\tau}\left(\sigma\backslash\tau\right)$. The following then hold:

(1) $\Delta^+=\left(\sum_{\tau\in X^{d-2}}\Delta_\tau^+\right)-\left(d-1\right)D$.

(2) $\left\langle\Delta_\tau^+f,f\right\rangle=\left\langle\Delta_{\operatorname{lk}\tau}^+f_\tau,f_\tau\right\rangle$.

(3) If $f\in Z_{d-1}$ then $f_\tau\in Z_0\left(\operatorname{lk}\tau\right)$.

(4) $\sum_{\tau\in X^{d-2}}\left\langle f_\tau,f_\tau\right\rangle=d\left\langle f,f\right\rangle$.

Proof. (1) By the definition of Δ_τ^+,

$$
\begin{aligned}
\sum_{\tau\in X^{d-2}}\Delta_\tau^+f\left(\sigma\right)-\left(d-1\right)Df\left(\sigma\right) &= \sum_{\substack{\tau\in X^{d-2}\\ \tau\subseteq\sigma}}\left(\deg_\tau\left(\sigma\right)f\left(\sigma\right)-\sum_{\substack{\sigma'\sim\sigma\\ \tau\subseteq\sigma'}}f\left(\sigma'\right)\right)-\left(d-1\right)\deg\left(\sigma\right)f\left(\sigma\right) \\
&= \left(\sum_{\substack{\tau\in X^{d-2}\\ \tau\subseteq\sigma}}\deg_\tau\left(\sigma\right)-\left(d-1\right)\deg\sigma f\left(\sigma\right)\right)-\sum_{\substack{\tau\in X^{d-2}\\ \tau\subseteq\sigma}}\sum_{\substack{\sigma'\sim\sigma\\ \tau\subseteq\sigma'}}f\left(\sigma'\right) \\
&= \deg\left(\sigma\right)f\left(\sigma\right)-\sum_{\sigma'\sim\sigma}f\left(\sigma'\right)=\Delta^+f\left(\sigma\right).
\end{aligned}
$$

(2) Let $f \in \Omega^{d-1}$ and $\tau \in X^{d-2}$. We first notice that $(\Delta_\tau^+ f)_\tau = \Delta_{\mathrm{lk}\,\tau}^+ f_\tau$, since

$$\left(\Delta_\tau^+ f\right)_\tau (v) = \left(\Delta_\tau^+ f\right)(v\tau) = \deg_\tau (v\tau) f(v\tau) - \sum_{\substack{\sigma' \sim v\tau \\ \tau \subseteq \sigma'}} f(\sigma')$$
$$= \deg_\tau (v\tau) f(v\tau) - \sum_{\substack{v' \sim \tau \\ v'\tau \sim v\tau}} f(v'\tau) = \deg_{\mathrm{lk}\,\tau}(v) f_\tau (v) - \sum_{v' \underset{\mathrm{lk}\,\tau}{\sim} v} f_\tau (v') = \Delta_{\mathrm{lk}\,\tau}^+ f_\tau (v).$$

Since $(\mathrm{lk}\,\tau)^0 = \{v \in V \mid v \sim \tau\}$, this gives

$$\left\langle \Delta_{\mathrm{lk}\,\tau}^+ f_\tau, f_\tau \right\rangle = \left\langle \left(\Delta_\tau^+ f\right)_\tau, f_\tau \right\rangle = \sum_{v \sim \tau} \left(\Delta_\tau^+ f\right)_\tau (v) f_\tau (v) = \sum_{v \sim \tau} \left(\Delta_\tau^+ f\right)(v\tau) f(v\tau) = \left\langle \Delta_\tau^+ f, f \right\rangle$$

where the last equality is since $\Delta_\tau^+ f$ is supported on $(d-1)$-cells containing τ.

(3) If $f \in Z_{d-1}$ and $\tau \in X^{d-2}$ then

$$\left(\partial_0^{\mathrm{lk}\,\tau} f_\tau\right)(\varnothing) = \sum_{v \in (\mathrm{lk}\,\tau)^0} f_\tau (v) = \sum_{v \in (\mathrm{lk}\,\tau)^0} f(v\tau) = \sum_{v \sim \tau} f(v\tau) = (\partial_{d-1} f)(\tau) = 0$$

implies that $f_\tau \in Z_0 (\mathrm{lk}\,\tau)$.

(4) This is by

$$\sum_{\tau \in X^{d-2}} \langle f_\tau, f_\tau \rangle = \sum_{\tau \in X^{d-2}} \sum_{v \sim \tau} f_\tau^2 (v) = \sum_{\tau \in X^{d-2}} \sum_{v \sim \tau} f^2 (v\tau) = d \sum_{\sigma \in X^{d-1}} f^2 (\sigma) = d \langle f, f \rangle.$$

□

Assume now that $f \in Z_{d-1}$ is a normalized eigenfunction for $\lambda(X)$, i.e. $\langle f, f \rangle = 1$ and $\Delta^+ f = \lambda(X) f$. Using the lemma we find that

$$\lambda(X) = \left\langle \Delta^+ f, f \right\rangle \overset{(1)}{=} \sum_{\tau \in X^{d-2}} \left\langle \Delta_\tau^+ f, f \right\rangle - (d-1)\langle Df, f \rangle \overset{(2)}{=} \sum_{\tau \in X^{d-2}} \left\langle \Delta_{\mathrm{lk}\,\tau}^+ f_\tau, f_\tau \right\rangle - (d-1)\langle Df, f \rangle$$
$$\geq \sum_{\tau \in X^{d-2}} \left\langle \Delta_{\mathrm{lk}\,\tau}^+ f_\tau, f_\tau \right\rangle - (d-1)k \overset{(3)}{\geq} \sum_{\tau \in X^{d-2}} \lambda(\mathrm{lk}\,\tau) \langle f_\tau, f_\tau \rangle - (d-1)k \overset{(4)}{=} d \min_{\tau \in X^{d-2}} \lambda(\mathrm{lk}\,\tau) - (d-1)k.$$

By (3.4) we obtain the bound

$$\frac{d\left(1 - \frac{d-1}{n}\right)^2}{8k} h^2 (X) - (d-1)k \leq \lambda(X).$$

Sadly, this bound is trivial, as it is not hard to show that the l.h.s. is non-positive for every complex X. The line of research which seems most promising is to find a stronger relation between the spectral gap of the complex and that of its links, for the case of complexes with a complete skeleton (Lemma 3.2 applies to general ones).

4 Mixing and pseudo-randomness

4.1 The complete skeleton case

Here we prove Theorem 1.4. We begin by formulating it precisely.

Theorem (1.4)**.** *Let X be a d-dimensional complex with a complete skeleton. Fix $\alpha \in \mathbb{R}$, and write* $\mathrm{Spec}\left(\alpha I - \Delta^{+}\right) = \{\mu_0 \geq \mu_1 \geq \ldots \geq \mu_m\}$ *(where $m = \binom{n}{d} - 1$). For any disjoint sets of vertices $A_0, \ldots, A_d$ (not necessarily a partition), one has*

$$\left| \left|F\left(A_0, \ldots, A_d\right)\right| - \frac{\alpha \cdot |A_0| \cdot \ldots \cdot |A_d|}{n} \right| \leq \rho_\alpha \cdot \left(|A_0| \cdot \ldots \cdot |A_d|\right)^{\frac{d}{d+1}}$$

where

$$\rho_\alpha = \max\left\{\left|\mu_{\binom{n-1}{d-1}}\right|, |\mu_m|\right\} = \left\|\left(\alpha I - \Delta^{+}\right)\big|_{Z_{d-1}}\right\|.$$

Remark 4.1. Which α should one take in practice? In the introduction we state the theorem for $\alpha = k$, the average degree of a $(d-1)$-cell, so that it generalize the familiar form of the Expander Mixing Lemma for k-regular graphs. However, the expectation of $|F\left(A_0, \ldots, A_d\right)|$ in the pseudo-random sense is actually $\mathfrak{D}\,|A_0| \cdot \ldots \cdot |A_d|$, where $\mathfrak{D}$ is the d-cell density $\frac{|X^d|}{\binom{n}{d}}$. By Proposition 2.5,$\alpha = n\mathfrak{D} = \frac{nk}{n-d}$ is therefore a more accurate choice. This becomes even clearer upon observing that we seek to minimize $\rho_\alpha = \left\|\left(\alpha I - \Delta^{+}\right)\big|_{Z_{d-1}}\right\|$, since Proposition 2.5 shows that the spectrum of $\Delta^{+}\big|_{Z_{d-1}}$ is centered around $\lambda_{avg} = n\mathfrak{D} = \frac{nk}{n-d}$. While for a fixed d the choice between k and $\frac{nk}{n-d}$ is negligible, this should be taken into account when d depends on n.

Proof. For any disjoint sets of vertices $A_0, \ldots, A_{d-1}$, define $\delta_{A_0,\ldots,A_{d-1}} \in \Omega^{d-1}$ by

$$\delta_{A_0,\ldots,A_{d-1}}(\sigma) = \begin{cases} \mathrm{sgn}(\pi) & \exists \pi \in \mathrm{Sym}_{\{0\ldots d-1\}} \text{ with } \sigma_i \in A_{\pi(i)} \text{ for } 0 \leq i \leq d-1 \\ 0 & \text{otherwise} \end{cases}.$$

Since the skeleton of X is complete,

$$\left\|\delta_{A_0,\ldots,A_{d-1}}\right\| = \sqrt{\sum_{\sigma \in X^{d-1}} \delta^2_{A_0,\ldots,A_{d-1}}(\sigma)} = \sqrt{|A_0| \cdot \ldots \cdot |A_{d-1}|}. \tag{4.1}$$

Now, let $A_0, \ldots, A_d$ be disjoint subsets of V (not necessarily a partition), and denote

$$\varphi = \delta_{A_0,A_1,A_2,\ldots,A_{d-1}}$$
$$\psi = \delta_{A_d,A_1,A_2,\ldots,A_{d-1}}.$$

Let σ be an oriented $(d-1)$-cell with one vertex in each of $A_0, A_1, \ldots, A_{d-1}$. We shall denote this by $\sigma \in F\left(A_0, \ldots, A_{d-1}\right)$, ignoring the orientation of σ. There is a correspondence between d-cells in $F\left(A_0, \ldots, A_d\right)$ containing σ, and neighbors of σ which lie in $F\left(A_d, A_1, \ldots, A_{d-1}\right)$. Furthermore, for such a neighbor σ' we have $\varphi(\sigma) = \psi(\sigma')$, since σ and σ' must share the vertices which belong to

$A_1, \ldots, A_{d-1}$. Therefore (cf. (2.5)),

$$\begin{aligned}\langle\varphi, \mathcal{A}^{\sim}\psi\rangle &= \sum_{\sigma\in X^{d-1}} \varphi(\sigma)(\mathcal{A}^{\sim}\psi)(\sigma) = \sum_{\sigma\in X^{d-1}}\sum_{\sigma'\sim\sigma}\varphi(\sigma)\psi(\sigma') \\ &= \sum_{\sigma\in F(A_0\ldots A_{d-1})}\sum_{\sigma'\sim\sigma}\varphi(\sigma)\psi(\sigma') = \sum_{\sigma\in F(A_0\ldots A_{d-1})} \#\left\{\sigma'\in F(A_d, A_1, \ldots, A_{d-1}) \,|\, \sigma'\sim\sigma\right\} \\ &= \sum_{\sigma\in F(A_0\ldots A_{d-1})} \#\left\{\tau\in F(A_0, A_1, \ldots, A_d) \,|\, \sigma\subseteq\tau\right\} = |F(A_0, A_1, \ldots, A_d)|. \end{aligned} \tag{4.2}$$

Notice that since the A_i are disjoint, φ and ψ are supported on different $(d-1)$-cells, so that for any $\alpha\in\mathbb{R}$

$$\langle\varphi, \mathcal{A}^{\sim}\psi\rangle = \left\langle\varphi, \left(D-\Delta^+\right)\psi\right\rangle = \left\langle\varphi, -\Delta^+\psi\right\rangle = \left\langle\varphi, \left(\alpha I-\Delta^+\right)\psi\right\rangle. \tag{4.3}$$

As Δ^+ decomposes w.r.t. the orthogonal decomposition $\Omega^{d-1} = B^{d-1}\oplus Z_{d-1}$, and since $B^{d-1}\subseteq Z^{d-1} = \ker\Delta^+$,

$$\begin{aligned}|F(A_0, A_1, \ldots, A_d)| &= \left\langle\varphi, \left(\alpha I-\Delta^+\right)\psi\right\rangle \\ &= \left\langle\varphi, \left(\alpha I-\Delta^+\right)\left(\mathbb{P}_{B^{d-1}}\psi + \mathbb{P}_{Z_{d-1}}\psi\right)\right\rangle \\ &= \left\langle\varphi, \alpha\mathbb{P}_{B^{d-1}}\psi + \left(\alpha I-\Delta^+\right)\mathbb{P}_{Z_{d-1}}\psi\right\rangle \\ &= \alpha\left\langle\varphi, \mathbb{P}_{B^{d-1}}\psi\right\rangle + \left\langle\varphi, \left(\alpha I-\Delta^+\right)\mathbb{P}_{Z_{d-1}}\psi\right\rangle. \end{aligned} \tag{4.4}$$

We proceed to evaluate each of these terms separately. Using (2.7) and (2.6) we find that

$$\alpha\left\langle\varphi, \mathbb{P}_{B^{d-1}}\psi\right\rangle = \frac{\alpha}{n}\left\langle\varphi, \Delta^-\psi\right\rangle = \frac{\alpha}{n}\left\langle\varphi, \left(nI-\Delta_X^+-\Delta_{\overline{X}}^+\right)\psi\right\rangle$$

and by (4.2) and (4.3) this implies

$$\begin{aligned}\alpha\left\langle\varphi, \mathbb{P}_{B^{d-1}}\psi\right\rangle &= \frac{\alpha}{n}\left\langle\varphi, \left(nI-\Delta_X^+\right)\psi\right\rangle + \frac{\alpha}{n}\left\langle\varphi, -\Delta_{\overline{X}}^+\psi\right\rangle \\ &= \frac{\alpha}{n}\left|F_X(A_0, A_1, \ldots, A_d)\right| + \frac{\alpha}{n}\left|F_{\overline{X}}(A_0, A_1, \ldots, A_d)\right| \\ &= \frac{\alpha\cdot|A_0|\cdot\ldots\cdot|A_d|}{n}. \end{aligned} \tag{4.5}$$

We turn to the second term in (4.4). First, we recall from Proposition 2.4 that $\dim B^{d-1} = \binom{n-1}{d-1}$. Since $B^{d-1}\subseteq\ker\Delta^+$, we can assume that in $\operatorname{Spec}\left(\alpha I-\Delta^+\right) = \{\mu_0\geq\mu_1\geq\ldots\geq\mu_m\}$ the first $\binom{n-1}{d-1}$ values correspond to B^{d-1}, and the rest to $\left(B^{d-1}\right)^{\perp} = Z_{d-1}$. Thus,

$$\rho_\alpha = \max\left\{\left|\mu_{\binom{n-1}{d-1}}\right|, |\mu_m|\right\} = \max\left\{|\mu| \,\middle|\, \mu\in\operatorname{Spec}\left(\alpha I-\Delta^+\right)|_{Z_{d-1}}\right\} = \left\|\left(\alpha I-\Delta^+\right)|_{Z_{d-1}}\right\|, \tag{4.6}$$

and therefore

$$\begin{aligned}\left|\left\langle\varphi, \left(\alpha I-\Delta^+\right)\mathbb{P}_{Z_{d-1}}\psi\right\rangle\right| &\leq \|\varphi\|\cdot\left\|\left(\alpha I-\Delta^+\right)\mathbb{P}_{Z_{d-1}}\psi\right\| \leq \|\varphi\|\cdot\left\|\left(\alpha I-\Delta^+\right)|_{Z_{d-1}}\right\|\cdot\left\|\mathbb{P}_{Z_{d-1}}\psi\right\| \\ &\leq \rho_\alpha\cdot\|\varphi\|\cdot\|\psi\| = \rho_\alpha\sqrt{|A_0|\,|A_d|}\,|A_1|\,|A_2|\ldots|A_{d-1}|, \end{aligned} \tag{4.7}$$

where the last step is by (4.1). Together (4.4), (4.5) and (4.7) give

$$\left| |F(A_0, A_1, \ldots, A_d)| - \frac{\alpha \cdot |A_0| \cdot \ldots \cdot |A_d|}{n} \right| \le \rho_\alpha \sqrt{|A_0| |A_d|} |A_1| |A_2| \ldots |A_{d-1}| .$$

Since $A_0, \ldots, A_d$ play the same role, one can also obtain the bound

$$\rho_\alpha \sqrt{\left|A_{\pi(0)}\right| \left|A_{\pi(d)}\right|} \left|A_{\pi(1)}\right| \left|A_{\pi(2)}\right| \ldots \left|A_{\pi(d-1)}\right| ,$$

for any $\pi \in \mathrm{Sym}_{\{0..d\}}$. Taking the geometric mean over all such π gives

$$\left| |F(A_0, A_1, \ldots, A_d)| - \frac{\alpha \cdot |A_0| \cdot \ldots \cdot |A_d|}{n} \right| \le \rho_\alpha \cdot (|A_0| |A_1| \ldots |A_d|)^{\frac{d}{d+1}} .$$

$\square$

Remark. The estimate (4.7) is somewhat wasteful. As is done in graphs, a slightly better one is

$$\left|\left\langle \varphi, \left(\alpha I - \Delta^+\right) \mathbb{P}_{Z_{d-1}} \psi \right\rangle\right| = \left|\left\langle \mathbb{P}_{Z_{d-1}} \varphi, \left(\alpha I - \Delta^+\right) \mathbb{P}_{Z_{d-1}} \psi \right\rangle\right| \le \rho_\alpha \cdot \left\|\mathbb{P}_{Z_{d-1}} \varphi\right\| \cdot \left\|\mathbb{P}_{Z_{d-1}} \psi\right\| ,$$

and we leave it to the curious reader to verify that this gives

$$\left|\left\langle \varphi, \left(\alpha I - \Delta^+\right) \mathbb{P}_{Z_{d-1}} \psi \right\rangle\right| \le \rho_\alpha \sqrt{|A_0| \left(1 - \frac{\sum_{i=0}^{d-1} |A_i|}{n}\right) |A_d| \left(1 - \frac{\sum_{i=1}^{d} |A_i|}{n}\right)} |A_1| \ldots |A_{d-1}| .$$

4.2 The general case

We move on the the case of complexes with non-complete skeleton. Recall that X is a (j, k, ε)-*expander* if $\varepsilon < 1$ and $\mathrm{Spec}\, \Delta_j^+\big|_{Z_j} \subseteq [k(1-\varepsilon), k(1+\varepsilon)]$, and that given $\overline{k} = (k_0, \ldots, k_{d-1})$ and $\overline{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{d-1})$, we say that X is a $\left(\overline{k}, \overline{\varepsilon}\right)$-*expander* if it is a (j, k_j, ε_j)-expander for all j. The restriction $\varepsilon_j < 1$ ensures that X has trivial j-th homology, i.e. $\beta_j = 0$. While some of our results hold for general ε (e.g. Lemma 1.7), or given any bound on it (e.g. Theorem 1.5), we shall need the stronger assumption for later applications.

In what follows we assume that X is a d-complex on n vertices, which is a (j, k_j, ε_j)-expander for $0 \le j < d$, and prove the descent lemma and the mixing lemmas it implies.

Proof of Lemma 1.7. As before, to any disjoint sets of vertices $A_0, \ldots, A_j$, we associate the characteristic j-form

$$\delta_{A_0 \ldots A_j}(\sigma) = \begin{cases} \mathrm{sgn}(\pi) & \exists \pi \in \mathrm{Sym}_{\{0 \ldots j\}} \text{ with } \sigma_i \in A_{\pi(i)} \text{ for } 0 \le i \le j \\ 0 & \text{otherwise.} \end{cases}$$

Restriction of j-forms to $F(A_0, \ldots, A_j)$ forms an orthogonal projection operator on Ω^j, which we denote by $\mathbb{P}_{A_0 \ldots A_j}$:

$$\left(\mathbb{P}_{A_0 \ldots A_j} \varphi\right)(\sigma) = \begin{cases} \varphi(\sigma) & \sigma \in F(A_0, \ldots, A_j) \\ 0 & \text{otherwise.} \end{cases}$$

As we have seen in the complete skeleton case, for disjoint sets $A_0,\ldots,A_{j+1}$ the form $(-1)^j\,\mathbb{P}_{A_0\ldots A_j}\mathcal{A}_j^{\sim}\delta_{A_1\ldots A_{j+1}}$ vanishes outside $F(A_0,\ldots,A_j)$, and to each j-cell therein it assigns the number of its $\sim$-neighbors in $F(A_1,\ldots,A_{j+1})$. As these neighbors are in correspondence with $(j+1)$-cells in $F(A_0,\ldots,A_{j+1})$, one obtains $\left|\left\langle\delta_{A_0\ldots A_j},\mathbb{P}_{A_0\ldots A_j}\mathcal{A}_j^{\sim}\delta_{A_1\ldots A_{j+1}}\right\rangle\right| = |F(A_0,\ldots,A_{j+1})|$.

Next, let φ be a j-form which is supported on $F(A_1,\ldots,A_{j+1})$, and which assigns to each j-cell σ the number of $(j+1)$-galleries in $A_1,\ldots,A_\ell$ whose first cell contains σ. By the same considerations as above, $(-1)^j\,\mathbb{P}_{A_0\ldots A_j}\mathcal{A}_j^{\sim}\varphi$ assigns to every j-cell τ in $F(A_0,\ldots,A_j)$ the number of $(j+1)$-galleries in $A_0,\ldots,A_\ell$ whose first $(j+1)$ cell contains τ. Therefore, $\left|\left\langle\delta_{A_0\ldots A_j},\mathbb{P}_{A_0\ldots A_j}\mathcal{A}_j^{\sim}\varphi\right\rangle\right| = \left|F^{j+1}(A_0,\ldots,A_\ell)\right|$, and we conclude by induction that

$$\left|F^{j+1}(A_0,\ldots,A_\ell)\right| = \left|\left\langle\delta_{A_0\ldots A_j},\left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\mathcal{A}_j^{\sim}\right)\delta_{A_{\ell-j}\ldots A_\ell}\right\rangle\right|. \tag{4.8}$$

Since the A_i are disjoint, $\delta_{A_i\ldots A_{i+j}}$ and $\delta_{A_{i+1}\ldots A_{i+j+1}}$ are supported on different cells, so that $\mathbb{P}_{A_i\ldots A_{i+j}}T\delta_{A_{i+1}\ldots A_{i+j+1}} = 0$ for any diagonal operator T. Thus, all the $\mathcal{A}_j^{\sim}$ in (4.8) can be replaced by $\mathcal{A}_j^{\sim}+T$, and taking $T = k_jI - D_j$ we obtain

$$\left|F^{j+1}(A_0,\ldots,A_\ell)\right| = \left|\left\langle\delta_{A_0\ldots A_j},\left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\left(k_jI-\Delta_j^+\right)\right)\delta_{A_{\ell-j}\ldots A_\ell}\right\rangle\right|. \tag{4.9}$$

Our next step is to approximate this quantity using the lower j-th Laplacian. Denoting $E = k_jI - \Delta_j^+ - \frac{k_j}{k_{j-1}}\Delta_j^-$, the orthogonal decomposition $\Omega^j = Z_j \oplus B^j$ gives

$$E = k_j\left(\mathbb{P}_{Z_j}+\mathbb{P}_{B^j}\right) - \Delta_j^+ - \frac{k_j}{k_{j-1}}\Delta_j^- = k_j\mathbb{P}_{Z_j} - \Delta_j^+ + \frac{k_j}{k_{j-1}}\left(k_{j-1}\mathbb{P}_{B^j}-\Delta_j^-\right).$$

We first observe that $\left\|k_j\mathbb{P}_{Z_j}-\Delta_j^+\right\| \le k_j\varepsilon_j$ follows from $\operatorname{Spec}\Delta_j^+\big|_{Z_j} \subseteq \left[k_j(1-\varepsilon_j),k_j(1+\varepsilon_j)\right]$ and $\Delta_j^+\big|_{B^j}\equiv 0$. For the lower Laplacian, we have

$$\begin{aligned}\operatorname{Spec}\Delta_j^-\big|_{B^j} &= \operatorname{Spec}\Delta_j^-\big|_{Z_j^\perp} = \operatorname{Spec}\Delta_j^-\setminus\{0\} \overset{(*)}{=} \operatorname{Spec}\Delta_{j-1}^+\setminus\{0\} = \operatorname{Spec}\Delta_{j-1}^+\big|_{(Z^{j-1})^\perp}\\ &= \operatorname{Spec}\Delta_{j-1}^+\big|_{B_{j-1}} \subseteq \operatorname{Spec}\Delta_{j-1}^+\big|_{Z_{j-1}} \subseteq \left[k_{j-1}(1-\varepsilon_{j-1}),k_{j-1}(1+\varepsilon_{j-1})\right],\end{aligned}$$

where $(*)$ follows from the fact that $\Delta_j^- = \partial_j^*\partial_j$ and $\Delta_{j-1}^+ = \partial_j\partial_j^*$. As Δ_j^- vanishes on Z_j, we have in total $\left\|k_{j-1}\mathbb{P}_{B^j}-\Delta_j^-\right\| \le k_{j-1}\varepsilon_{j-1}$, so that

$$\|E\| \le \left\|k_j\mathbb{P}_{Z_j}-\Delta_j^+\right\| + \frac{k_j}{k_{j-1}}\left\|k_{j-1}\mathbb{P}_{B^j}-\Delta_j^-\right\| \le k_j\left(\varepsilon_{j-1}+\varepsilon_j\right). \tag{4.10}$$

We proceed to expand (4.9), using $\frac{k_j}{k_{j-1}}\Delta_j^- + E = k_jI - \Delta_j^+$, and on occasions translating Δ_j^- by some diagonal (in fact, scalar) operators:

$$
\begin{aligned}
\left|F^{j+1}\left(A_0,\ldots,A_\ell\right)\right| &= \left|\left\langle \delta_{A_0\ldots A_j}, \left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\left(\frac{k_j}{k_{j-1}}\Delta_j^- + E\right)\right)\delta_{A_{\ell-j}\ldots A_\ell}\right\rangle\right| \\
&= \left|\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\left\langle \delta_{A_0\ldots A_j}, \left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\Delta_j^-\right)\delta_{A_{\ell-j}\ldots A_\ell}\right\rangle\right. \\
&\quad \left. + \sum_{m=1}^{\ell-j}\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left\langle \delta_{A_0\ldots A_j}, \begin{array}{l}\left(\prod_{i=0}^{\ell-j-m-1}\mathbb{P}_{A_i\ldots A_{i+j}}\Delta_j^-\right)\mathbb{P}_{A_{\ell-j-m}\ldots A_{\ell-m}}E\cdot \\ \cdot\left(\prod_{i=\ell-j-m+1}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\left(\frac{k_j}{k_{j-1}}\Delta_j^- + E\right)\right)\delta_{A_{\ell-j}\ldots A_\ell}\end{array}\right\rangle\right| \\
&= \left|\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\left\langle \delta_{A_0\ldots A_j}, \left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\mathcal{A}_j^{\pitchfork}\right)\delta_{A_{\ell-j}\ldots A_\ell}\right\rangle\right. \qquad (4.11) \\
&\quad \left. + \sum_{m=1}^{\ell-j}\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left\langle \delta_{A_0\ldots A_j}, \begin{array}{l}\left(\prod_{i=0}^{\ell-j-m-1}\mathbb{P}_{A_i\ldots A_{i+j}}\left(\Delta_j^- - k_{j-1}I\right)\right)\mathbb{P}_{A_{\ell-j-m}\ldots A_{\ell-m}}E\cdot \\ \cdot\left(\prod_{i=\ell-j-m+1}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\left(k_jI - \Delta_j^+\right)\right)\delta_{A_{\ell-j}\ldots A_\ell}\end{array}\right\rangle\right|.
\end{aligned}
$$

We first study the summand in line (4.11). Note that the form $(-1)^j\,\mathbb{P}_{A_0\ldots A_j}\mathcal{A}_j^{\pitchfork}\delta_{A_1\ldots A_{j+1}}$ assigns to every j-cell in $F\left(A_0,\ldots,A_j\right)$ the number of j-cells in $F\left(A_1,\ldots,A_{j+1}\right)$ with which it intersects, so that $\left|\left\langle\delta_{A_0\ldots A_j},\mathbb{P}_{A_0\ldots A_j}\mathcal{A}_j^{\pitchfork}\delta_{A_1\ldots A_{j+1}}\right\rangle\right| = \left|F^j\left(A_0,\ldots,A_{j+1}\right)\right|$ (recall that for $A_j^{\sim}$ in place of $A_j^{\pitchfork}$ we obtained $\left|F^{j+1}\left(A_0,\ldots,A_{j+1}\right)\right|$). By the same arguments as before one sees that

$$\left|F^j\left(A_0,\ldots,A_\ell\right)\right| = \left|\left\langle \delta_{A_0\ldots A_j}, \left(\prod_{i=0}^{\ell-j-1}\mathbb{P}_{A_i\ldots A_{i+j}}\mathcal{A}_j^{\pitchfork}\right)\delta_{A_{\ell-j}\ldots A_\ell}\right\rangle\right|,$$

so that line (4.11) is precisely $\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}\left|F^j\left(A_0,\ldots,A_\ell\right)\right|$, our estimate for $\left|F^{j+1}\left(A_0,\ldots,A_\ell\right)\right|$. Denoting by $\mathcal{E}$ the error term (the line below (4.11)), we bound it using (4.10) together with $\left\|\Delta_j^- - k_{j-1}I\right\| \le k_{j-1}$ and $\left\|k_jI - \Delta_j^+\right\| \le k_j$ (both follow from the discussion preceding (4.10)):

$$
\begin{aligned}
\mathcal{E} &\le \sum_{m=1}^{\ell-j}\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j-m}\left\|\delta_{A_0\ldots A_j}\right\| k_{j-1}^{\ell-j-m}k_j\left(\varepsilon_{j-1}+\varepsilon_j\right)k_j^{m-1}\left\|\delta_{A_{\ell-j}\ldots A_\ell}\right\| \\
&= (\ell-j)\,k_j^{\ell-j}\left(\varepsilon_{j-1}+\varepsilon_j\right)\sqrt{\left|F\left(A_0,\ldots,A_j\right)\right|\left|F\left(A_{\ell-j},\ldots,A_\ell\right)\right|},
\end{aligned}
$$

which concludes the proof.

We remark that a slightly better bound is possible here: As $\operatorname{Spec}\Delta_j^+ \subseteq \left[0,k_j\left(1+\varepsilon_j\right)\right]$, we can replace $k_jI - \Delta_j^+$ in the line below (4.11) by $\frac{k_j(1+\varepsilon_j)}{2}I - \Delta_j^+$, which is bounded by $\frac{k_j(1+\varepsilon_j)}{2}$, and likewise for Δ_j^- (whose spectrum lies within $\left[0,k_{j-1}\left(1+\varepsilon_{j-1}\right)\right]$). For example, putting $\varepsilon = \max\varepsilon_i$ this gives

$$\mathcal{E} \le (\ell-j)\,k_j^{\ell-j}2\varepsilon\left(\frac{1+\varepsilon}{2}\right)^{\ell-j-1}\sqrt{\left|F\left(A_0,\ldots,A_j\right)\right|\left|F\left(A_{\ell-j},\ldots,A_\ell\right)\right|}$$

which might be useful when all ε_i are small. $\square$

Using the Descent Lemma repeatedly gives:

Proposition 4.2. *For any $j < \ell$, there exists $c_{j,\ell}$ such that any disjoint sets of vertices $A_0, \ldots, A_\ell$ in a $\left(\overline{k}, \overline{\varepsilon}\right)$-expander satisfy*

$$\left|\left|F^{j+1}\left(A_0, \ldots, A_\ell\right)\right| - \frac{k_0 k_1 \ldots k_{j-1} k_j^{\ell-j}}{n^\ell} \prod_{i=0}^{\ell} |A_i|\right| \leq c_{j,\ell} k_0 k_1 \ldots k_{j-1} k_j^{\ell-j} \left(\varepsilon_0 + \ldots + \varepsilon_j\right) \max |A_i| .$$

In particular, for $j = d-1$, $\ell = d$ we obtain Theorem 1.5:

Theorem (1.5)**.** *Any disjoint sets of vertices $A_0, \ldots, A_d$ in a $\left(\overline{k}, \varepsilon\right)$-expander of dimension d satisfy*

$$\left|\left|F\left(A_0, \ldots, A_d\right)\right| - \frac{k_0 \ldots k_{d-1}}{n^d} |A_0| \cdot \ldots \cdot |A_d|\right| \leq c_d k_0 \ldots k_{d-1} \left(\varepsilon_0 + \ldots + \varepsilon_{d-1}\right) \max |A_i| ,$$

for some constant c_d which depends only on d.

Proof of Proposition 4.2. We denote $m = \max |A_i|$ and assume by induction that the proposition holds for $j-1$ (and any ℓ), i.e. that

$$\left|F^j\left(A_0, \ldots, A_\ell\right) - \frac{k_0 \ldots k_{j-2} k_{j-1}^{\ell-j+1}}{n^\ell} \prod_{i=0}^{\ell} |A_i|\right| \leq c_{j-1,\ell} m k_0 k_1 \ldots k_{j-2} k_{j-1}^{\ell-j+1} \left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right). \quad (4.12)$$

For $j = 0$ this indeed holds, in the sense that

$$\left|F^0\left(A_0, \ldots, A_\ell\right) - \frac{k_{-1}^\ell}{n^\ell} \prod_{i=0}^{\ell} |A_i|\right| = 0. \quad (4.13)$$

Let us denote by $\mathcal{E}$ the discrepancy $\left|\left|F^{j+1}\left(A_0, \ldots, A_\ell\right)\right| - \frac{k_0 k_1 \ldots k_{j-1} k_j^{\ell-j}}{n^\ell} \prod_{i=0}^{\ell} |A_i|\right|$. Combining the Descent Lemma with (4.12) (or (4.13), for $j = 0$) multiplied by $\left(\frac{k_j}{k_{j-1}}\right)^{\ell-j}$ gives

$$\begin{aligned}\mathcal{E} &\leq (\ell - j) k_j^{\ell-j} \left(\varepsilon_j + \varepsilon_{j-1}\right) \sqrt{\left|F\left(A_0, \ldots, A_j\right)\right| \left|F\left(A_{\ell-j}, \ldots, A_\ell\right)\right|} \\ &\quad + c_{j-1,\ell} m k_0 k_1 \ldots k_{j-1} k_j^{\ell-j} \left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right).\end{aligned}$$

To bound $\left|F\left(A_0, \ldots, A_j\right)\right|$ we use (4.12) with $\ell = j$, which gives

$$\begin{aligned}\left|F^j\left(A_0, \ldots, A_j\right)\right| &\leq \frac{k_0 \ldots k_{j-1}}{n^j} \prod_{i=0}^{j} |A_i| + c_{j-1,j} m k_0 \ldots k_{j-1} \left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right) \\ &\leq \left[1 + c_{j-1,j} \left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right)\right] m k_0 \ldots k_{j-1} \\ &\leq \left(1 + j c_{j-1,j}\right) m k_0 \ldots k_{j-1}.\end{aligned}$$

(here we have used $\varepsilon_i < 1$, but any bound on the ε_i would do). The same holds for $\left|F\left(A_{\ell-j}, \ldots, A_\ell\right)\right|$, hence

$$
\begin{aligned}
\mathcal{E} &\leq (\ell - j)\, k_j^{\ell - j} \left(\varepsilon_j + \varepsilon_{j-1}\right) \left(1 + j c_{j-1,j}\right) m k_0 \ldots k_{j-1} \\
&\quad + c_{j-1,\ell} m k_0 k_1 \ldots k_{j-1} k_j^{\ell - j} \left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right) \\
&= m k_0 k_1 \ldots k_{j-1} k_j^{\ell - j} \left[c_{j-1,\ell} \left(\varepsilon_0 + \ldots + \varepsilon_{j-1}\right) + (\ell - j) \left(1 + j c_{j-1,j}\right) \left(\varepsilon_j + \varepsilon_{j-1}\right)\right] \\
&\leq \underbrace{\left[c_{j-1,\ell} + (\ell - j) \left(1 + j c_{j-1,j}\right)\right]}_{c_{j,\ell}} m k_0 k_1 \ldots k_{j-1} k_j^{\ell - j} \left(\varepsilon_0 + \ldots + \varepsilon_j\right).
\end{aligned}
$$

as desired. □

5 Examples and Applications

5.1 Gromov's geometric overlap

Recall from Definition 1.8 that X has overlap $\geq \varepsilon$ if for every simplicial mapping of X into $\mathbb{R}^d$, some point in $\mathbb{R}^d$ is covered by at least an ε-fraction of the d-cells of X. A theorem of Pach relates geometric overlap to combinatorial expansion:

Theorem 5.1 ([Pac98])**.** *For every $d \geq 1$, there exists $\mathcal{P}_d > 0$ such that for every $d+1$ disjoint subsets $P_0, \ldots, P_d$ of n points in $\mathbb{R}^d$, there exist $z \in \mathbb{R}^d$ and subsets $Q_i \subseteq P_i$ with $|Q_i| \geq \mathcal{P}_d \cdot n$, such that every d-simplex with one vertex in each Q_i contains z.*

Combining Pach's theorem with Theorem 1.4 gives a bound on the geometric overlap of a complex in terms of the width of its spectrum:

Corollary 5.2. *Let X be a d-complex with a complete skeleton, and denote the average degree of a $(d-1)$-cell in X by k. If the nontrivial spectrum of the Laplacian of X is contained in $[k-\varepsilon, k+\varepsilon]$, then*

$$\operatorname{overlap}(X) \geq \frac{\mathcal{P}_d^d}{e^{d+1}} \left(\mathcal{P}_d - \frac{\varepsilon (d+1)}{k} \right),$$

where $\mathcal{P}_d$ is Pach's constant from Theorem 5.1.

Proof. Given $\varphi : V \to \mathbb{R}^d$, choose arbitrarily some partition of V into equally sized parts $P_0, \ldots, P_d$. By Pach's theorem, there exist $\mathcal{P}_d > 0$ and $Q_i \subseteq P_i$ of sizes $|Q_i| = \mathcal{P}_d |P_i|$ such that for some $x \in \mathbb{R}^{d+1}$ we have $x \in \operatorname{conv} \{\varphi(v) \,|\, v \in \sigma\}$ for any $\sigma \in F(Q_0, \ldots, Q_d)$. By the Mixing Lemma (Theorem 1.4),

$$|F(Q_0, \ldots, Q_d)| \geq \frac{k \cdot |Q_0| \cdot \ldots \cdot |Q_d|}{n} - \varepsilon \cdot (|Q_0| \cdot \ldots \cdot |Q_d|)^{\frac{d}{d+1}} = \left(\frac{\mathcal{P}_d n}{d+1} \right)^d \left(\frac{\mathcal{P}_d k}{d+1} - \varepsilon \right).$$

On the other hand,

$$\left| X^d \right| = \left| X^{d-1} \right| \frac{k}{d+1} = \binom{n}{d} \frac{k}{d+1} \leq \left(\frac{en}{d} \right)^d \frac{k}{d+1}.$$

As this holds for every φ,

$$\operatorname{overlap}(X) \geq \left(\frac{\mathcal{P}_d d}{e(d+1)} \right)^d \left(\mathcal{P}_d - \frac{\varepsilon (d+1)}{k} \right) \geq \frac{\mathcal{P}_d^d}{e^{d+1}} \left(\mathcal{P}_d - \frac{\varepsilon (d+1)}{k} \right).$$

$\square$

Remark 5.3. Following Remark 4.1, if $\operatorname{Spec} \Delta^+\big|_{Z_{d-1}} \subseteq [\lambda_{avg} - \varepsilon', \lambda_{avg} + \varepsilon']$ then using the Mixing Lemma with $\alpha = \lambda_{avg} = \frac{nk}{n-d}$ one has

$$|F(Q_0, \ldots, Q_d)| \geq \frac{k \cdot |Q_0| \cdot \ldots \cdot |Q_d|}{n-d} - \varepsilon' \cdot (|Q_0| \cdot \ldots \cdot |Q_d|)^{\frac{d}{d+1}} \geq \left(\frac{\mathcal{P}_d n}{d+1} \right)^d \left(\frac{nk\mathcal{P}_d}{(n-d)(d+1)} - \varepsilon' \right)$$

so that

$$\operatorname{overlap}(X) \geq \frac{\mathcal{P}_d^d n}{e^{d+1}(n-d)} \left(\mathcal{P}_d - \frac{\varepsilon' (d+1)}{\lambda_{avg}} \right).$$

In §5.4 we study the spectrum of random Linial-Meshulam complexes, and use this theorem to deduce that for suitable parameters they have the geometric overlap property.

For complexes with a non-complete skeleton we can use Theorem 1.5 to show the following:

Proposition 5.4. *If X is a d-dimensional $(\overline{k},\overline{\varepsilon})$-expander then*

$$\operatorname{overlap} X > \frac{\mathcal{P}_d d!}{2^d}\left[\left(\frac{\mathcal{P}_d}{d+1}\right)^d - c_d\left(\varepsilon_0+\ldots+\varepsilon_{d-1}\right)\right],$$

where $\mathcal{P}_d$ is Pach's constant from Theorem 5.1, and c_d is the constant from Theorem 1.5 (both depend only on d).

In particular, a family of d-complexes which have $\varepsilon_0+\ldots+\varepsilon_{d-1}$ small enough is a family of geometric expanders. For the proof of Proposition 5.4 we shall need the following lemma, which relates the Laplace spectrum to cell density:

Lemma 5.5. *Let X be a d-complex with $\beta_j = 0$ for $j < d$, and let $\widetilde{\lambda}_j$ be the average nontrivial eigenvalue of Δ_j^+, for $-1 \le j < d$ (in particular $\lambda_{-1} = n$). For any $0 \le m < d$ the average degree of an m-cell is*

$$\operatorname{avg}\left\{\deg\sigma\,|\,\sigma\in X^m\right\} = \lambda_m\left(1-\frac{m+1}{\lambda_{m-1}}\right), \tag{5.1}$$

and the number of m-cells is

$$|X^m| = \frac{\widetilde{\lambda}_{m-1}}{m+1}\cdot\prod_{j=-1}^{m-2}\left(\frac{\widetilde{\lambda}_j}{j+2}-1\right) = \frac{\widetilde{\lambda}_{m-1}(n-1)}{m+1}\cdot\prod_{j=0}^{m-2}\left(\frac{\widetilde{\lambda}_j}{j+2}-1\right). \tag{5.2}$$

Proof. Since the trivial spectrum of Δ_j^+ consists of zeros,

$$|X^m| = \frac{1}{m+1}\sum_{\sigma\in X^{m-1}}\deg\sigma = \frac{1}{m+1}\operatorname{trace} D_{m-1} = \frac{1}{m+1}\operatorname{trace}\Delta_{m-1}^+ = \frac{\widetilde{\lambda}_{m-1}}{m+1}\dim Z_{m-1}.$$

Thus, (5.2) is equivalent to the assertion that

$$\dim Z_{m-1} = \prod_{j=-1}^{m-2}\left(\frac{\widetilde{\lambda}_j}{j+2}-1\right).$$

This is true for $m=0$, and by induction, together with the triviality of the $(m-2)$-th homology we find that

$$\begin{aligned}\dim Z_{m-1} &= \dim\Omega^{m-1} - \dim B_{m-2} = \left|X^{m-1}\right| - \dim Z_{m-2}\\ &= \frac{\widetilde{\lambda}_{m-2}}{m}\prod_{j=-1}^{m-3}\left(\frac{\widetilde{\lambda}_j}{j+2}-1\right) - \prod_{j=-1}^{m-3}\left(\frac{\widetilde{\lambda}_j}{j+2}-1\right) = \prod_{j=-1}^{m-2}\left(\frac{\widetilde{\lambda}_j}{j+2}-1\right)\end{aligned}$$

as desired. Formula (5.1) follows from (5.2), as $\operatorname{avg}\left\{\deg\sigma\,|\,\sigma\in X^m\right\} = (m+2)\left|X^{m+1}\right| / |X^m|$. □

We can now proceed:

Proof of Proposition 5.4. Let φ be a simplicial map $X \to \mathbb{R}^d$. As in the proof of Corollary 5.2, there exist disjoint $Q_i \subseteq V$ of size $|Q_i| = \frac{\mathcal{P}_d n}{d+1}$, and a point $x \in \mathbb{R}^{d+1}$, such that $x \in \operatorname{conv}\{\varphi(v) \mid v \in \sigma\}$ for all $\sigma \in F(Q_0, \ldots, Q_d)$. Denoting $\mathcal{K} = k_0 \cdot \ldots \cdot d_{d-1}$ and $\mathcal{E} = \varepsilon_0 + \ldots + \varepsilon_{d-1}$, we have by Theorem 1.5

$$|F(Q_0, \ldots, Q_d)| \geq \frac{\mathcal{K}}{n^d} \left(\frac{\mathcal{P}_d n}{d+1}\right)^{d+1} - \frac{c_d \mathcal{P}_d n \mathcal{K}\mathcal{E}}{d+1} = \frac{\mathcal{K}\mathcal{P}_d n}{d+1}\left[\left(\frac{\mathcal{P}_d}{d+1}\right)^d - c_d \mathcal{E}\right],$$

and by the lemma above

$$\left|X^d\right| = \frac{\widetilde{\lambda}_{d-1}}{d+1} \cdot \prod_{j=-1}^{d-2}\left(\frac{\widetilde{\lambda}_j}{j+2} - 1\right) \leq \prod_{j=-1}^{d-1} \frac{\widetilde{\lambda}_j}{j+2} \leq n \prod_{j=0}^{d-1} \frac{k_j(1+\varepsilon_j)}{j+2} < \frac{2^d n \mathcal{K}}{(d+1)!}.$$

This means x is covered by at least a $\frac{\mathcal{P}_d d!}{2^d}\left(\left(\frac{\mathcal{P}_d}{d+1}\right)^d - c_d\mathcal{E}\right)$-fraction of the d-cells, and the proposition follows. $\square$

5.2 Chromatic bounds

We turn our attention to colorings. We say that a d-complex X is c-colorable if there is a coloring of its vertices by c colors so that no d-cell is monochromatic. The *chromatic number* of X, denoted $\chi(X)$, is the smallest c for which X is c-colorable. We will use the mixing property to show that spectral expansion implies a chromatic bound, as is done for graphs in [LPS88]. These results are weaker than Hoffman's chromatic bound for graphs [Hof70], as they require a two-sided spectral bound, and the bound obtained is not optimal. A chromatic bound for complexes which does generalize Hoffman's result was recently obtained in [Gol13].

Proposition (1.9)**.** *If X is a d-dimensional $(\overline{k}, \overline{\varepsilon})$-expander, then*

$$\chi(X) \geq \frac{1}{(d+1)\sqrt[d]{c_d(\varepsilon_0 + \ldots + \varepsilon_{d-1})}},$$

where c_d is the constant from Theorem 1.5.

Proof. Coloring X by $\chi = \chi(X)$ colors, there is necessarily a monochromatic set of vertices of size at least $\frac{n}{\chi}$. Take $\frac{n}{\chi}$ of these vertices and partition them arbitrarily to $d+1$ sets $A_0, \ldots, A_d$ of equal size. As in a coloring there are no monochromatic d-cells we have $F(A_0, \ldots, A_d) = \varnothing$, so that Theorem 1.5 reads

$$\frac{k_0 \ldots k_{d-1}}{n^d} \prod_{i=0}^{d} |A_i| \leq c_d k_0 \ldots k_{d-1}(\varepsilon_0 + \ldots + \varepsilon_{d-1}) \max |A_i|,$$

and since $|A_i| = \frac{n}{\chi \cdot (d+1)}$, the conclusion follows. $\square$

5.3 Ideal expanders

Let us say that X is an *ideal* $\overline{k}$*-expander* if it is a $(j, k_j, 0)$-expander for $0 \leq j < d$. In this case, the Descent Lemma tell us that

$$F^{j+1}(A_0, \ldots, A_\ell) = \left(\frac{k_j}{k_{j-1}}\right)^{\ell-j} \left|F^j(A_0, \ldots, A_\ell)\right|,$$

and the number of j-galleries between disjoint sets of vertices is completely determined by their sizes:

$$\left|F^j(A_0, \ldots, A_\ell)\right| = \frac{k_0 k_1 \ldots k_{j-2} k_{j-1}^{\ell-j+1}}{n^\ell} \prod_{i=0}^{\ell} |A_i| \tag{5.3}$$

(in particular, $|F(A_0, \ldots, A_d)| = \frac{k_0 \ldots k_{d-1}}{n^d} |A_0| \ldots |A_d|$). For $\overline{k} = \big(\overbrace{n, \ldots, n}^{m}, \overbrace{0, \ldots, 0}^{d-m}\big)$, an example of an ideal $\overline{k}$-expander is given by $K_n^{(m)}$, the m-th skeleton of the complete complex on n vertices. For this complex (5.3) holds trivially, and perhaps disappointingly, these are the only examples of ideal expanders: if X is an ideal $\overline{k}$-expander on n vertices, and $X^{(j)} = K_n^{(j)}$ (which holds for $j = 0$), one has $k_0 = \ldots = k_{j-1} = n$, and also $k_j \leq n$ by Proposition 2.3. For vertices $v_0, \ldots, v_{j+1}$, $|F(\{v_0\}, \ldots, \{v_{j+1}\})| = \frac{k_0 \ldots k_j}{n^{j+1}} \in \{0, 1\}$ then forces either $k_j = n$, which implies that $X^{(j+1)} = K_n^{(j+1)}$ as well, or $k_j = 0$, which means that X has no $(j+1)$-cells at all.

While ideal $\overline{k}$-expanders do not actually exist, save for the trivial examples $\overline{k} = (n, \ldots, n, 0, \ldots)$, they provide a conceptual way to think of expanders in general: $(\overline{k}, \overline{\varepsilon})$-expanders spectrally approximate the ideal (nonexistent) $\overline{k}$-expander, and the mixing lemma asserts that they also combinatorially approximate it. This point of view seems close in spirit to that of *spectral sparsification* [ST11], which proved to be fruitful in both graphs and complexity theory.

5.4 Linial-Meshulam complexes

In this section we study expansion in Linial-Meshulam complexes: recall that $X(d, n, p)$ is a d-dimensional simplicial complex on n vertices, with a complete skeleton, and with every d-cell being included independently with probability p. The main idea is the following lemma, which is a variation on the analysis in [GW12] of the spectrum of $\mathcal{A}^{\sim} = D - \Delta^+$.

Lemma 5.6. *Let* $c > 0$*, and* $X = X\left(d, n, \frac{C \cdot \log n}{n}\right)$*. There exists* $\gamma = O\left(\sqrt{C}\right)$ *such that* X *is a* $\left(d-1, C \log n, \frac{\gamma}{C}\right)$*-expander, i.e.*

$$\operatorname{Spec}\left(\Delta^+\big|_{Z_{d-1}}\right) \subseteq \left[(C - \gamma)\log n, (C + \gamma)\log n\right],$$

with probability at least $1 - n^{-c}$*.*

Proof. We denote $p = \frac{C \cdot \log n}{n}$. For C large enough we shall find $\gamma = O\left(\sqrt{C}\right)$ such that

$$\left\|\left(\Delta^+ - pn \cdot I\right)\big|_{Z_{d-1}}\right\| \leq \gamma \log n \tag{5.4}$$

holds with probability at least $1-n^{-c}$. This implies the Lemma, as

$$\operatorname{Spec}\left(\Delta^{+}\big|_{Z_{d-1}}\right) \subseteq [pn-\gamma\log n, pn+\gamma\log n] = [(C-\gamma)\log n, (C+\gamma)\log n].$$

To show (5.4) we use

$$\begin{aligned}\left\|\left(\Delta^{+}-pn\cdot I\right)\big|_{Z_{d-1}}\right\| &= \left\|\left(\Delta^{+}-p(n-d)I-pdI+D-D\right)\big|_{Z_{d-1}}\right\| \\ &\le \left\|\left(D-p(n-d)I\right)\big|_{Z_{d-1}}\right\| + \left\|\left(\mathcal{A}^{\sim}+pdI\right)\big|_{Z_{d-1}}\right\| \end{aligned} \tag{5.5}$$

and we will treat each term separately. For the first, we have

$$\left\|\left(D-(n-d)pI\right)\big|_{Z_{d-1}}\right\| \le \left\|D-(n-d)pI\right\| = \max_{\sigma\in X^{d-1}} \left|\deg\sigma-(n-d)p\right|.$$

Since $\deg\sigma \sim B(n-d,p)$, a Chernoff type bound (e.g. [Jan02, Theorem 1]) gives that for every $t>0$

$$\operatorname{Prob}\left(\left|\deg\sigma-(n-d)p\right|>t\right) \le 2e^{-\frac{t^2}{2(n-d)p+\frac{2t}{3}}}.$$

By a union bound on the degrees of the $(d-1)$-cells we get

$$\operatorname{Prob}\left(\max_{\sigma\in X^{d-1}}\left|\deg\sigma-(n-d)p\right|>t\right) \le 2\binom{n}{d}e^{-\frac{t^2}{2(n-d)p+\frac{2t}{3}}}, \tag{5.6}$$

and a straightforward calculation shows that there exists $\alpha=\alpha(c,d)>0$ such that for $t=\alpha\sqrt{np\log n}$, the r.h.s. in (5.6) is bounded by $\frac{1}{2n^c}$ for large enough C and n. In total this implies

$$\operatorname{Prob}\left(\left\|\left(D-(n-d)pI\right)\big|_{Z_{d-1}}\right\| \le \alpha\sqrt{C}\log n\right) \ge 1-\frac{1}{2n^c}. \tag{5.7}$$

In order to understand the last term in (5.5) we follow [GW12], which shows that $\mathcal{A}_X^{\sim}\big|_{Z_{d-1}}$ is close to p times $\mathcal{A}_{K_n^d}^{\sim}\big|_{Z_{d-1}} = \left(D_{K_n^d}-\Delta_{K_n^d}^{+}\right)\big|_{Z_{d-1}}$, where K_n^d is the complete d-complex on n vertices. Note that $D_{K_n^d}=(n-d)\cdot I$ and $\Delta_{K_n^d}^{+}\big|_{Z_{d-1}}=n\cdot I$, and that $Z_{d-1}(X)=Z_{d-1}\left(K_n^d\right)$ as both have the same $(d-1)$-skeleton. In the proof of Theorem 7 in [GW12] (which uses an idea from [Oli10]), it is shown that

$$\operatorname{Prob}\left(\left\|\left(\mathcal{A}_X^{\sim}+pdI\right)\big|_{Z_{d-1}}\right\|\ge t\right) = \operatorname{Prob}\left(\left\|\mathcal{A}_X^{\sim}\big|_{Z_{d-1}}-p\mathcal{A}_{K_n^d}^{\sim}\big|_{Z_{d-1}}\right\|\ge t\right) \le 2\binom{n}{d}e^{-\frac{t^2}{8pnd+4t}}.$$

Again, there exists $\beta=\beta(c,d)>0$ such that for $t=\beta\sqrt{np\log n}$, the r.h.s. is bounded by $\frac{1}{2n^c}$ for large enough C and n. Consequently,

$$\operatorname{Prob}\left(\left\|\left(\mathcal{A}_X^{\sim}+pdI\right)\big|_{Z_{d-1}}\right\| \le \beta\sqrt{C}\log n\right) \ge 1-\frac{1}{2n^c},$$

so that by (5.5)

$$\operatorname{Prob}\left(\left\|\left(\Delta^{+}-pnI\right)\big|_{Z_{d-1}}\right\| \le (\alpha+\beta)\sqrt{C}\log n\right) \ge 1-n^{-c},$$

and $\gamma = (\alpha + \beta)\sqrt{C}$ gives the required result. $\square$

We obtain the following:

Corollary 5.7. *Observe* $X = X\left(d, n, \frac{C\cdot\log n}{n}\right)$.

(1) There exist $H = C - O\left(\sqrt{C}\right)$ *and* $\Xi = \Omega\left(\sqrt[2d]{C}\right)$ *such that the Cheeger constant and the chromatic number satisfy*

$$h(X) \geq H \cdot \log n \qquad \textit{and} \qquad \chi(X) \geq \Xi \tag{5.8}$$

asymptotically almost surely w.r.t. n.

(2) For any $\vartheta < \left(\frac{\mathcal{P}_d}{e}\right)^{d+1}$ *(where* $\mathcal{P}_d$ *is Pach's constant from Theorem 5.1), for* C *and* n *large enough a.a.s.*

$$\operatorname{overlap}(X) > \vartheta.$$

(3) If $C < 1$ *then* $\operatorname{Prob}(h(X) = 0) \overset{n\to\infty}{\longrightarrow} 1$.

Proof. (1) Since $\lambda(X) \leq h(X)$ (Theorem 1.2), $h(X) \geq H \cdot \log n$ follows from Lemma 5.6 with $H = C - \gamma$ (recall that $\gamma = O\left(\sqrt{C}\right)$). The same Lemma, together with Proposition 1.9 yield $\chi(X) \geq \Xi$ for

$$\Xi = \frac{1}{(d+1)\sqrt[d]{c_d(\varepsilon_0 + \ldots + \varepsilon_{d-1})}} = \frac{1}{(d+1)\sqrt[d]{c_d \cdot \frac{\gamma}{C}}} = \Omega\left(\sqrt[2d]{C}\right).$$

(2) Again by Lemma 5.6 a.a.s. $\operatorname{Spec}\left(\Delta^+\big|_{Z_{d-1}}\right) \subseteq [\lambda_{avg} - \varepsilon', \lambda_{avg} + \varepsilon']$ with $\varepsilon' = 2\gamma\log n$. By Remark 5.3,

$$\operatorname{overlap}(X) \geq \frac{\mathcal{P}_d^d n}{e^{d+1}(n-d)}\left(\mathcal{P}_d - \frac{2\gamma\log n\,(d+1)}{\lambda_{avg}}\right) \geq \frac{\mathcal{P}_d^d}{e^{d+1}}\left(\mathcal{P}_d - \frac{2\gamma(d+1)}{C-\gamma}\right) \overset{C\to\infty}{\longrightarrow} \left(\frac{\mathcal{P}_d}{e}\right)^{d+1}.$$

(3) Choose some $\tau \in X^{d-2}$. It was observed in [GW12] that $\operatorname{lk}\tau \sim G\left(n-d+1, \frac{C\cdot\log n}{n}\right)$ (where $G(n,p) = X(1,n,p)$ is the Erdős–Rényi model), and $G\left(n, \frac{C\cdot\log n}{n}\right)$ has isolated vertices a.a.s. for $C < 1$ [ER59, Erd61]. These correspond to isolated $(d-1)$-cells in X (cells of degree zero), whose existence implies $h(X) = 0$ (and thus also $\lambda(X) = 0$). $\square$

5.5 Ramanujan triangle complexes

Let F be a nonarchimedean local field (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_{p^e}((t))$) with ring of integers $\mathcal{O}$, uniformizer π, and residue field $\mathbb{F}_q = \mathcal{O}/\pi\mathcal{O}$ of size q. For every d there exist an infinite $(d-1)$-dimensional complex denoted $\mathcal{B} = \mathcal{B}_d(F)$, the affine Bruhat-Tits building of type $\widetilde{A}_{d-1}$ associated with F. The vertices of $\mathcal{B}$ are in correspondence with the left K-cosets in G, where $G = \mathrm{PGL}_d(F)$, $K = \mathrm{PGL}_d(\mathcal{O})$, and they admit a coloring in $\mathbb{Z}/d\mathbb{Z}$, defined by $\operatorname{col}(gK) = \operatorname{ord}_\pi(\det g) + d\mathbb{Z}$. We refer the reader to [Lub12b, §2.1] for the definition of $\mathcal{B}$ and its basic properties.

The group $G = \mathrm{PGL}_d(F)$ acts on $\mathcal{B}$, and given a lattice $\Gamma \leq G$ the quotient $\Gamma \backslash \mathcal{B}$ is a finite complex. For $d = 2$, $\mathcal{B}$ is a $(q+1)$-regular tree (see, e.g. [Ser80, p. 70]), and its quotients by lattices in G are $(q+1)$-regular graphs. A special family among these quotients form excellent expanders[(†)]:

Theorem ([LPS88])**.** *If Γ is an arithmetic lattice in G then the nontrivial spectrum of $\Delta_0^+(\Gamma\backslash\mathcal{B})$ is contained within $\left[q+1-2\sqrt{q}, q+1+2\sqrt{q}\right]$ (hence $\Gamma\backslash\mathcal{B}$ is a $\left(0, q+1, \frac{2\sqrt{q}}{q+1}\right)$-expander).*

Remark. In [LPS88], the eigenvalue $2(q+1)$ is also considered as trivial. This eigenvalue exists in the spectrum iff the quotient is bipartite, which happens when Γ preserves colors (i.e. $2 \mid \mathrm{ord}_\pi(\det\gamma)$ for all $\gamma \in \Gamma$).

The cited theorem makes use of a special case of the "Peterson-Ramanujan conjecture" for GL_2, which is due to Eichler in characteristic zero, and to Drinfeld in positive characteristic. Thus, the corresponding graphs were baptized "Ramanujan graphs", and the term was broadened to include all $(q+1)$-regular graphs whose nontrivial spectrum lies in the $\left[q+1-2\sqrt{q}, q+1+2\sqrt{q}\right]$ strip. We refer the reader to the monograph [Lub10] for more details.

In recent years, several authors have turned to study the quotients of higher dimensional Bruhat-Tits building, i.e. $\Gamma\backslash\mathcal{B}_d$ for Γ a lattice in $G = \mathrm{PGL}_d(F)$: we refer the reader to [Bal00, CSŻ03, Li04, KLW10], and especially to [LSV05a, LSV05b], which study the consequences of the Ramanujan conjecture for GL_d in positive characteristic, proved by Lafforgue [Laf02]. However, it seems that as of now, none of these works studies the simplicial Hodge Laplacian on these complexes, and furthermore, that there are many open questions regarding their combinatorial properties. In an ongoing joint research with Konstantin Golubev we pursuit this line of study. At present, we focus on triangle complexes, i.e. quotients of $\mathcal{B}_3$. These are regular complexes, with vertex degrees $k_0 = 2\left(q^2+q+1\right)$ and edge degrees $k_1 = q+1$. There are two slightly different cases, according to whether or not X is 3-colorable (equivalently, Γ preserves colors). We have the following Theorem:

Theorem 5.8 ([GP13])**.** *Let X be a Ramanujan triangle complex on n vertices, which is not 3-colorable. Then*

(1) The nontrivial spectrum of Δ_0^+ is contained within $[k_0 - 6q, k_0 + 3q]$.

(2) The nontrivial spectrum of Δ_1^+ consists of:

- *(a) $n\left(q^2+q-2\right)+2$ eigenvalues in the strip $\left[k_1 - 2\sqrt{q}, k_1 + 2\sqrt{q}\right]$.*
- *(b) For every $\lambda \in \mathrm{Spec}\,\Delta_0^+\big|_{Z_0}$, the eigenvalues $\frac{3k_1}{2} \pm \sqrt{\left(\frac{3k_1}{2}\right)^2 - \lambda}$. This amounts to $n-1$ eigenvalues in each of the strips*

$$\begin{aligned} I_- &= \left[\frac{1}{2}\left(3k_1 - \sqrt{k_1^2 + 32q}\right), \frac{1}{2}\left(3k_1 - \sqrt{k_1^2 - 4q}\right)\right] \\ I_+ &= \left[\frac{1}{2}\left(3k_1 + \sqrt{k_1^2 - 4q}\right), \frac{1}{2}\left(3k_1 - \sqrt{k_1^2 + 32q}\right)\right]. \end{aligned} \tag{5.9}$$

- *(c) The eigenvalue $3k_1$, corresponding to the form $\varphi([vw]) = (-1)^{\mathrm{col}(w)-\mathrm{col}(v)}$ (this is a disorientation, see Definition 6.6).*

(†) In fact, they are spectrally optimal expanders, in the sense of the Alon-Boppana theorem (Theorem 7.7).

*If X **is** 3-colorable, then there are only $n-3$ eigenvalues of Δ_0^+ in $[k_0-6q,k_0+3q]$. The two other nontrivial eigenvalues equal $\frac{3k_0}{2}$, and correspond to the eigenforms $\omega^{\pm\operatorname{col}(v)}$, where $\omega=e^{\frac{2\pi i}{3}}$. The $n-3$ eigenvalues in $[k_0-6q,k_0+3q]$ then account for $n-3$ eigenvalues of Δ_1^+ in each of I_+ and I_-, and there are now $n\left(q^2+q-2\right)+6$ eigenvalues in $\left[k_1-2\sqrt{q},k_1+2\sqrt{q}\right]$. Unlike in the case of graphs, the $3k_1$ eigenvalue of Δ_1^+ appears even if X is not 3-colorable, as the difference $\operatorname{col}(w)-\operatorname{col}(v)\in\mathbb{Z}/3\mathbb{Z}$ is still well defined for neighboring vertices.*

Note that a Ramanujan triangle complex is "almost" a $\left(1,q+1,\frac{2\sqrt{q}}{q+1}\right)$-expander: the eigenvalues which ruin this are the the disorientation, and the $n-1$ eigenvalues in I_+. Therefore, a mixing lemma for these complexes, along the lines of Theorem 1.5, could perhaps be established by using only forms which are orthogonal to the "bad eigenforms". This is not hard for the disorientation, but at present open for the eigenforms which correspond to I_+. Nevertheless, the fact that the spectrum is bounded from below suffices us to deduce an isoperimetric bound:

Theorem (Theorem 1.10, extended)**.** *If X is a non-3-colorable Ramanujan triangle complex with n vertices, vertex degree $k_0=2\left(q^2+q+1\right)$ and edge degree $k_1=q+1$, then*

$$\frac{|F(A,B,C)|}{|A|\,|B|\,|C|}\geq\frac{1}{n^2}\left(q+1-2\sqrt{q}\right)\left(2q^2+2q+2-6q\left(1+\frac{10}{9\,|A|\,|B|\,|C|}\right)\right)$$

holds for any partition $V(X)=A\coprod B\coprod C$. Thus, if we fix $\vartheta>0$ and define

$$h_\vartheta(X)=\min_{\substack{V=A\coprod B\coprod C\\|A|,|B|,|C|\geq\vartheta n}}\frac{|F(A,B,C)|\,n^2}{|A|\,|B|\,|C|},\tag{5.10}$$

then

$$h_\vartheta(X)\geq 2q^3-O_\vartheta\left(q^{2.5}\right).\tag{5.11}$$

Remarks. (1) This corresponds to the pseudo-random intuition of expansion: X has $\frac{1}{6}nk_0k_1$ triangles, so its triangle density is indeed $\dfrac{\frac{1}{3}n\left(q^2+q+1\right)(q+1)}{\binom{n}{3}}\approx\dfrac{2q^3}{n^2}$.

(2) The restriction $|A|,|B|,|C|\geq\vartheta n$ is unavoidable, for the following reason: let us take any sub-linear function $f(n)$ (i.e. $\frac{f(n)}{n}\xrightarrow{n\to\infty}0$) and define $h_f(X)$ by replacing ϑn with $f(n)$ in (5.10). If X_i is any sequence of triangle complexes with $n_i=\left|X_i^0\right|\to\infty$ and with globally bounded vertex degrees, then one can take $A\subseteq X_i^0$ to be any set of size $f(n_i)$, B to be $\partial A=\{v\,|\,\operatorname{dist}(v,A)=1\}$ (if $|B|<f(n_i)$ enlarge it by adding any vertices), and C the rest of the vertices. Assuming i is large enough one has $|A|,|B|,|C|\geq f(n_i)$, and $F(A,B,C)=\varnothing$ since all triangles with a vertex in A have their other vertices in either A or B. Therefore, $h_f(X_i)=0$ for all large enough i.

Theorem 1.10 is an immediate corollary from Theorem 5.8 and the following Cheeger-type inequality:

Theorem 5.9. *Let X be a triangle complex on n vertices V, with*

$$\operatorname{Spec}\Delta_0^+\big|_{Z_0}\subseteq[k_0-\mu_0,k_0+\mu_0]$$
$$\operatorname{Spec}\Delta_1^+\big|_{Z_1}\subseteq[\lambda_1,\infty)$$

for some k_0, μ_0, λ_1*. Then*

$$\frac{|F(A,B,C)|\, n^2}{|A|\,|B|\,|C|} \geq \lambda_1 \left(k_0 - \mu_0 \left(1 + \frac{10\, n^3}{9\,|A|\,|B|\,|C|}\right)\right) \tag{5.12}$$

holds for any partition $V = A \coprod B \coprod C$*, so that*

$$h_\vartheta(X) \geq \lambda_1 \left(k_0 - \mu_0 \left(1 + \frac{10}{9\vartheta^3}\right)\right).$$

Note that if X has a complete skeleton then $\operatorname{Spec} \Delta_0^+\big|_{Z_0} = \{n\}$, so that $k_0 = n$ and $\mu_0 = 0$. Therefore, (5.12) reads $\frac{|F(A,B,C)|n^2}{|A||B||C|} \geq \lambda_1 \cdot n$, so that Theorem 1.2 is obtained as a special case. We give now the proofs of Theorems 5.8 and 5.9:

Proof of Theorem 5.8. The element $\sigma = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pi & 0 & 0 \end{smallmatrix}\right) \in G$ acts by rotation on the triangle consisting of the vertices K, σK, and $\sigma^2 K$ (note that $\sigma^3 = id$ in G). Let us choose fundamental vertex $v_0 = K$, edge $e_0 = [K, \sigma K]$, and triangle $t_0 = \left[K, \sigma K, \sigma^2 K\right]$. If $X = \Gamma \backslash \mathcal{B}$, we can identify $\Omega^0 = \Omega^0(\Gamma\backslash\mathcal{B})$ with $L^2(\Gamma\backslash G/K)$. As G preserves the orientation of edges, and acts transitively on the nonoriented edges of $\mathcal{B}$, we can identify $\Omega^1 = L^2(\Gamma\backslash G/E)$, where

$$E = \operatorname{stab} e_0 = K \cap \sigma K \sigma^{-1} = \left\{ \left(\begin{smallmatrix} * & * & * \\ * & * & * \\ x & y & * \end{smallmatrix}\right) \in K \,\middle|\, x, y \in \pi\mathcal{O} \right\}$$

(E is sometimes called a "parahoric subgroup"). If I is the Iwahori subgroup

$$I = K \cap \sigma K \sigma^{-1} \cap \sigma^2 K \sigma^{-2} = \left\{ \left(\begin{smallmatrix} * & * & * \\ x & * & * \\ y & z & * \end{smallmatrix}\right) \in K \,\middle|\, x, y, z \in \pi\mathcal{O} \right\},$$

which fixes all vertices of t_0, then I and $\langle\sigma\rangle$ commute, and Ω^2 can be identified with $L^2(T\backslash G/E)$, where

$$T \overset{def}{=} \operatorname{stab}_G t_0 = \langle\sigma\rangle I = I \cup \sigma I \cup \sigma^2 I.$$

The space $L^2(\Gamma\backslash G/K)$ corresponds naturally to $L^2(\Gamma\backslash G)^K$, the space of K-fixed vectors in the right G-representation $L^2(\Gamma\backslash G)$, and similarly for E and T. As $I \subseteq E \subseteq K$ and $I \subseteq T$, the three of $L^2(\Gamma\backslash G)^K$, $L^2(\Gamma\backslash G)^E$ and $L^2(\Gamma\backslash G)^T$ are contained in $L^2(\Gamma\backslash G)^I$ - the space of Iwahori-fixed vectors. We now turn to express the Laplacians and adjacency operators on $\mathcal{B}$ and its quotients as Hecke, and generalized Hecke operators on these spaces.

In dimension zero, $\mathcal{A}_0^{\sim} = \mathcal{A}_{0,1}^{\sim} + \mathcal{A}_{0,2}^{\sim}$ where $\mathcal{A}_{0,1}^{\sim}$ and $\mathcal{A}_{0,2}^{\sim}$ are the classical Hecke operators studied in [CSŻ03, Li04, LSV05a, LSV05b], and which correspond, respectively, to

$$K\sigma K = \bigcup_{x,y \in \mathbb{F}_q} \left(\begin{smallmatrix} \pi & x & y \\ & 1 & \\ & & 1 \end{smallmatrix}\right) K \cup \bigcup_{z \in \mathbb{F}_q} \left(\begin{smallmatrix} 1 & & \\ & \pi & z \\ & & 1 \end{smallmatrix}\right) K \cup \left(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & \pi \end{smallmatrix}\right) K \quad \text{and}$$

$$K\sigma^2 K = \bigcup_{x,y \in \mathbb{F}_q} \left(\begin{smallmatrix} \pi & & x \\ & \pi & y \\ & & 1 \end{smallmatrix}\right) K \cup \bigcup_{z \in \mathbb{F}_q} \left(\begin{smallmatrix} \pi & z & \\ & 1 & \\ & & \pi \end{smallmatrix}\right) K \cup \left(\begin{smallmatrix} 1 & & \\ & \pi & \\ & & \pi \end{smallmatrix}\right) K.$$

Similarly, $\mathcal{A}_1^{\sim} = \mathcal{A}_{1,1}^{\sim} + \mathcal{A}_{1,2}^{\sim}$ where $\mathcal{A}_{0,1}^{\sim}$ and $\mathcal{A}_{0,2}^{\sim}$ correspond to

$$E\sigma E = \left(\begin{smallmatrix} & & 1 \\ & 1 & \\ \pi & & \end{smallmatrix}\right) E \cup \bigcup_{x\in\mathbb{F}_q} \left(\begin{smallmatrix} & 1 & \\ & x & 1 \\ \pi & & \end{smallmatrix}\right) E \quad \text{and} \quad E\sigma^2 E = \left(\begin{smallmatrix} \pi & & \\ & & 1 \\ & \pi & \end{smallmatrix}\right) E \cup \bigcup_{x\in\mathbb{F}_q} \left(\begin{smallmatrix} & & 1 \\ \pi & & x \\ & \pi & \end{smallmatrix}\right) E,$$

respectively. In dimension two, $\mathcal{A}_2^{\pitchfork}$ corresponds to

$$T \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & \pi & 0 \\ \pi^2 & 0 & 0 \end{smallmatrix}\right) T = \bigcup_{j=0}^{2} \bigcup_{x\in\mathbb{F}_q} \left(\begin{smallmatrix} & 1 & \\ & & 1 \\ \pi & & \end{smallmatrix}\right)^j \left(\begin{smallmatrix} & 1 & \\ \pi & & \\ & \pi x & \pi \end{smallmatrix}\right) T$$

(this can be used for verification purposes, since $\operatorname{Spec} \Delta_1^+\big|_{B_1} = \operatorname{Spec} \Delta_2^-\big|_{B^1}$).

The representation $L^2(\Gamma\backslash G)$ decomposes as a sum of irreducible unitary representations, $L^2(\Gamma\backslash G) = \bigoplus_i V_i$, and $L^2(\Gamma\backslash G)^I = \bigoplus_i V_i^I$. It is therefore enough to study the V_i which contain Iwahori-fixed vectors. Such representations are called *Iwahori-spherical*, and a theorem of Casselman [Cas80] states that they are embeddable in the *principal series representations*. The principal series representation $V_{\mathfrak{z}}$ with *Satake parameters* $\mathfrak{z} = (z_1, z_2, z_3)$, where $z_i \in \mathbb{C}$ and $z_1 z_2 z_3 = 1$, is obtained as follows: The Borel group $B = \left\{ \left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{smallmatrix}\right) \in G \right\}$ admits a character $\chi_{\mathfrak{z}}(b) = \prod_{i=1}^3 z_i^{\operatorname{ord}_\pi b_{ii}}$ (here we need $z_1 z_2 z_3 = 1$ to assure that $\chi_{\mathfrak{z}}$ factors modulo $Z(\mathrm{GL}_3(F))$), and $V_{\mathfrak{z}}$ is the unitary induction of $\chi_{\mathfrak{z}}$ from B to G, namely

$$V_{\mathfrak{z}} = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{c} f(bg) = \delta^{-\frac{1}{2}}(b)\,\chi_{\mathfrak{z}}(b)\, f(g) \;\; \forall b \in B \\ \int_K |f(k)|\, dk < \infty \end{array} \right\},$$

where $\delta(b) = \operatorname{ord}_\pi^2(b_{11}) \operatorname{ord}_\pi^{-2}(b_{33})$ is the unimodular character of B.

By the Iwasawa decomposition $G = BK$, for $f \in V_{\mathfrak{z}}^K$ we have $f(g) = f(bk) = \delta^{-\frac{1}{2}}(b)\,\chi_{\mathfrak{z}}(b)\, f(k) = \delta^{-\frac{1}{2}}(b)\,\chi_{\mathfrak{z}}(b)\, f(id)$, so that $V_{\mathfrak{z}}^K$ is at most one-dimensional. In fact, it is one-dimensional, since $b \in B \cap K$ implies that $b_{11}, b_{22}, b_{33} \in \mathcal{O}^\times$, so that $f(bk) = \delta^{-\frac{1}{2}}(b)\,\chi_{\mathfrak{z}}(b)$ is well defined. Letting a permutation $w \in S_3$ stand for the the permutation matrix $(T)_{i,j} = \delta_{w(i),j}$ in G, the so-called Iwahori-Bruhat decomposition $G = \bigcup_{w\in S_3} BwI$ shows that $\dim V_{\mathfrak{z}}^I = 6$, with basis $\{f_w\}_{w\in S_3}$ defined by $f_w(w') = \left\{\begin{smallmatrix} 1 & w=w' \\ 0 & w\neq w' \end{smallmatrix}\right.$. The subspace $V_{\mathfrak{z}}^K \subseteq V_{\mathfrak{z}}^I$ is spanned by $\mathscr{F} \overset{def}{=} \sum_{w\in S_3} f_w$. Also, $(1\,2) = \left(\begin{smallmatrix} & 1 & \\ 1 & & \\ & & 1 \end{smallmatrix}\right) \in E$ implies that $G = \bigcup_{w\in A_3} BwE$, and that $\dim V_{\mathfrak{z}}^E = 3$ with basis $\{h_w\}_{w\in A_3}$, where $h_w \overset{def}{=} f_w + f_{w\cdot(1\,2)}$. With some more work, one can show that $G = BT \bigcup B\,(1\,2)\,T$, and that $\dim V_{\mathfrak{z}}^T = 2$ with basis

$$f_{(\,)} + \frac{z_1}{q} f_{(1\,3\,2)} + \frac{1}{z_3 q} f_{(1\,2\,3)}, \quad f_{(1\,2)} + z_2 f_{(2\,3)} + \frac{1}{z_3 q} f_{(1\,3)}.$$

As G acts both on $V_{\mathfrak{z}}$ and on $L^2(\Gamma\backslash G)$ by right translation, the action of the (generalized) Hecke operators on both of them is given by the corresponding decomposition to right cosets. For example,

$$\left(\mathcal{A}_2^{\pitchfork} f\right)(g) = \sum_{j=0}^{2} \sum_{x\in\mathbb{F}_q} f\left(g \left(\begin{smallmatrix} & 1 & \\ & & 1 \\ \pi & & \end{smallmatrix}\right)^j \left(\begin{smallmatrix} & 1 & \\ \pi & & \\ & \pi x & \pi \end{smallmatrix}\right) \right)$$

for $f \in L^2(\Gamma\backslash G)^T \cong \Omega^2(X)$. Using the coset decompositions corresponding to our Hecke operators, we find that $\mathcal{A}_0^{\sim}$ acts on $V_{\mathfrak{z}}^K = \mathbb{C}\mathscr{F}$ as multiplication by $q \sum_{i=1}^3 \left(z_i + z_i^{-1}\right)$. Denoting $\widetilde{\mathfrak{z}} = \sum_{i=1}^3 \left(z_i + z_i^{-1}\right)$, this gives (as $\Delta_0^+ = k_0 I - \mathcal{A}_0^\sim$)

$$\operatorname{Spec}\Delta_0^+\big|_{V_\mathfrak{z}^K} = \{k_0 - q\widetilde{\mathfrak{z}}\}\,. \tag{5.13}$$

Similarly, $\mathcal{A}_1^\sim$ acts on $V_\mathfrak{z}^E$ with respect to our chosen basis by

$$[\mathcal{A}_1^\sim]_{\{h_{()},h_{(1\,2\,3)},h_{(1\,3\,2)}\}} = \begin{pmatrix} 0 & qz_3 + qz_1z_3 & q^2z_3 + qz_2z_3 \\ z_2 + z_1z_2 & 0 & qz_2 + qz_2z_3 \\ \frac{z_1}{q} + z_1z_2 & z_1 + z_1z_3 & 0 \end{pmatrix}.$$

Using $\Delta_1^+ = k_1 I - \mathcal{A}_1^\sim$ one obtains that $\Delta_1^+\big|_{V_\mathfrak{z}^E}$ has a zero eigenvalue (which corresponds to $\partial^*\mathscr{F}$), and the eigenvalues

$$\lambda_\pm = \frac{3k_1 \pm \sqrt{k_1^2 + 4q\prod_{i=1}^3 (1+z_i)}}{2} = \frac{3k_1 \pm \sqrt{k_1^2 + 8q + 4q\widetilde{\mathfrak{z}}}}{2} = \frac{3k_1}{2} \pm \sqrt{\left(\frac{3k_1}{2}\right)^2 - \lambda_\mathscr{F}}, \tag{5.14}$$

where $\lambda_\mathscr{F} = k_0 - q\widetilde{\mathfrak{z}}$ is the eigenvalue of Δ_0^+ acting on $V_\mathfrak{z}^K = \mathbb{C}\mathscr{F}$.

In general, a Iwahori-spherical representation is only a subrepresentation of $V_\mathfrak{z}$. Let us denote by $W_\mathfrak{z}$ this subrepresentation (there is only one such). Tadic [Tad86] has classified the Satake triples $\mathfrak{z}$ for which the representation $W_\mathfrak{z}$ admits a unitary structure. In [KLW10] the possible $\mathfrak{z}$ for $\mathrm{PGL}_3(F)$ are explicitly computed, and $W_\mathfrak{z}$ is identified within $V_\mathfrak{z}$ using [Bor76, Zel80]. Furthermore, the $\mathfrak{z}$ for which $W_\mathfrak{z}$ cannot appear in a Ramanujan complexes are singled out [KLW10, Theorem 2]. It turns out that a unitary $W_\mathfrak{z}$ which can appear in a Ramanujan quotient, and which is E-spherical (i.e. have E-fixed vectors), is of one of the following types:

(a) $|z_i| = 1$ for $i = 1, 2, 3$. In this case $V_\mathfrak{z}$ is irreducible, hence $W_\mathfrak{z} = V_\mathfrak{z}$. Here $\widetilde{\mathfrak{z}} = 2\Re\left(\sum z_i\right) \in [-3, 6]$ gives $\lambda_\mathscr{F} \in [k_0 - 6q, k_0 + 3q]$, and $\lambda_\pm \in I_\pm$ (see (5.14) and (5.9)).

(b) $\mathfrak{z} = \left(\frac{c}{\sqrt{q}}, c\sqrt{q}, c^{-2}\right)$ for some $|c| = 1$. In this case $W_\mathfrak{z}^E$ is one-dimensional, and it is spanned by $\mathscr{H} = h_{(1\,3\,2)} - qh_{(1\,2\,3)}$, which corresponds to to λ_- in (5.14). For this $\mathfrak{z}$ we have

$$\begin{aligned}\lambda_- &= \frac{1}{2}\left(3k_1 \pm \sqrt{k_1^2 + 8q + 4q\left(\frac{c}{\sqrt{q}} + \overline{c}\sqrt{q} + c\sqrt{q} + \frac{\overline{c}}{q} + c^{-2} + c^2\right)}\right) \\ &= \frac{1}{2}\left(3k_1 - \sqrt{q^2 + 8q\sqrt{q}\Re(c) + 2q + 16q\Re(c)^2 + 1 + 8\sqrt{q}\Re(c)}\right) \\ &= \frac{1}{2}\left(3k_1 - (q + 4\sqrt{q}\Re(c) + 1)\right) = k_1 - 2\sqrt{q}\Re(c)\end{aligned}$$

which lies in $\left[k_1 - 2\sqrt{q}, k_1 + 2\sqrt{q}\right]$. As $\mathscr{H}$ is not K-fixed, $W_\mathfrak{z}^K = 0$.

(c) $\mathfrak{z} = \left(q, 1, \frac{1}{q}\right)$. In this case $W_\mathfrak{z}$ is the trivial representation $\rho : G \to \mathbb{C}^\times$, and $W_\mathfrak{z}^E = W_\mathfrak{z}^K$ are one-dimensional (spanned by $\mathscr{F}$). As $\mathscr{F}$ is constant, $\Delta_0^+\mathscr{F} = 0$ and $\Delta_1^+\mathscr{F} = 3k_1\mathscr{F}$ (this can also be verified using (5.13) and (5.14)).

(d) $\mathfrak{z} = \left(\omega q, \omega, \frac{\omega}{q}\right)$ where $\omega = e^{\frac{2\pi i}{3}}$ or $\omega = e^{-\frac{2\pi i}{3}}$. Here $W_\mathfrak{z}$ is the one-dimensional representation $\rho(g) = \omega^{\mathrm{col}(g)}$, and again $W_\mathfrak{z}^E = W_\mathfrak{z}^K = \mathbb{C}\cdot\mathscr{F}$. This time $\Delta_0^+\mathscr{F} = \frac{3k_0}{2}\mathscr{F}$, and $\Delta_1^+\mathscr{F} = 0$.

Let $X = \Gamma\backslash\mathcal{B}$ be a Ramanujan complex with $L^2(\Gamma\backslash G) = \bigoplus_i W_{\mathfrak{z}_i}$, which is not 3-colorable. In this case the representations of type (d) do not appear in $L^2(\Gamma\backslash G)$. The representation (c) appears once in $L^2(\Gamma\backslash G)$, and corresponds to B^0, the constant functions in Ω^0, and to the disorientation forms in Ω^1. Since representations of type (b) contain no K-fixed vectors, $n = \dim\Omega^0(X) = \dim L^2(\Gamma\backslash G)^K = \sum_i \dim W_{\mathfrak{z}_i}^K$ shows that that there are $(n-1)$ representations of type (a) in $L^2(\Gamma\backslash G)$, each contributing three eigenvalues to Δ_1^+ (one of which is trivial). The representations of type (c) and (a) account so far for $1+3(n-1)$ eigenforms in Ω^1, and by $n\left(q^2+q+1\right) = \dim\Omega^1 = \sum_i \dim W_{\mathfrak{z}_i}^E$ it follows that $L^2(\Gamma\backslash G)$ must contain $n\left(q^2+q+1\right) - 1 - 3(n-1) = n\left(q^2+q-2\right)+2$ representations of type (b).

If X is 3-colorable, there are two representations of type (d) in $L^2(\Gamma\backslash G)$. These correspond to the eigenfunctions $f_\pm(v) = \omega^{\pm\operatorname{col} v}$ in Ω^0, which have eigenvalue $q\left(q^2+q+1\right) = \frac{3k_0}{2}$, and to the coboundary forms $\partial^* f_\pm$ in Ω^1. The computation of the number of representations of type (a), (b) and (c) then continues analogously to the non 3-colorable case. $\square$

Proof of Theorem 5.9. Denote $|A|, |B|, |C|$ by a, b, c, respectively, and define

$$f \in \Omega^1, \qquad f(vw) = \begin{cases} c & v \in A, w \in B \\ a & v \in B, w \in C \\ b & v \in C, w \in A \\ 0 & else \end{cases}$$

(implying $f(vw) = -c$ for $v \in B, w \in A$, etc.). Let $f_B = \mathbb{P}_{B^1} f$ and $f_Z = \mathbb{P}_{Z_1} f$. Then

$$\begin{aligned} |F(A,B,C)|\, n^2 &= \sum_{t\in T} (\delta f)^2(t) = \|\delta f\|^2 = \|\delta f_Z\|^2 \\ &= \left\langle \Delta^+ f_Z, f_Z \right\rangle \geq \lambda_1 \|f_Z\|^2 = \lambda_1 \left(\|f\|^2 - \|f_B\|^2\right). \end{aligned}$$

Let us denote $E = k_0\mathbb{P}_{B^1} - \Delta_1^-$. Since

$$\begin{aligned} \operatorname{Spec}\Delta_1^-\big|_{B^1} &= \operatorname{Spec}\Delta_0^+\big|_{B_0} \subseteq \operatorname{Spec}\Delta_0^+\big|_{Z_0} \subseteq [k_0-\mu_0, k_0+\mu_0], \\ \text{and } \operatorname{Spec}\Delta_1^-\big|_{Z_1} &= 0 \end{aligned}$$

we have $\|E\| \leq \mu_0$, so that

$$\|f_B\|^2 = \langle \mathbb{P}_{B^1} f, \mathbb{P}_{B^1} f\rangle = \langle \mathbb{P}_{B^1} f, f\rangle \leq \frac{|\langle Ef, f\rangle| + \left|\left\langle \Delta_1^- f, f\right\rangle\right|}{k_0} \leq \frac{\mu_0 \|f\|^2 + \|\partial f\|^2}{k_0}.$$

We would like to bound $\|\partial f\|^2$. Let us begin with

$$\begin{aligned}
\sum_{\alpha\in A}(\partial f)^2(\alpha) &= \sum_{\alpha\in A}\Big(c\sum_{\beta\in B}\delta_{\alpha\beta}-b\sum_{\gamma\in C}\delta_{\alpha\gamma}\Big)^2\\
&= c^2\sum_{\substack{\alpha\in A\\ \beta,\beta'\in B}}\delta_{\alpha\beta}\delta_{\alpha\beta'}-2bc\sum_{\alpha,\beta,\gamma}\delta_{\alpha\beta}\delta_{\alpha\gamma}+b^2\sum_{\alpha,\gamma,\gamma'}\delta_{\alpha\gamma}\delta_{\alpha\gamma'}\\
&= c^2\left|F^1(B,A,B)\right|-2bc\left|F^1(B,A,C)\right|+b^2\left|F^1(C,A,C)\right|.
\end{aligned}$$

By [Par13a, Lem. 1.3] with $\ell=2$ and $j=0$, we have (recall that $k_{-1}=n$ and $\varepsilon_{-1}=0$):

$$\begin{aligned}
\left|\left|F^1(B,A,B)\right|-\left(\frac{k_0}{n}\right)^2 b^2a\right| &\le 2k_0\mu_0 b\\
\left|\left|F^1(B,A,C)\right|-\left(\frac{k_0}{n}\right)^2 bac\right| &\le 2k_0\mu_0\sqrt{bc}\\
\left|\left|F^1(C,A,C)\right|-\left(\frac{k_0}{n}\right)^2 c^2a\right| &\le 2k_0\mu_0 c.
\end{aligned}$$

Therefore,

$$\sum_{\alpha\in A}(\partial f)^2(\alpha)\le 2k_0\mu_0\left[c^2b+2(bc)^{\frac{3}{2}}+b^2c\right]=2k_0\mu_0 bc\left(\sqrt{b}+\sqrt{c}\right)^2.$$

and repeating this for $\sum_{\beta\in B}$ and $\sum_{\gamma\in C}$ gives

$$\|\partial f\|^2\le 2k_0\mu_0\left[bc\left(\sqrt{b}+\sqrt{c}\right)^2+ac\left(\sqrt{a}+\sqrt{c}\right)^2+ab\left(\sqrt{a}+\sqrt{b}\right)^2\right]\le k_0\mu_0 n^3.$$

Using the classic expander mixing lemma for $E(A,B)$, $E(B,C)$ and $E(C,A)$ we have

$$\begin{aligned}
\|f\|^2 &= |E(A,B)|\,c^2+|E(B,C)|\,a^2+|E(C,A)|\,b^2\\
&\ge\left(\frac{k_0}{n}ab-\mu_0\sqrt{ab}\right)c^2+\left(\frac{k_0}{n}bc-\mu_0\sqrt{bc}\right)a^2+\left(\frac{k_0}{n}ca-\mu_0\sqrt{ca}\right)b^2\\
&= k_0abc-\mu_0\left[\sqrt{ab}c^2+\sqrt{bc}a^2+\sqrt{ac}b^2\right]\ge k_0abc-\frac{\mu_0n^3}{9}.
\end{aligned}$$

Combining everything now gives

$$\begin{aligned}
&\frac{|F(A,B,C)|\,n^2}{abc}\ge\frac{\lambda_1}{abc}\left(\|f\|^2-\|f_B\|^2\right)\ge\frac{\lambda_1}{abc}\left(\|f\|^2\left(1-\frac{\mu_0}{k_0}\right)-\mu_0n^3\right)\\
&\ge\frac{\lambda_1}{abc}\left(\left(k_0abc-\frac{\mu_0n^3}{9}\right)\left(1-\frac{\mu_0}{k_0}\right)-\mu_0n^3\right)\ge\lambda_1\left(k_0-\mu_0\left(1+\frac{10n^3}{9abc}\right)\right).
\end{aligned}$$

□

6 High dimensional random walk

In this section we study the stochastic process on complexes which was demonstrated in §1.4. For this purpose we introduce a different inner product structure on Ω^j than the one used so far, and this will change the Laplacians and the spectral gap as well (see §6.2). We do not assume that X has a complete skeleton in this section, but we are again primarily interested in the $(d-1)$ dimension, hence we will write again $\Delta^{\pm} = \Delta^{\pm}_{d-1}$ and $\lambda = \lambda_{d-1}$. Throughout this section we assume that X is *uniform*, meaning that every cell is contained in some cell of dimension d.

6.1 The $(d-1)$-walk and expectation process

Let X be a uniform d-dimensional complex and $0 \le p < 1$. Recall that two oriented $(d-1)$-cells $\sigma, \sigma' \in X^{d-1}_{\pm}$ are said to be *neighbors* (denoted $\sigma \sim \sigma'$) if there exists an oriented d-cell τ, such that both σ and $\overline{\sigma'}$ are faces of τ with the orientations induced by it (see Figure 1.1). The following process is the generalization of the edge walk from §1.4:

Definition 6.1. The p-lazy $(d-1)$-walk on a d-complex X is defined as follows: The walk starts at an initial oriented $(d-1)$-cell σ_0, and at each step the walker stays in place with probability p, and with probability $(1-p)$ chooses uniformly one of its neighbors and moves to it.

Put differently, it is the Markov chain on $X^{d-1}_{\pm}$ with transition probabilities

$$\mathrm{Prob}\left(X_{n+1} = \sigma' | X_n = \sigma\right) = \begin{cases} p & \sigma' = \sigma \\ \frac{1-p}{d \deg(\sigma)} & \sigma' \sim \sigma \\ 0 & \text{otherwise} \end{cases},$$

For $j = 0$ Definition 6.1 gives the standard p-lazy random walk on a graph.

Definition 6.2. We say that X is $(d-1)$-*connected* if the $(d-1)$-walk on it is irreducible, i.e., for every pair of oriented $(d-1)$-cells σ and σ' there exist a chain $\sigma = \sigma_0 \sim \sigma_1 \sim \ldots \sim \sigma_n = \sigma'$. Moreover, having such a chain defines an equivalence relation on the $(d-1)$-cells of X, whose classes we call the $(d-1)$-*components* of X.

Remark. Due to the assumption of uniformity, it is enough to observe unoriented cells - X is $(d-1)$-connected iff for every $\sigma, \sigma' \in X^{d-1}$ there exists a chain of unoriented $(d-1)$-cells $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma'$ with $\sigma_i \cup \sigma_{i+1} \in X^d$ for all i. This is also equivalent to the assertion that for any $\tau, \tau' \in X^d$ there is a chain $\tau = \tau_0, \tau_1, \ldots, \tau_m = \tau'$ of d-cells with $\tau_i \cap \tau_{i-1} \in X^{d-1}$ for all i (this is sometimes referred to as a *chamber complex*). We note that it follows from uniformity that a $(d-1)$-connected complex is connected as a topological space. The other direction does not hold: the complex $\blacktriangleright\!\!\blacktriangleleft$ is not 1-connected, even though it is connected (and uniform).

Observing the $(d-1)$-walk on X, we denote by $\mathbf{p}_n^{\sigma_0}(\sigma)$ the probability that the random walk starting at σ_0 reaches σ at time n. We then have:

Definition 6.3. For $d \ge 2$, the *expectation process* on X starting at σ_0 is the sequence of $(d-1)$-forms $\{\mathcal{E}_n^{\sigma_0}\}_{n=0}^{\infty}$ defined by

$$\mathcal{E}_n^{\sigma_0}(\sigma) = \mathbf{p}_n^{\sigma_0}(\sigma) - \mathbf{p}_n^{\sigma_0}(\overline{\sigma}).$$

For $d=1$ (i.e. graphs) we simply define $\mathcal{E}_n^{v_0}=\mathbf{p}_n^{v_0}$.[(†)] The *normalized expectation process* on X is

$$\widetilde{\mathcal{E}}_n^{\sigma_0}(\sigma)\overset{def}{=}\left(\frac{d}{p(d-1)+1}\right)^n\mathcal{E}_n^{\sigma_0}(\sigma)=\left(\frac{d}{p(d-1)+1}\right)^n\left[\mathbf{p}_n^{\sigma_0}(\sigma)-\mathbf{p}_n^{\sigma_0}(\overline{\sigma})\right],$$

where p is the laziness of the walk. In particular, for $d=1$ one has $\widetilde{\mathcal{E}}_n^{v_0}=\mathcal{E}_n^{v_0}=\mathbf{p}_n^{v_0}$ for all p.

The reason for this particular normalization is that for a lazy enough process (in particular for $p\geq\frac{1}{2}$) one has $\|\mathcal{E}_n^{\sigma_0}\|=\Theta\left(\left(\frac{p(d-1)+1}{d}\right)^n\right)$ (see (6.8)). Note that $\widetilde{\mathcal{E}}_0^{\sigma_0}=\mathcal{E}_0^{\sigma_0}=\mathbb{1}_{\sigma_0}$.

Remark 6.4. The name "expectation process" is due to the fact that for any $(d-1)$-form f the expected value of f at time n is

$$\mathbb{E}_n^{\sigma_0}[f]=\sum_{\sigma\in X_{\pm1}^{d-1}}\mathbf{p}_n^{\sigma_0}(\sigma)f(\sigma)=\sum_{\sigma\in X^{d-1}}\mathcal{E}_n^{\sigma_0}(\sigma)f(\sigma)$$

where, as implied by the notation, $\mathcal{E}_n^{\sigma_0}(\sigma)f(\sigma)$ does not depend on the orientation of σ.

The evolution of the expectation process over time is given by $\mathcal{E}_{n+1}^{\sigma_0}=A\mathcal{E}_n^{\sigma_0}$, where $A=A(X,p)$ is the *transition operator* acting on Ω^{d-1} by

$$(Af)(\sigma)=pf(\sigma)+\frac{(1-p)}{d}\sum_{\sigma'\sim\sigma}\frac{f(\sigma')}{\deg(\sigma')}\qquad\left(f\in\Omega^{d-1},\sigma\in X^{d-1}\right). \tag{6.1}$$

In terms of the adjacency operator we have $A=pI+(1-p)\mathcal{A}_{d-1}^{\sim}D^{-1}$, but we will not use this in what follows. Note that the evolution of $\mathbf{p}_n^{\sigma_0}$ is given by the same A, acting on all functions from $X_{\pm}^{d-1}$ to $\mathbb{R}$, and not only on forms.

It is sometimes useful to observe the Markov operator $M=M(X,p)$ associated with this evolution, which is characterized by

$$\mathbb{E}_{n+1}^{\sigma_0}[f]=\mathbb{E}_n^{\sigma_0}[Mf],$$

and is given explicitly by

$$(Mf)(\sigma)=pf(\sigma)+\frac{1-p}{d\deg(\sigma)}\sum_{\sigma'\sim\sigma}f(\sigma')\qquad\left(f\in\Omega^{d-1},\sigma\in X^{d-1}\right).$$

This is the transpose of A, w.r.t. to a natural choice of basis for $\Omega^{d-1}(X)$.

6.2 Normalized Laplacians

Given any weight function $w:X\to(0,\infty)$, Ω^k become inner product spaces (for $-1\leq k\leq d$) with

$$\langle f,g\rangle=\sum_{\sigma\in X^k}w(\sigma)f(\sigma)g(\sigma)\qquad\forall f,g\in\Omega^k.$$

[(†)]The results which follow hold for graphs as well, using this definition of $\mathcal{E}_n^{v_0}$, but they are all well known. In some cases the proofs are slightly different, and we will not trouble to handle this special case.

Recall that $v \triangleleft \sigma$ if $v \notin \sigma$ and $v\sigma = \{v\} \cup \sigma$ is a cell in X, and $(\partial_k f)(\sigma) = \sum_{v \triangleleft \sigma} f(v\sigma)$. The adjoint coboundary operators w.r.t. the weighted inner products are given by

$$(\delta_k f)(\sigma) = (\partial_k^* f)(\sigma) = \frac{1}{w(\sigma)} \sum_{i=0}^{k} (-1)^i w(\sigma \backslash \sigma_i) f(\sigma \backslash \sigma_i) \quad 0 \le k \le d.$$

We will adhere to the notation ∂_k^* until we discuss infinite complexes, where sometimes δ_k is defined even though ∂_k is not.

The following weight functions will be used throughout §6 and §7:

$$w(\sigma) = \begin{cases} \frac{1}{\deg \sigma} & \sigma \in X^{d-1} \\ 1 & \sigma \in X \backslash X^{d-1} \end{cases}.$$

Notice that for $\sigma \in X^{d-1}$

$$\frac{1}{w(\sigma)} = \deg(\sigma) = \left|\left\{\tau \in X^d \,\middle|\, \sigma \subset \tau\right\}\right| = |\{v \mid v \triangleleft \sigma\}| = \frac{1}{d} \left|\left\{\sigma' \in X^{d-1} \,\middle|\, \sigma' \sim \sigma\right\}\right|.$$

Due to our choice of weights, the inner product and coboundary operators in §6, §7 are given by

$$\langle f, g \rangle = \begin{cases} \sum\limits_{\sigma \in X^k} f(\sigma) g(\sigma) & f, g \in \Omega^k, k \ne d-1 \\ \sum\limits_{\sigma \in X^{d-1}} \frac{f(\sigma) g(\sigma)}{\deg \sigma} & f, g \in \Omega^{d-1} \end{cases} \tag{6.2}$$

$$(\delta_k f)(\sigma) = (\partial_k^* f)(\sigma) = \begin{cases} \sum\limits_{i=0}^{k} (-1)^i f(\sigma \backslash \sigma_i) & k \le d-2 \\ \deg(\sigma) \sum\limits_{i=0}^{d-1} (-1)^i f(\sigma \backslash \sigma_i) & k = d-1 \\ \sum\limits_{i=0}^{d} \dfrac{(-1)^i f(\sigma \backslash \sigma_i)}{\deg(\sigma \backslash \sigma_i)} & k = d, \end{cases} \tag{6.3}$$

and the Laplacians by

$$\begin{aligned} \left(\Delta^+ f\right)(\sigma) &= \sum_{v \triangleleft \sigma} (\partial_d^* f)(v\sigma) = \sum_{v \triangleleft \sigma} \sum_{i=0}^{d} \frac{(-1)^i f(v\sigma \backslash (v\sigma)_i)}{\deg(v\sigma \backslash (v\sigma)_i)} \\ &= f(\sigma) - \sum_{v \triangleleft \sigma} \sum_{i=0}^{d-1} \frac{(-1)^i f(v(\sigma \backslash \sigma_i))}{\deg(v(\sigma \backslash \sigma_i))} = f(\sigma) - \sum_{\sigma' \sim \sigma} \frac{f(\sigma')}{\deg(\sigma')} \end{aligned} \tag{6.4}$$

and

$$\left(\Delta^- f\right)(\sigma) = \deg \sigma \sum_{i=0}^{d-1} (-1)^i \sum_{v \triangleleft \sigma \backslash \sigma_i} f(v\sigma \backslash \sigma_i). \tag{6.5}$$

Several properties of the Laplacians are independent from the inner product chosen for $\Omega^\bullet$. For example, the spectrum is real and non-negative, and the spectral gap $\lambda(X) = \min \operatorname{Spec}\left(\Delta^+|_{Z_{d-1}}\right)$ vanishes iff X has nontrivial $(d-1)$-th homology. Note that while zero is still obtained precisely on closed forms, i.e. $\ker \Delta^+ = Z^{d-1}$, these are not the same as for the non-normalized Laplacian. For example, B^0 consists of the scalar multiples of the degree function, and not of the constant functions. Let us define the following variant of the spectral gap:

Definition 6.5. The *essential gap* of X, denoted $\widetilde{\lambda}(X)$, is

$$\widetilde{\lambda}(X) = \min \operatorname{Spec}\left(\Delta^+|_{B_{d-1}}\right) = \min \operatorname{Spec}\left(\Delta|_{B_{d-1}}\right).$$

While λ vanishes iff X has nontrivial $(d-1)$-homology, $\widetilde{\lambda}$ never vanishes, as $B_{d-1} = \left(Z^{d-1}\right)^{\perp} = (\ker \Delta^+)^{\perp}$. If the $(d-1)$-homology is trivial then $B_{d-1} = Z_{d-1}$ implies $\lambda = \widetilde{\lambda}$, hence $\widetilde{\lambda}$ is only of additional interest when there is nontrivial homology. In a disconnected graph $\widetilde{\lambda}$ controls the mixing rate of the random walk as λ does for a connected graph, and we will see that the same happens in higher dimension (see (6.10)).

In order to understand the other extremity of $\operatorname{Spec} \Delta^+$ we introduce the following definition:

Definition 6.6. A *disorientation* of a d-complex X is a choice of orientation X_+^d of its d-cells, so that whenever $\sigma, \sigma' \in X_+^d$ intersect in a $(d-1)$-cell they induce the same orientation on it. If X has a disorientation we say it is *disorientable*.

Remarks. (1) A disorientable 1-complex is precisely a bipartite graph, and thus disorientability should be thought of as a high-dimensional analogue of bipartiteness. Another natural analogue is "$(d+1)$-partiteness": having some partition $A_0, \ldots, A_d$ of V so that every d-cell contains one vertex from each A_i. A $(d+1)$-partite complex is easily seen to be disorientable, but the converse does not hold for $d \geq 2$.

(2) Notice the similarity to the notion of *orientability*: a d-complex is orientable if there is a choice of orientations of its d-cells, so that cells intersecting in a codimension one cell induce *opposite* orientations on it. However, orientability implies that $(d-1)$-cells have degrees at most two, whereas disorientability imposes no such restrictions. Note that a complex can certainly be both orientable and disorientable (e.g. Figure 6.1(a)).

Proposition 6.7. *Let X be a finite complex of dimension d.*

(1) $\operatorname{Spec} \Delta^+(X)$ *is the union of* $\operatorname{Spec} \Delta^+(X_i)$ *where X_i are the $(d-1)$-components of X.*

(2) The spectrum of $\Delta^+ = \Delta^+(X)$ is contained in $[0, d+1]$.

(3) Zero is achieved on the closed $(d-1)$-forms, Z^{d-1}.

(4) If X is $(d-1)$-connected, then $d+1$ is in the spectrum iff X is disorientable, and is achieved on the boundaries of disorientations (see (6.6)*).*

Proof. (1) follows from the observation that Δ^+ decomposes w.r.t. the decomposition $\Omega^{d-1}(X) = \bigoplus_i \Omega^{d-1}(X_i)$, as $\sim$-neighbors are necessarily in the same $(d-1)$-component. We already know *(3)*, and the fact that the spectrum is nonnegative. Now, assume that $\Delta^+ f = \lambda f$. Choose $\sigma \in X^{d-1}$ which maximize $\frac{|f(\sigma)|}{\deg(\sigma)}$. By (6.4),

$$\lambda f(\sigma) = \left(\Delta^+ f\right)(\sigma) = f(\sigma) - \sum_{\sigma' \sim \sigma} \frac{f(\sigma')}{\deg(\sigma')}$$

and therefore

$$|\lambda f(\sigma)| \le |f(\sigma)| + \sum_{\sigma'\sim\sigma} \frac{|f(\sigma')|}{\deg(\sigma')} \le (d+1)|f(\sigma)|,$$

(since $\#\{\sigma' \,|\, \sigma' \sim \sigma\} = d \deg \sigma$), hence $\lambda \le d+1$ and *(2)* is obtained. Next, assume that X is $(d-1)$-connected and that X_+^d is a disorientation. Define

$$F(\tau) = \begin{cases} 1 & \tau \in X_+^d \\ -1 & \tau \in X_\pm^d \backslash X_+^d \end{cases}, \tag{6.6}$$

and $f = \partial_d F$. For any $\sigma \in X_\pm^{d-1}$, there exists some vertex v with $v \triangleleft \sigma$ (since X is uniform). Furthermore, by the assumption on X_+^d, if $v \triangleleft \sigma$ and $v' \triangleleft \sigma$ for vertices v, v' then $v\sigma \in X_+^d$ if and only if $v'\sigma \in X_+^d$, and thus

$$f(\sigma) = (\partial_d F)(\sigma) = \sum_{v \triangleleft \sigma} F(v\sigma) = \deg(\sigma) F(\tau)$$

where τ is any d-cell containing σ. If σ and σ' are neighboring $(d-1)$-faces in $X_\pm^{d-1}$, then by definition, for some $\tau \in X_\pm^d$, σ is a face of τ and σ' is a face of $\overline{\tau}$, so that

$$\frac{f(\sigma)}{\deg \sigma} + \frac{f(\sigma')}{\deg \sigma'} = F(\tau) + F(\overline{\tau}) = 0,$$

and consequently for any $\sigma \in X_\pm^{d-1}$

$$\left(\Delta^+ f\right)(\sigma) = f(\sigma) - \sum_{\sigma'\sim\sigma} \frac{f(\sigma')}{\deg(\sigma')} = f(\sigma) - \sum_{\sigma'\sim\sigma} \frac{-f(\sigma)}{\deg(\sigma)} = (d+1) f(\sigma),$$

so that f is a Δ^+-eigenform with eigenvalue $d+1$.

In the other direction, assume that X is $(d-1)$-connected and that $\Delta^+ f = (d+1) f$ for some $f \in \Omega^{d-1}(X) \setminus \{0\}$. Fix some $\widetilde{\sigma} \in X_\pm^{d-1}$ which maximize $\frac{|f(\sigma)|}{\deg \sigma}$, normalize f so that $\frac{|f(\widetilde{\sigma})|}{\deg \widetilde{\sigma}} = 1$, and define

$$F = \frac{\partial_d^* f}{d+1}, \quad X_+^d = \left\{\tau \in X_\pm^d \,\middle|\, F(\tau) > 0\right\}.$$

We have $f = \frac{\Delta^+ f}{d+1} = \frac{\partial_d \partial_d^* f}{d+1} = \partial_d F$ by assumption, and we proceed to show that X_+^d is a disorientation with F the corresponding form as in (6.6). By the definition of Δ^+

$$\deg \widetilde{\sigma} = |f(\widetilde{\sigma})| = \frac{1}{d}\left|\sum_{\sigma\sim\widetilde{\sigma}} \frac{f(\sigma)}{\deg(\sigma)}\right| \le \frac{1}{d}\sum_{\sigma\sim\widetilde{\sigma}} \frac{|f(\sigma)|}{\deg(\sigma)} \le \frac{1}{d}\sum_{\sigma\sim\widetilde{\sigma}} 1 = \deg \widetilde{\sigma},$$

so that $\frac{|f(\sigma)|}{\deg \sigma} = 1$ for every $\sigma \sim \widetilde{\sigma}$. Continuing in this manner, $(d-1)$-connectedness implies that $\frac{|f(\sigma)|}{\deg \sigma} \equiv 1$ on all $X_\pm^d$. Using again the definition of Δ^+, for any σ in $X_\pm^d$

$$\frac{f(\sigma)}{\deg \sigma} = -\frac{1}{\deg \sigma \cdot d} \sum_{\sigma'\sim\sigma} \frac{f(\sigma')}{\deg(\sigma')}.$$

Since the r.h.s is an average over terms whose absolute value is that of the l.h.s this gives $\frac{f(\sigma')}{\deg \sigma'} = -\frac{f(\sigma)}{\deg \sigma}$

whenever $\sigma \sim \sigma'$, hence

$$F(\tau) = \frac{1}{d+1} \sum_{i=0}^{d} \frac{(-1)^i f(\tau \backslash \tau_i)}{\deg(\tau \backslash \tau_i)} = \frac{f(\tau \backslash \tau_0)}{\deg(\tau \backslash \tau_0)}$$

is always of absolute value one. Furthermore, if $\tau, \tau' \in X_{\pm}^{d}$ intersect in a face σ and induce opposite orientations on it, then $\tau = v\sigma$ and $\tau' = \overline{v'\sigma}$ for some vertices v, v', hence

$$F(\tau) = F(v\sigma) = \frac{f(\sigma)}{\deg \sigma} = F(v'\sigma) = -F\left(\overline{v'\sigma}\right) = -F(\tau')$$

which concludes the proof. $\square$

6.3 Walk and spectrum

By equations (6.1) and (6.4), the transition operator $A = A(X,p)$ of the $(d-1)$-walk defined in §6.1 relates to the normalized Laplacian by

$$A = \frac{p(d-1)+1}{d} \cdot I - \frac{1-p}{d} \cdot \Delta^{+}, \tag{6.7}$$

so that the expectation process is given by:

$$\mathcal{E}_n^{\sigma_0} = A^n \mathcal{E}_0^{\sigma_0} = \left(\frac{p(d-1)+1}{d} \cdot I - \frac{1-p}{d} \cdot \Delta^{+} \right)^n \mathcal{E}_0^{\sigma_0}.$$

This gives the asymptotic behavior of the expectation process:

Proposition 6.8. *Let $A = A(X,p)$ denote the transition operator of the p-lazy $(d-1)$-walk on X. Then:*

(1) The spectrum of A is contained in $\left[2p-1, \frac{p(d-1)+1}{d}\right]$, with $2p-1$ achieved by disorientations, and $\frac{p(d-1)+1}{d}$ by closed forms (Z^{d-1}).

(2) The expectation process satisfies

$$\frac{1}{\sqrt{K_{d-2}K_{d-1}}} \left(\frac{p(d-1)+1}{d} \right)^n \leq \|\mathcal{E}_n^{\sigma_0}\| \leq \max \left(|2p-1|, \frac{p(d-1)+1}{d} \right)^n \tag{6.8}$$

where K_j is the maximal degree of a j-cell in X.

Proof. *(1)* follows trivially from (6.7) and Proposition 6.7. The upper bound in *(2)* follows from *(1)* by $\mathcal{E}_n^{\sigma_0} = A^n \mathcal{E}_0^{\sigma_0}$ and $\|\mathcal{E}_0^{\sigma_0}\| = \|\mathbb{1}_{\sigma_0}\| = \frac{1}{\sqrt{\deg \sigma_0}} \leq 1$. For the lower bound, let v be a vertex in σ_0, and $\sigma_0, \ldots, \sigma_k$ the $(d-1)$-cells containing $\sigma_0 \backslash v$. Define $f = \partial_d^* \mathbb{1}_{\sigma_0 \backslash v} = \sum_{i=0}^{k} \deg \sigma_i \cdot \mathbb{1}_{\sigma_i}$, so that $f \in Z^{d-1}$ and $\|f\|^2 = \sum_{i=0}^{k} \deg \sigma_i \leq K_{d-2}K_{d-1}$. Since Δ^{+} decomposes w.r.t. the orthogonal sum

$\Omega^{d-1} = Z^{d-1} \oplus B_{d-1}$ so does $A = \frac{p(d-1)+1}{d} \cdot I - \frac{1-p}{d} \cdot \Delta^{+}$, hence by *(1)*

$$\begin{aligned}\|\mathcal{E}_n^{\sigma_0}\| = \|A^n \mathbb{1}_{\sigma_0}\| &\geq \left(\tfrac{p(d-1)+1}{d}\right)^n \|\mathbb{P}_{Z^{d-1}}(\mathbb{1}_{\sigma_0})\| \geq \left(\tfrac{p(d-1)+1}{d}\right)^n \left|\left\langle \tfrac{f}{\|f\|}, \mathbb{1}_{\sigma_0}\right\rangle\right| \\ &= \left(\tfrac{p(d-1)+1}{d}\right)^n \frac{|f(\sigma_0)|}{\|f\| \deg \sigma_0} \geq \frac{1}{\sqrt{K_{d-2}K_{d-1}}} \left(\tfrac{p(d-1)+1}{d}\right)^n.\end{aligned}$$

□

This proposition leads to the connection between the asymptotic behavior of the $(d-1)$-walk and the homology and spectrum of the complex:

Theorem 6.9. *Let $\widetilde{\mathcal{E}}_n^{\sigma}$ be the normalized expectation process associated with the p-lazy $(d-1)$-walk on X starting from σ (see Definitions 6.1, 6.3). Then $\widetilde{\mathcal{E}}_\infty^{\sigma} = \lim_{n\to\infty} \widetilde{\mathcal{E}}_n^{\sigma}$ exists and satisfies the following:*

(1) If $\frac{d-1}{3d-1} < p < 1$, then $\widetilde{\mathcal{E}}_\infty^{\sigma}$ is exact for every starting point σ if and only if $H_{d-1}(X) = 0$.[†] If furthermore $p \geq \frac{1}{2}$ then

$$\operatorname{dist}\left(\widetilde{\mathcal{E}}_n^{\sigma}, B^{d-1}\right) = O\left(\left(1 - \frac{1-p}{p(d-1)+1}\lambda(X)\right)^n\right). \tag{6.9}$$

(2) More generally, the dimension of $H_{d-1}(X)$ equals the dimension of $\operatorname{Span}\left\{\mathbb{P}_{Z_{d-1}}\left(\widetilde{\mathcal{E}}_\infty^{\sigma}\right) \,\middle|\, \sigma \in X^{d-1}\right\}$.

(3) If $p = \frac{d-1}{3d-1}$ then $\widetilde{\mathcal{E}}_\infty^{\sigma}$ is exact for all σ if and only if X has a trivial $(d-1)$-homology and no disorientable $(d-1)$-components.

(4) More generally, if $\frac{d-1}{3d-1} < p < 1$ then $\widetilde{\mathcal{E}}_\infty^{\sigma}$ is closed, and likewise for $p = \frac{d-1}{3d-1}$, unless X has a disorientable $(d-1)$-component. If $p \geq \frac{1}{2}$ then

$$\operatorname{dist}\left(\widetilde{\mathcal{E}}_n^{\sigma}, Z^{d-1}\right) = O\left(\left(1 - \frac{1-p}{p(d-1)+1}\widetilde{\lambda}(X)\right)^n\right). \tag{6.10}$$

Proof. **Case (*i*) – $\boldsymbol{\frac{d-1}{3d-1} < p < 1}$:** We have $|2p-1| < \frac{p(d-1)+1}{d}$, so that $\|A\| = \max \operatorname{Spec} A = \frac{p(d-1)+1}{d}$. Thus,

$$\operatorname{Spec} A\big|_{B_{d-1}} \subseteq \left[2p-1, \frac{p(d-1)+1}{d}\right) \subseteq (-\|A\|, \|A\|).$$

Since A decomposes w.r.t. $\Omega^{d-1} = B_{d-1} \oplus Z^{d-1}$, and $A\big|_{Z^{d-1}} = \|A\| \cdot I\big|_{Z^{d-1}}$, this means that $\left(\frac{A}{\|A\|}\right)^n$ converges to the orthogonal projection $\mathbb{P}_{Z^{d-1}}$. Now $\widetilde{\mathcal{E}}_n^{\sigma} = \left(\frac{d}{p(d-1)+1}\right)^n \mathcal{E}_n^{\sigma} = \left(\frac{A}{\|A\|}\right)^n \mathcal{E}_0^{\sigma}$, which shows that

$$\widetilde{\mathcal{E}}_\infty^{\sigma} = \mathbb{P}_{Z^{d-1}}\left(\widetilde{\mathcal{E}}_0^{\sigma}\right) = \mathbb{P}_{Z^{d-1}}(\mathcal{E}_0^{\sigma}) = \mathbb{P}_{Z^{d-1}}(\mathbb{1}_{\sigma}). \tag{6.11}$$

In particular $\widetilde{\mathcal{E}}_\infty^{\sigma}$ is closed, so that if the homology of X is trivial then it is exact. On the other hand, assume that $\widetilde{\mathcal{E}}_\infty^{\sigma}$ is exact for all σ: then

$$\widetilde{\mathcal{E}}_\infty^{\sigma} = \mathbb{P}_{Z^{d-1}}(\mathcal{E}_0^{\sigma}) = \mathbb{P}_{Z^{d-1}}(\mathbb{1}_{\sigma}) = \mathbb{P}_{B^{d-1}}(\mathbb{1}_{\sigma}) + \mathbb{P}_{\mathcal{H}^{d-1}}(\mathbb{1}_{\sigma})$$

[†] Note that the first value of p for which the homology can be studied via the walk in every dimension is $p = \frac{1}{3}$.

so that $\mathbb{P}_{\mathcal{H}^{d-1}}(\mathbb{1}_\sigma)=0$ by (2.2). As $\{\mathbb{1}_\sigma\}$ span Ω^{d-1}, this shows that $H_{d-1}\cong\mathcal{H}^{d-1}=0$. To further understand the dimension of the homology, observe that

$$\operatorname{Span}\left\{\mathbb{P}_{Z_{d-1}}\left(\widetilde{\mathcal{E}}_\infty^\sigma\right)\,\middle|\,\sigma\in X^{d-1}\right\}=\mathcal{H}^{d-1}(X),$$

which follows from

$$\mathbb{P}_{Z_{d-1}}\left(\widetilde{\mathcal{E}}_\infty^\sigma\right)=\mathbb{P}_{Z_{d-1}}\left(\mathbb{P}_{Z^{d-1}}(\mathbb{1}_\sigma)\right)=\mathbb{P}_{\mathcal{H}^{d-1}}(\mathbb{1}_\sigma).$$

If $p\geq\frac{1}{2}$ then we know not only that $\|A\|=\max\operatorname{Spec}A$ but also that $\left\|A\big|_{Z_{d-1}}\right\|=\max\operatorname{Spec}\left(A\big|_{Z_{d-1}}\right)$, which allows us to say more: In this case A is positive semidefinite, so that (6.10) follows by

$$\begin{aligned}\left\|\left(\frac{d}{p(d-1)+1}A\right)^n-\mathbb{P}_{Z^{d-1}}\right\|&=\left\|\left(\frac{d}{p(d-1)+1}A\big|_{B_{d-1}}\right)^n\right\|\\&=\left\|\left(I-\frac{1-p}{p(d-1)+1}\cdot\Delta^+\right)^n\big|_{B_{d-1}}\right\|=\left(1-\frac{1-p}{p(d-1)+1}\widetilde{\lambda}(X)\right)^n,\end{aligned}$$

which gives (6.9) as well when the homology is trivial.

Case *(ii)* – $p=\frac{d-1}{3d-1}$: Now, $|2p-1|=\frac{p(d-1)+1}{d}=\|A\|$. If X has no disorientable $(d-1)$-components then again $\operatorname{Spec}A\big|_{B_{d-1}}\subseteq(-\|A\|,\|A\|)$, which gives (6.11), and everything is as before. On the other hand, let us assume that $\widetilde{\mathcal{E}}_\infty^\sigma$ is closed for all σ. Denoting by Ω_λ^{d-1} the λ-eigenspace of A, now $\left(\frac{d}{p(d-1)+1}A\right)^{2n}$ converges to $\mathbb{P}_{Z^{d-1}}+\mathbb{P}_{\Omega_{2p-1}^{d-1}}$ (Δ^+ is diagonalizable and consequently so is A). Since $\widetilde{\mathcal{E}}_\infty^\sigma$ is closed this shows that $\mathbb{P}_{\Omega_{2p-1}^{d-1}}(\mathbb{1}_\sigma)=0$, and consequently that $\Omega_{2p-1}^{d-1}=0$, i.e. X has no disorientable $(d-1)$-components. □

Remarks.

(1) The study of complexes via $(d-1)$-walk gives a conceptual reason to the fact that the high-dimensional case is harder than that of graphs: while graphs are studied by the evolution of probabilities, analogue properties of high-dimensional complexes are reflected in the expectation process. As the latter is given by the difference of two probability vectors, it is much harder to analyze. Several examples of this appear in the open questions in §9.

(2) In order to study the connectedness of a graph it is enough to observe the walk starting at one vertex. If $\mathbf{p}_\infty^{v_0}$ is not exact (i.e. not proportional to the degree function) for even one v_0, then the graph is necessarily disconnected. In general dimension, however, this is not enough: there are complexes (even $(d-1)$-connected ones!) with nontrivial $(d-1)$-homology, such that $\widetilde{\mathcal{E}}_\infty^\sigma$ is exact for a carefully chosen σ.

(3) If one starts the process with a general initial distribution $\mathbf{p}_0$ instead of the Dirac probability $\mathbb{1}_\sigma$, then Theorem 6.9 holds for the corresponding expectation process (i.e. $\mathcal{E}_0(\sigma)=\mathbf{p}_0(\sigma)-\mathbf{p}_0(\overline{\sigma})$, $\mathcal{E}_{n+1}=A\mathcal{E}_n$). Furthermore, in these settings a disorientable component corresponds to a distribution for which $\widetilde{\mathcal{E}}_n$ is 2-periodic for $p=\frac{d-1}{3d-1}$ (see Figure 6.1(a)); a nontrivial homology corresponds to a distribution which induces a stationery non-exact $\widetilde{\mathcal{E}}_n$ for $p\geq\frac{d-1}{3d-1}$ (see Figure 6.1(b)).

(a) (b)

Figure 6.1: Two distributions on the edges of 2-complexes (the orientations drawn have uniform probability, and their inverses probability zero). (a) is a distribution for which $\widetilde{\mathcal{E}}_n = \left(\frac{5}{3}\right)^n \mathcal{E}_n$ is 2-periodic under the $\frac{1}{5}$-lazy walk; (b) is a distribution for which $\widetilde{\mathcal{E}}_n$ is stable and non exact (under the p-lazy walk, for any $p > \frac{1}{5}$).

7 Infinite complexes

In this section we move on to infinite complexes. We use the weighted inner products and normalized Laplacians introduced in §6, but some necessary adjustments are due: they are explained in the next two sections.

7.1 Infinite graphs

Recall that for a finite graph $G = (V, E)$, we observed $\Delta^+ = \Delta^+(G)$, and defined

$$\lambda(G) = \min \operatorname{Spec} \Delta^+\big|_{(B^0)^\perp} = \min \operatorname{Spec} \Delta^+\big|_{Z_0}.$$

In contrast, when G is an infinite graph (i.e. $|V| = \infty$) one usually restricts his attention to $L^2(V)$ and defines

$$\lambda(G) = \min \operatorname{Spec} \Delta^+\big|_{L^2(V)}. \tag{7.1}$$

Here there is no restriction to Z_0, nor to $\left(B^0\right)^\perp$. These two spaces, which coincide in the finite dimensional case, since

$$Z_0 = \ker \partial_0 = (\operatorname{im} \partial_0^*)^\perp = \left(B^0\right)^\perp, \tag{7.2}$$

fail to do so in the infinite settings. In fact, Z_0 is not even defined, as $(\partial_0 f)(\varnothing) = \sum_{v\in V} f(v)$ has no meaning for general $f \in L^2(V)$. One can observe $B^0 = \operatorname{im} \delta_0$, taking (6.3) as the definition of δ_0 (as ∂_0 is not defined). With this definition, B^0 consists of the scalar multiples of the degree function. Since these are never in $L^2(V)$ (assuming that there are no isolated vertices), we have $B^0 = 0$ and $\left(B^0\right)^\perp = L^2(V)$, justifying (7.1). Another thing which fails here is the chain complex property $\partial_0\partial_1 = 0$: there may exist $f \in \Omega^1(G)$ such that $\partial_0\partial_1 f$ is defined and nonzero. For example, take $V = \mathbb{Z}$, $E = \{\{i, i+1\} \,|\, i \in \mathbb{Z}\}$, and $f([i, i+1]) = \begin{cases} 0 & i < 0 \\ 1 & 0 \le i \end{cases}$. Here $\partial_1 f = \mathbb{1}_0$, and thus $(\partial_0\partial_1 f)(\varnothing) = 1$. If G is transient, e.g. the $\mathbb{Z}^3$ graph, or a k-regular tree with $k \ge 3$, then there are even such f in L^2 - see §7.8.

7.2 Infinite complexes of general dimension

For a complex X of dimension d, and $-1 \le k \le d$, we denote

$$\Omega^k_{L^2} = \Omega^k_{L^2}(X) = \left\{ f \in \Omega^k(X) \,\middle|\, \|f\|^2 < \infty \right\} \subseteq \Omega^k(X),$$

where we recall that

$$\|f\|^2 = \sum_{\sigma\in X^k} w(\sigma) f(\sigma)^2 = \begin{cases} \sum_{\sigma\in X^k} f(\sigma)^2 & k \ne d-1 \\ \sum_{\sigma\in X^k} \frac{f(\sigma)^2}{\deg \sigma} & k = d-1 \end{cases}.$$

Whenever referring to infinite complexes, the domain of all operators (i.e. $\partial, \delta, \Delta^+, \Delta^-, \Delta$) is assumed to be $\Omega^k_{L^2}$, unless explicitly stated that we are interested in Ω^k.

Let us examine these operators. We shall always assume that the $(d-1)$-cells in X have globally bounded degrees, which ensures that the boundary and coboundary operators $\partial_d : \Omega^d \to \Omega^{d-1}$, $\delta_d : \Omega^{d-1} \to \Omega^d$ are defined, bounded, and adjoint to one another, so that $\Delta^+ = \partial_d \delta_d = \partial_d \partial_d^*$ is bounded and self-adjoint. We do not assume that the degrees in other dimensions are bounded, as this would rule out infinite graphs, for example. This means that in general δ_k does not take $\Omega_{L^2}^{k-1}$ into $\Omega_{L^2}^k$ but only to Ω^k, and ∂_k need not even be defined. In particular, one cannot always define Δ^-.

The cochain property $\delta_k \delta_{k-1} = 0$ always holds, whereas in general $\partial_{k-1}\partial_k(f)$ can be defined and nonzero for some $f \in \Omega_{L^2}^k$. If the degrees of $(k-1)$-cells are bounded, then δ_k and ∂_k are bounded and $\delta_k = \partial_k^*$. Thus, if the degrees of $(k-1)$-cells and $(k-2)$-cells are globally bounded one has $\partial_{k-1}\partial_k = (\delta_k \delta_{k-1})^* = 0^* = 0$ as well.

In contrast with infinite graphs, an infinite d-complex may have $(d-2)$-cells of finite degree, so that the image of δ_{d-1} may contain L^2-coboundaries. For example, if v is a vertex of finite degree in an infinite triangle complex, then the "star" $\delta_1 \mathbb{1}_v$ is an L^2-coboundary. We denote by B^{d-1} the L^2-coboundaries, i.e. $B^{d-1} = \operatorname{im} \delta_{d-1} \cap \Omega_{L^2}^{d-1}$. In order to avoid trivial zeros in the spectrum of Δ^+, we define $Z_{d-1} = \left(B^{d-1}\right)^\perp$ (the orthogonal complement in $\Omega_{L^2}^{d-1}$), and

$$\lambda(X) = \min \operatorname{Spec} \Delta^+\big|_{Z_{d-1}}.$$

We stress out that Z_{d-1} is not necessarily the kernel of ∂_{d-1} (which is not even defined in general). If the $(d-2)$-degrees are globally bounded then ∂_{d-1} is defined and dual to δ_{d-1}, and this gives inclusion in one direction:

$$Z_{d-1} = \left(B^{d-1}\right)^\perp = \left(\operatorname{im} \delta_{d-1}\right)^\perp \subseteq \ker \partial_{d-1}. \tag{7.3}$$

For finite complexes there is an equality here (as in (7.2)) due to dimension considerations.

In infinite graphs we had $B^0 = 0$, $Z_0 = \Omega_{L^2}^0 = L^2(V)$ and $\lambda = \min \operatorname{Spec} \Delta^+\big|_{L^2(V)}$. The following lemma shows that this happens whenever all $(d-2)$-cells are of infinite degree:

Lemma 7.1. *If X is a d-complex whose $(d-2)$-cells are all of infinite degree, then $B^{d-1} = 0$ and thus $\lambda(X) = \min \operatorname{Spec} \Delta^+$.*

Proof. Let $f \in \Omega^{d-2}$ be such that $\delta_{d-1} f \in \Omega_{L^2}^{d-1} \setminus \{0\}$. Choose $\tau \in X_\pm^{d-2}$ for which $f(\tau) > 0$, and let $\{\sigma_i\}_{i=1}^\infty$ be a sequence of $(d-1)$-cells containing τ. Since $\sum_{i=1}^\infty (\delta_{d-1} f)^2 (\sigma_i) \le \|\delta_{d-1} f\|^2 < \infty$, for infinitely many i we have $|(\delta_{d-1} f)(\sigma_i)| \le \frac{f(\tau)}{2}$. Since τ contributes $f(\tau)$ to $(\delta_{d-1} f)(\sigma_i)$, one of the other faces of σ_i must be of absolute value at least $\frac{f(\tau)}{2(d-1)}$. Since these faces are all different $(d-2)$-cells (if $\sigma_i \cap \sigma_j$ contains τ and another $(d-2)$-cell, then $\sigma_i = \sigma_j$), we have $\|f\| = \infty$. $\square$

7.3 Example - arboreal complexes

Definition 7.2. We say that a d-complex is *arboreal* if it is $(d-1)$-connected, and has no simple "d-loops". That is, there are no non-backtracking closed chains of d-cells, $\sigma_0, \sigma_1, \ldots, \sigma_n = \sigma_0$ s.t. $\dim(\sigma_i \cap \sigma_{i+1}) = d-1$ (σ_i and σ_{i+1} are adjacent) and $\sigma_i \neq \sigma_{i+2}$ (the chain is non-backtracking).

For $d = 1$, these are simply trees. As in trees, there is a unique k-regular arboreal d-complex for every $k \in \mathbb{N}$, and we denote it by T_k^d. It can be constructed as follows: start with a d-cell, and attach to each of its faces $k-1$ new d-cells. Continue by induction, adding to each face of a d-cell in the boundary $k-1$ new d-cells at every step. For example, the 2-regular arboreal triangle complex T_2^2 can be thought of as an ideal triangulation of the hyperbolic plane, depicted in Figure 7.1.

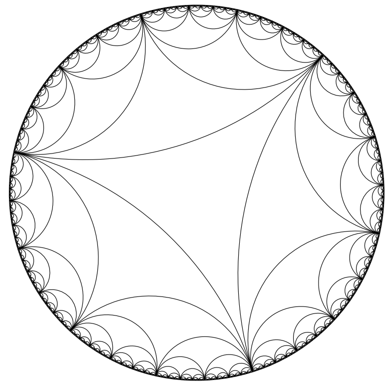


Figure 7.1: The 2-regular arboreal triangle complex T_2^2.

Theorem 7.3. *The spectrum of the non-lazy transition operator on the k-regular arboreal d-complex is*

$$\operatorname{Spec} A\left(T_k^d, 0\right) = \begin{cases} \left[\frac{1-d-2\sqrt{d(k-1)}}{kd}, \frac{1-d+2\sqrt{d(k-1)}}{kd}\right] \cup \left\{\frac{1}{d}\right\} & 2 \le k \le d \\ \left[\frac{1-d-2\sqrt{d(k-1)}}{kd}, \frac{1-d+2\sqrt{d(k-1)}}{kd}\right] & d < k. \end{cases} \tag{7.4}$$

Remarks.

(1) For $d = 1$ this gives the spectrum of the k-regular tree, which is a famous result of Kesten [Kes59]:

$$\operatorname{Spec} A\left(T_k^1, 0\right) = \left[-\frac{2\sqrt{k-1}}{k}, \frac{2\sqrt{k-1}}{k}\right].$$

(2) Since for $2 \le k \le d$ the value $\frac{1}{d}$ is an isolated value of the spectrum of T_k^d, it follows that it is in fact an eigenvalue. This is a major difference from the case of graphs, where the value $\frac{1}{d} = 1$ cannot be an eigenvalue for infinite graphs. This phenomena will play a crucial role in the counterexample for the Alon-Boppana theorem in general dimension (see §7.5-7.6).

(3) Another phenomena which does not occur in the case of graphs, is that in the region $2 \le k \le d$ the spectrum expands as k becomes larger. The spectrum is maximal (as a set) for $k = d+1$, where $\operatorname{Spec} A\left(T_{d+1}^d, 0\right) = \left[-\frac{3d-1}{d(d+1)}, \frac{1}{d}\right]$, merging with the isolated eigenvalue which appears for smaller k.

(4) The spectra of the Laplacian $\Delta^+ = \Delta^+\left(T_k^d\right)$, and of the p-lazy transition operator $A_p = A\left(T_k^d, p\right)$, are obtained from (7.4) using $\Delta^+ = I - d \cdot A$ and $A_p = p \cdot I + (1-p) \cdot A$.

In order to prove Theorem 7.3 we will need the following lemma, for the idea of which we are indebted to Jonathan Breuer:

Lemma 7.4. *Let X be any set, and $L^2(X)$ the Hilbert space of complex functions of finite L^2-norm on X (with respect to the counting measure). Let A be a bounded self adjoint operator on $L^2(X)$, and $a < b \in \mathbb{R}$, such that the following hold:*

(1) For every $x \in X$ and $a \le \lambda \le b$, there exists $\psi_x^\lambda \in L^2(X)$ such that $(A - \lambda I)\, \psi_x^\lambda = \mathbb{1}_x$.

(2) The integral $\int_a^b c(\lambda)^2\, d\lambda$ is finite, where $c(\lambda) = \sup\limits_{x\in X} \left\|\psi_x^\lambda\right\|$.

Then $(a,b) \cap \operatorname{Spec}(A) = \varnothing$.

Proof. We show that $\mathbb{P}_{[a,b]}$, the spectral projection of A on the interval $[a,b]$, is zero, and the conclusion $(a,b) \cap \operatorname{Spec}(A) = \varnothing$ follows by the spectral theorem. Stone's formula states that

$$(s)\lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_a^b \left[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\right] d\lambda = \mathbb{P}_{(a,b)} + \frac{1}{2}\mathbb{P}_{\{a,b\}}$$

where $\mathbb{P}_{(a,b)}$ and $\mathbb{P}_{\{a,b\}}$ the spectral projections of A on (a,b) and $\{a,b\}$ respectively, and $(s)\lim$ denotes a limit in the strong sense. Denoting $\mathbb{P} = \mathbb{P}_{(a,b)} + \frac{1}{2}\mathbb{P}_{\{a,b\}}$, this gives for every $x \in X$

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_a^b \left\langle \left[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\right] \mathbb{1}_x, \mathbb{1}_x \right\rangle d\lambda = \langle \mathbb{P}\mathbb{1}_x, \mathbb{1}_x\rangle$$

Evaluating the right hand side we get

$$\begin{aligned}
\langle \mathbb{P}\mathbb{1}_x, \mathbb{1}_x\rangle &= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_a^b \left\langle \left[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\right] \mathbb{1}_x, \mathbb{1}_x \right\rangle d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_a^b \left\langle \left[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\right] (A-\lambda)\,\psi_x^\lambda, (A-\lambda)\,\psi_x^\lambda \right\rangle d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_a^b \left\langle (A-\lambda+i\varepsilon)^{-1} \left[A-\lambda+i\varepsilon - A+\lambda+i\varepsilon\right] (A-\lambda-i\varepsilon)^{-1} (A-\lambda)^2\, \psi_x^\lambda, \psi_x^\lambda \right\rangle d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \left\langle \left((A-\lambda)^2+\varepsilon^2\right)^{-1} (A-\lambda)^2\, \psi_x^\lambda, \psi_x^\lambda \right\rangle d\lambda \\
&\le \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \left\| \left((A-\lambda)^2+\varepsilon^2\right)^{-1} (A-\lambda)^2 \right\| c(\lambda)^2\, d\lambda.
\end{aligned}$$

Defining $f_{\varepsilon,\lambda}(t) = \frac{(t-\lambda)^2}{(t-\lambda)^2+\varepsilon^2}$, we have $|f_{\varepsilon,\lambda}(t)| \le 1$ for every $t,\lambda \in \mathbb{R}$ and $\varepsilon > 0$, and thus $\|f_{\varepsilon,\lambda}(A)\| \le 1$. Therefore, using *(2)*, the last limit above is zero. Consequently, for any $x, y \in X$

$$|\langle \mathbb{P}\mathbb{1}_x, \mathbb{1}_y\rangle| = |\langle \mathbb{P}\mathbb{1}_x, \mathbb{P}\mathbb{1}_y\rangle| \le \langle \mathbb{P}\mathbb{1}_x, \mathbb{P}\mathbb{1}_x\rangle^{\frac{1}{2}} \cdot \langle \mathbb{P}\mathbb{1}_y, \mathbb{P}\mathbb{1}_y\rangle^{\frac{1}{2}} = 0.$$

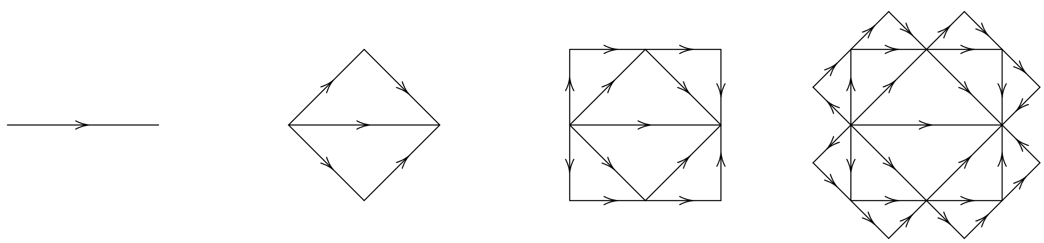
It follows that for general $f \in L^2(X)$

$$\langle \mathbb{P}f, f\rangle = \left\langle \mathbb{P}\left(\sum_{x\in X} f(x)\mathbb{1}_x\right), \sum_{y\in X} f(y)\mathbb{1}_y \right\rangle = \sum_{x,y\in X} f(v)f(w)\,\langle \mathbb{P}\mathbb{1}_x, \mathbb{1}_y\rangle = 0,$$

which implies that $\mathbb{P} = 0$, hence also $\mathbb{P}_{(a,b)}$ and $\mathbb{P}_{\{a,b\}}$, and therefore also $\mathbb{P}_{[a,b]}$. $\square$

Proof of Theorem 7.3. Let $X = T_k^d$, and $\Lambda_\pm = \frac{1-d\pm 2\sqrt{d(k-1)}}{kd}$. The proof is separated into two parts. First we prove that every $\Lambda_- \leq \lambda \leq \Lambda_+$, and also $\lambda = \frac{1}{d}$ when $k \leq d$, is in the spectrum, by exhibiting an appropriate eigenform or an approximate one. In the second part we use Lemma 7.4 to prove that there are no other points in the spectrum.

Define an orientation X_+^{d-1} as follows: choose an arbitrary $(d-1)$-cell $\sigma_0 \in X_\pm^{d-1}$ and place it in X_+^{d-1}. Then add to X_+^{d-1} all the $k \cdot d$ neighbors of σ_0. Next, for every neighbor τ of the recently added $k \cdot d$ cells, add τ to X_+^{d-1}, unless τ or $\overline{\tau}$ is already there. Continue expanding in this manner, adding at each stage the neighbors of the last "layer" which are further away from the starting cell σ_0. Apart from orientation, this process gives X_+^{d-1} a layer structure: $\{\sigma_0\}$ is the 0^{th} layer, its neighbors the 1^{st} layer, and so on. We denote by $S_n(X,\sigma_0)$ the n^{th} layer, and also write $B_n(X,\sigma_0) = \bigcup_{k\leq n} S_k(X,\sigma_0)$ for the "n^{th} ball" around σ_0. Figure 7.2 demonstrates this for the first four layers of T_2^2.



$B_0\left(T_2^2,\sigma_0\right)$ $B_1\left(T_2^2,\sigma_0\right)$ $B_2\left(T_2^2,\sigma_0\right)$ $B_3\left(T_2^2,\sigma_0\right)$

Figure 7.2: The orientation at the zeroth, first, second, and third layers of $X = T_2^2$.

We shall study X_+^{d-1}-spherical forms, i.e. forms in $\Omega^{d-1}(X)$ which are constant on each layer of X_+^{d-1}. For such a form f, we abuse notation and write $f(n)$ for the value of f on the cells in the n^{th} layer of X_+^{d-1}. As in regular trees, if one allows forms which are not in L^2, then for every $\lambda \in \mathbb{R}$ there is a unique (up to a constant) X_+^{d-1}-spherical eigenform f with eigenvalue λ. This form is given explicitly by

$$f(n) = \left(\frac{\lambda-\alpha_-}{\alpha_+-\alpha_-}\right)\cdot\alpha_+^n + \left(\frac{\alpha_+-\lambda}{\alpha_+-\alpha_-}\right)\cdot\alpha_-^n,$$

where

$$\alpha_\pm = \frac{d-1+dk\lambda \pm \sqrt{(d-1+dk\lambda)^2 - 4d(k-1)}}{2d(k-1)}, \tag{7.5}$$

except for the case $\alpha_+ = \alpha_-$, which happens when $\lambda \in \{\Lambda_-,\Lambda_+\}$. In this case f is given by

$$f(n) = (1-n)\left(\frac{(d-1)+dk\lambda}{2d(k-1)}\right)^n + \lambda n\left(\frac{(d-1)+dk\lambda}{2d(k-1)}\right)^{n-1},$$

but this will not concern us as the spectrum is closed, and it is therefore enough to show that (Λ_-,Λ_+) is contained in it to deduce this for $[\Lambda_-,\Lambda_+]$.

The term inside the root in (7.5) is negative for $\Lambda_- < \lambda < \Lambda_+$, hence in this case $|\alpha_+| = |\alpha_-| = \frac{1}{\sqrt{d(k-1)}}$. We claim the following: for any $\Lambda_- < \lambda < \Lambda_+$ there exist $0 < c_1 < c_2 < \infty$ (which depend on λ) such that

(1) For all $n \in \mathbb{N}$,

$$|f(n)| \leq c_2 \left(\frac{1}{\sqrt{d(k-1)}}\right)^n. \tag{7.6}$$

(2) For infinitely many $n \in \mathbb{N}$,

$$c_1 \left(\frac{1}{\sqrt{d(k-1)}}\right)^n \leq |f(n)|. \tag{7.7}$$

Indeed, (1) follows from $|f(n)| \leq \left[\left|\frac{\lambda-\alpha_-}{\alpha_+-\alpha_-}\right| + \left|\frac{\alpha_+-\lambda}{\alpha_+-\alpha_-}\right|\right] \left(\frac{1}{\sqrt{d(k-1)}}\right)^n$ (as $\alpha_+ \neq \alpha_-$ for $\Lambda_- < \lambda < \Lambda_+$). Next, denote $\gamma = \frac{\lambda-\alpha_-}{\alpha_+-\alpha_-}$ and observe that

$$|f(n)| \left[d(k-1)\right]^{\frac{n}{2}} = \left|\gamma\alpha_+^n + \overline{\gamma}\alpha_-^n\right| \left[d(k-1)\right]^{\frac{n}{2}} = 2\Re\left(\gamma\left(\alpha_+\sqrt{d(k-1)}\right)^n\right).$$

If (2) fails, then $|f(n)| \left[d(k-1)\right]^{\frac{n}{2}} \overset{n\to\infty}{\longrightarrow} 0$. Since $\left|\alpha_+\sqrt{d(k-1)}\right| = 1$, this means that $n \arg \alpha_+ \overset{n\to\infty}{\longrightarrow} \frac{\pi}{2} - \arg\gamma \pmod{\pi}$, hence $\alpha_+ \in \mathbb{R}$, which is false.

Even though f is not in $\Omega_{L^2}^{d-1}(X)$ it induces a natural sequence of approximate eigenforms:

$$f_n(\sigma) = \begin{cases} f(k) & \sigma \in S_k(X, \sigma_0) \text{ and } k \leq n \\ -f(k) & \overline{\sigma} \in S_k(X, \sigma_0) \text{ and } k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

To see this, observe that $(A_0 - \lambda) f = 0$, and that f_n coincides with f on $B_n(X, \sigma_0)$ for $k \leq n$ and vanishes on $\left(T_d^k\right)^{d-1} \setminus B_n(X, \sigma_0)$. It follows that $(A_0 - \lambda) f_n$ is supported on $S_n(X, \sigma_0) \cup S_{n+1}(X, \sigma_0)$, and by $|S_n(X, \sigma_0)| = d^n k(k-1)^{n-1}$, the definition of A_0, and (7.6)

$$\begin{aligned}
\frac{\|(A_0 - \lambda) f_n\|^2}{\|f_n\|^2} &= \frac{|S_n(X, \sigma_0)| \left(\frac{1}{dk}\left[f(n-1) - (d-1) f(n)\right] - \lambda f(n)\right)^2 + |S_{n+1}(X, \sigma_0)| \left(\frac{1}{dk} f(n)\right)^2}{\sum_{j=0}^n |S_j(X, \sigma_0)| f^2(j)} \\
&= \frac{d^n k(k-1)^{n-1} \cdot \left(-\frac{k-1}{k} f(n+1)\right)^2 + d^{n+1} k(k-1)^n \left(\frac{1}{dk} f(n)\right)^2}{f^2(0) + \sum_{j=1}^n d^j k(k-1)^{j-1} f^2(j)} \\
&= \frac{d^n k^{-1}(k-1)^{n+1} f(n+1)^2 + d^{n-1} k^{-1} (k-1)^n f(n)^2}{f^2(0) + \sum_{j=1}^n d^j k(k-1)^{j-1} f^2(j)} \\
&\leq \frac{\frac{2c_2^2}{dk}}{f^2(0) + \frac{k}{k-1}\sum_{j=1}^n \left[d(k-1)\right]^j f(j)^2}.
\end{aligned}$$

By (7.7), the denominator becomes arbitrarily large as n grows, and therefore $\frac{\|(A_0-\lambda) f_n\|^2}{\|f_n\|^2} \to 0$ and $\lambda \in \operatorname{Spec} A_0$.

Turning to the isolated eigenvalues in (7.4), one can easily check that $f(n) = \frac{1}{d^n}$ is an eigenform with eigenvalue $\frac{1}{d}$, and for $2 \leq k \leq d$ it is in L^2. This concludes the first part of the proof.

Next assume that $\lambda \in \left(-1, \frac{1}{d}\right) \setminus [\Lambda_-, \Lambda_+]$. We show that in this case Lemma 7.4 can be applied. Let

σ_0 and X_+^{d-1} be as before, including the layer structure. Define the following X_+^{d-1}-spherical forms:

$$\psi_{\sigma_0}^{\lambda}(n) = \frac{\alpha_+^n}{\alpha_+ - \lambda}, \qquad \varphi_{\sigma_0}^{\lambda}(n) = \frac{\alpha_-^n}{\alpha_- - \lambda}. \tag{7.8}$$

The functions $\psi_{\sigma_0}^{\lambda}$ is defined whenever $\lambda \neq \alpha_+$, which holds unless $\lambda = \frac{1}{d}$ and $k \leq d+1$ (see (7.5)). Similarly, $\varphi_{\sigma_0}^{\lambda}$ is defined unless $\lambda = -1$, or $\lambda = \frac{1}{d}$ and $k \leq d+1$. It is straightforward to verify that

$$(A_0 - \lambda I)\, \psi_{\sigma_0}^{\lambda} = (A_0 - \lambda I)\, \varphi_{\sigma_0}^{\lambda} = \mathbb{1}_{\sigma_0}$$

whenever the functions are defined. For every X_+^{d-1}-spherical form f one has

$$\|f\|^2 = \sum_{n=0}^{\infty} |S_n(X, \sigma_0)|\, f^2(n) = f^2(0) + \frac{k}{k-1} \sum_{n=1}^{\infty} [(k-1)\, d]^n\, f^2(n). \tag{7.9}$$

One can verify that $0 < d(k-1)\,\alpha_+^2 < 1$ holds for all $\lambda < \Lambda_-$, and thus by (7.8) and (7.9) $\left\|\psi_{\sigma_0}^{\lambda}\right\|$ is finite. In fact, $\left\|\psi_{\sigma_0}^{\lambda}\right\|$ is continuous w.r.t. λ in this region, so that it is bounded on every interval $[a,b] \subseteq (-\infty, \Lambda_-)$. Furthermore, for any $\sigma \in X^{d-1}$ there is an isometry of T_k^d which takes σ_0 to σ, and thus $\psi_{\sigma_0}^{\lambda}$ to a form ψ_{σ}^{λ} with the same L^2-norm as $\psi_{\sigma_0}^{\lambda}$, and which satisfies $(A_0 - \lambda I)\, \psi_{\sigma}^{\lambda} = \mathbb{1}_{\sigma}$. We can now invoke Lemma 7.4 for $[a,b] \subseteq (-\infty, \Lambda_-)$, using $\psi_{\sigma_0}^{\lambda}$ and its translations by isometries, and obtain that $(a,b) \cap \operatorname{Spec} A_0 = \varnothing$. Thus, $\operatorname{Spec} A_0$ does not intersect $(-\infty, \Lambda_-)$.

Similarly, $0 < d(k-1)\,\alpha_-^2 < 1$ holds for all $\lambda > \Lambda_+$, so that the same argumentation for $\varphi_{\sigma_0}^{\lambda}$ shows that $\operatorname{Spec} A_0$ does not intersect (Λ_+, ∞), provided that $d+1 < k$. When $k \leq d+1$ we know that $\frac{1}{d} \in \operatorname{Spec} A_0$, and we need to show that $\operatorname{Spec} A_0$ does not intersect $(\Lambda_+, \infty) \setminus \left\{\frac{1}{d}\right\}$. This is done in the same manner, observing all intervals $[a,b]$ s.t. $[a,b] \subseteq \left(\Lambda_+, \frac{1}{d}\right)$ or $[a,b] \subseteq \left(\frac{1}{d}, \infty\right)$. $\square$

7.4 Continuity of the spectral measure

In this section we generalize parts of Grigorchuk and Żuk's work on graphs [GŻ99] to general simplicial complexes. We assume throughout the section that all d-complexes referred to are $(d-1)$-connected, and that families and sequences of d-complexes we encounter have globally bounded $(d-1)$-degrees.

For a uniform d-complex X we define the distance between two $(d-1)$-cells to be the minimal length of a $(d-1)$-chain connecting them:

$$\operatorname{dist}(\sigma, \sigma') = \min \left\{ n \,\middle|\, \begin{array}{c} \exists \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma_0 \in X^{d-1} \text{ s.t.} \\ \sigma_i \cup \sigma_{i+1} \in X^d \quad \forall i \end{array} \right\}.$$

We denote by $B_n(X, \sigma)$ the ball of radius n around σ in X, which is the maximal uniform subcomplex of X all of whose $(d-1)$-cells are of distance at most n from $\sigma^{(\dagger)}$. A *marked d-complex* (X, σ) is a d-complex with a choice of a $(d-1)$-cell σ. On the space of marked d-complexes with finite $(d-1)$-

$^{(\dagger)}$this is similar to $B_n(X, \sigma)$ defined in the proof of theorem 7.3, but there $B_n(X, \sigma)$ referred only to the $(d-1)$-cells, and here to the entire subcomplex degrees one can define a metric by

$$\operatorname{dist}\left(\left(X_1,\sigma_1\right),\left(X_2,\sigma_2\right)\right)=\inf\left\{\frac{1}{n+1}\;:\;B_n\left(X_1,\sigma_1\right)\text{ is isometric to }B_n\left(X_2,\sigma_2\right)\right\}.$$

Remarks.

(1) A limit (X,σ) of a sequence (X_n,σ_n) in this space is unique up to isometry.

(2) For every $K\in\mathbb{N}$, the subspace of d-complexes with $(d-1)$-degrees bounded by K is compact. This is due to the fact that there is only a finite number of possibilities for a ball of radius n, so that every sequence has a converging subsequence by a diagonal argument (see [GŻ99] for details).

Our next goal is to study the relation of this metric to the spectra of complexes. We use some standard spectral theoretical results which we summarize as follows: Let X be a countable set with a weighted counting measure w, i.e., $\int_X f=\sum_{x\in X}w(x)f(x)$, and A a self-adjoint operator on $L^2(X,w)$. For every $x\in X$, the spectral measure μ_x is the unique regular Borel measure on $\mathbb{C}$ such that for every polynomial $P(t)\in\mathbb{C}[t]$

$$\langle P(A)\mathbb{1}_x,\mathbb{1}_x\rangle=\int_{\mathbb{C}}P(z)d\mu_x(z),$$

where $\mathbb{1}_x$ is the Dirac function of the point x. For $x,y\in X$ the spectral measure $\mu_{x,y}$ is the unique regular Borel measure on $\mathbb{C}$ such that for every polynomial P

$$\langle P(A)\mathbb{1}_x,\mathbb{1}_y\rangle=\int_{\mathbb{C}}P(z)d\mu_{x,y}(z).$$

The spectrum of A can be inferred from the spectral measures by

$$\operatorname{Spec}A=\bigcup_{x,y\in X}\operatorname{supp}\mu_{x,y}=\bigcup_{x\in X}\operatorname{supp}\mu_x. \tag{7.10}$$

We wish to apply this mechanism to the analysis of the action of $A=A(X,0)=\frac{I-\Delta^+}{d}$ on $\Omega_{L^2}^{d-1}$ (with the inner product as in (6.2)), and this is justified by observing that for any choice of orientation X_+^{d-1} of X^{d-1}, we have an isometry $\Omega_{L^2}^{d-1}\cong L^2\left(X_+^{d-1},w\right)$, where $w(\sigma)=\frac{1}{\deg\sigma}$. For any $\sigma\in X_\pm^{d-1}$ we denote by μ_σ^X the spectral measure of A w.r.t. $\mathbb{1}_\sigma$. Similarly, $\mu_{\sigma,\sigma'}^X$ denotes the spectral measure of A w.r.t. $\mathbb{1}_\sigma$ and $\mathbb{1}_{\sigma'}$.

Lemma 7.5. *If* $\lim\limits_{n\to\infty}(X_n,\sigma_n)=(X,\sigma)$ *then* $\mu_{\sigma_n}^{X_n}$ *converges weakly to* μ_σ^X.

Proof. For regular finite Borel measures on $\mathbb{R}$ with compact support, weak convergence follows from convergence of the moments of the measures (see e.g. [Fel66, §VIII.1]). For $m\geq 0$ the m^{th} moment of μ_σ^X, denoted $\left(\mu_\sigma^X\right)^{(m)}$, is given by

$$\left(\mu_\sigma^X\right)^{(m)}=\int_{\mathbb{C}}z^m d\mu_\sigma^X(z)=\langle A^m\mathbb{1}_\sigma,\mathbb{1}_\sigma\rangle=\langle A^m\mathcal{E}_0^\sigma,\mathbb{1}_\sigma\rangle=\langle\mathcal{E}_m^\sigma,\mathbb{1}_\sigma\rangle=\frac{\mathcal{E}_m^\sigma(\sigma)}{\deg\sigma},$$

where $\mathcal{E}_m^\sigma$ is the 0-lazy expectation process starting at σ, at time m. However,

$$\mathcal{E}_m^\sigma(\sigma) = \mathbf{p}_m^\sigma(\sigma) - \mathbf{p}_m^\sigma(\overline{\sigma})$$

is determined by the structure of the complex in the ball $B_m(X,\sigma)$. For large enough n, $B_m(X,\sigma)$ is isometric to $B_m(X_n,\sigma_n)$, which implies that $\left(\mu_{\sigma_n}^{X_n}\right)^{(m)} = \left(\mu_\sigma^X\right)^{(m)}$. $\square$

7.5 Alon-Boppana type theorems

Definition 7.6. A sequence of d-complexes X_n, whose $(d-1)$-degrees are bounded globally, is said to converge to the complex X (written $X_n \overset{n\to\infty}{\longrightarrow} X$) if (X_n,σ_n) converges to (X,σ) for some choice of $\sigma_n \in X_n^{d-1}$ and $\sigma \in X^{d-1}$.

In particular, if X is an infinite d-complex with bounded $(d-1)$-degrees, and $\{X_n\}$ is a sequence of quotients of X whose injectivity radii approach infinity, then $X_n \overset{n\to\infty}{\longrightarrow} X$.

The following is (one form of) the classic Alon-Boppana theorem:

Theorem 7.7 (Alon-Boppana, cf. [GŻ99])**.** *Let G_n be a sequence of graphs whose degrees are globally bounded, and G a graph s.t. $G_n \overset{n\to\infty}{\longrightarrow} G$. Then*

$$\liminf_{n\to\infty} \lambda(G_n) \leq \lambda(G).$$

In the literature one encounters many variations on this formulation: some refer only to quotients of G, some only to regular graphs, and some are quantitative (e.g. [Nil91]).

In this section we study the analogue question for complexes of general dimension. We start with the following:

Theorem 7.8. *If $X_n \overset{n\to\infty}{\longrightarrow} X$ and $\lambda \in \operatorname{Spec} A(X,0)$, there exist $\lambda_n \in \operatorname{Spec} A(X_n,0)$ with $\lim\limits_{n\to\infty} \lambda_n = \lambda$. The same holds for the corresponding Laplacians Δ_X^+ and $\Delta_{X_n}^+$.*

Proof. Let σ_n, σ be as in Definition 7.6. Since $\lambda \in \operatorname{Spec} A(X,0)$, for every $\varepsilon > 0$ there exists $\sigma' \in X^{d-1}$ such that $\mu_{\sigma'}^X((\lambda-\varepsilon,\lambda+\varepsilon)) > 0$. We denote $r = \operatorname{dist}(\sigma,\sigma')$, and restrict our attention to the tail of $\{(X_n,\sigma_n)\}$ in which $B_r(X_n,\sigma_n)$ is isometric to $B_r(X,\sigma)$. If σ'_n is the image of σ' under such an isometry, and $d_n = \max\{k \mid B_k(X_n,\sigma_n) \cong B_k(X,\sigma)\}$, then $B_{d_n-r}(X_n,\sigma'_n) \cong B_{d_n-r}(X,\sigma')$, and since $d_n - r \to \infty$ we have $(X_n,\sigma'_n) \to (X,\sigma')$. By Lemma 7.5, $\mu_{\sigma'_n}^{X_n}((\lambda-\varepsilon,\lambda+\varepsilon)) > 0$ for large enough n and therefore $\operatorname{Spec} A(X_n,0)$ intersects $(\lambda-\varepsilon,\lambda+\varepsilon)$. The result for the Laplacians follows from the fact that $\Delta^+ = I - d\cdot A$. $\square$

In particular this gives:

Corollary 7.9. *If $X_n \overset{n\to\infty}{\longrightarrow} X$ then $\operatorname{Spec} A_X \subseteq \overline{\bigcup_n \operatorname{Spec} A_{X_n}}$.*

This is an analogue of [Li04, Thm. 4.3], which is also regarded sometimes as an Alon-Boppana theorem. In [Li04] the same statement is proved for the Hecke operators acting on the vertices of $X = \mathcal{B}_n$, the Bruhat-Tits building of type $\widetilde{A}_{n-1}$ (see §5.5), and on a sequence of quotients of X whose injectivity radii approach infinity.

Returning to the spectral gap formulation of the Alon-Boppana Theorem, Theorem 7.8 yields as an immediate result that if $X_n \overset{n\to\infty}{\longrightarrow} X$ then

$$\liminf_{n\to\infty} \min \operatorname{Spec} \Delta^+_{X_n} \leq \min \operatorname{Spec} \Delta^+_X \leq \lambda\left(X\right). \tag{7.11}$$

In order to obtain the higher dimensional analogue of the Alon-Boppana theorem one would like to verify that this holds also when the spectrum of $\Delta^+_{X_n}$ is restricted to $Z_{d-1} = \left(B^{d-1}\right)^{\perp}$. But while this holds for graphs, the situation is more involved in general dimension. First of all, it does not hold in general:

Theorem 7.10. *Let T^2_2 be the arboreal 2-regular triangle complex (Figure 7.1), and $X_r = B_r\left(T^2_2, e_0\right)$ be the ball of radius r around an edge in it (as in Figure 7.2). Then $\lim_{r\to\infty} \lambda\left(X_r\right) = \frac{3}{2} - \sqrt{2}$, while $\lambda\left(T^2_2\right) = 0$.*

The proof follows in the next section. Before we delve into this counterexample, let us exhibit first several cases in which the Alon-Boppana analogue does hold:

Theorem (1.12)**.** *If $X_n \overset{n\to\infty}{\longrightarrow} X$, and one of the following holds:*

(1) Zero is not in $\operatorname{Spec} \Delta^+_X\big|_{Z_{d-1}}$ (i.e. $\lambda\left(X\right) \neq 0$),

(2) zero is a non-isolated point in $\operatorname{Spec} \Delta^+_X\big|_{Z_{d-1}}$, or

(3) the $(d-1)$-skeletons of the complexes X_n form a family of $(d-1)$-expanders,

then $\liminf_{n\to\infty} \lambda\left(X_n\right) \leq \lambda\left(X\right)$.

Proof. By Theorem 7.8 there exist $\lambda_n \in \operatorname{Spec} \Delta^+_{X_n}$ with $\lambda_n \to \lambda\left(X\right)$. If *(1)* holds, then $\lambda_n > 0$ for large enough n, which implies that $\lambda_n \in \operatorname{Spec} \Delta^+_{X_n}\big|_{Z_{d-1}}$, hence $\lambda\left(X_n\right) = \min \operatorname{Spec} \Delta^+_{X_n}\big|_{Z_{d-1}} \leq \lambda_n$. Thus, $\liminf_{n\to\infty} \lambda\left(X_n\right) \leq \liminf_{n\to\infty} \lambda_n = \lambda\left(X\right)$. If *(2)* holds then there are $\mu_n \in \operatorname{Spec} \Delta^+_X \setminus \{0\}$ with $\mu_n \to \lambda\left(X\right)$. For every μ_n there is a sequence $\lambda_{n,m} \in \operatorname{Spec} \Delta^+_{X_m}\big|_{Z_{d-1}}$ with $\lambda_{n,m} \overset{m\to\infty}{\longrightarrow} \mu_n$, and $\lambda_{n,n} \to \lambda\left(X\right)$.

In *(3)* we mean that the $(d-2)$-cells in X_n have globally bounded degrees, and the $(d-2)$-dimensional spectral gaps

$$\lambda_{d-2}\left(X_n\right) = \min \operatorname{Spec} \Delta^+_{d-2}\big|_{Z_{d-2}(X_n)}$$

are bounded away from zero (see Remark (1) after the proof). For example, if X_n are triangle complexes, this means that their underlying graphs form a family of expander graphs in the classical sense. By the previous cases, we can assume that $\lambda\left(X\right) = 0$, and furthermore that zero is an isolated point in $\operatorname{Spec} \Delta^+_X\big|_{Z_{d-1}}$. This implies that it is an eigenvalue, so that there exists $0 \neq f \in Z_{d-1}\left(X\right) = B^{d-1}\left(X\right)^{\perp}$ with $\Delta^+_X f = 0$.

Since $X_n \overset{n\to\infty}{\longrightarrow} X$ there exist $\sigma_n \in X_n$, $\sigma_\infty \in X$, a sequence $r\left(n\right) \to \infty$, and isometries $\psi_n : B_{r(n)}\left(X_n, \sigma_n\right) \overset{\cong}{\longrightarrow} B_{r(n)}\left(X, \sigma_\infty\right)$. Define $f_n \in \Omega^{d-1}_{L^2}\left(X_n\right)$ by

$$f_n\left(\tau\right) = \begin{cases} f\left(\psi_n\left(\tau\right)\right) & \operatorname{dist}\left(\tau, \sigma_n\right) \leq r\left(n\right) \\ 0 & r\left(n\right) < \operatorname{dist}\left(\tau, \sigma_n\right). \end{cases}$$

We first claim that $\|\Delta^+ f_n\|$ and $\|\Delta^- f_n\|$ converge to zero ($\Delta^- = \Delta^-(X_n)$ are defined since the $(d-2)$-degrees are bounded). Since f_n is zero outside $B_{r(n)}(X_n, \sigma_n)$ and coincide with f on it, by $\Delta^+ f = 0$ we have

$$\begin{aligned}\left\|\Delta^+ f_n\right\|^2 &= \sum_{\sigma \in X_n^{d-1}} \left|\Delta^+ f_n(\sigma)\right|^2 = \sum_{\sigma : r(n) \le \operatorname{dist}(\sigma, \sigma_n) \le r(n)+1} \left|\Delta^+ f_n(\sigma)\right|^2 \\ &= \sum_{\sigma : r(n) \le \operatorname{dist}(\sigma, \sigma_n) \le r(n)+1} \left| f_n(\sigma) - \sum_{\sigma' \sim \sigma} \frac{f_n(\sigma')}{\deg \sigma'}\right|^2.\end{aligned}$$

Using $\left(\sum_{i=1}^k a_i\right)^2 \le k \sum_{i=1}^k a_i^2$ this gives

$$\left\|\Delta^+ f_n\right\|^2 \le (dK+1) \sum_{\sigma : r(n) \le \operatorname{dist}(\sigma, \sigma_n) \le r(n)+1} \left[\left|f_n(\sigma)\right|^2 + \sum_{\sigma' \sim \sigma} \left|f_n(\sigma')\right|^2\right],$$

where K is a bound on the degree of $(d-1)$-cells in X and X_n. Since every $(d-1)$-cell has at most dK neighbors, we have

$$\begin{aligned}\left\|\Delta^+ f_n\right\|^2 &\le dK(dK+1) \sum_{\sigma : r(n)-1 \le \operatorname{dist}(\sigma, \sigma_n) \le r(n)+2} \left|f_n(\sigma)\right|^2 \\ &\le dK(dK+1) \left\| f\big|_{X \setminus B_X(\sigma, r(n)-2)}\right\|^2 \xrightarrow{n\to\infty} 0.\end{aligned}$$

The reasoning for $\|\Delta^- f_n\| \to 0$ (see (6.5)) is analogous: (7.3) gives $\Delta^- f = 0$, and the assumptions that $(d-2)$-degrees are globally bounded yields similar bounds as done for Δ^+.

For every n write $f_n = z_n + b_n$, with $z_n \in Z_{d-1}(X_n)$ and $b_n \in B^{d-1}(X_n)$. It is enough to show that $\|z_n\|$ are bounded away from zero, since then $\frac{\|\Delta^+ z_n\|}{\|z_n\|} = \frac{\|\Delta^+ f_n\|}{\|z_n\|} \to 0$, showing that $\lambda(X_n) = \min \operatorname{Spec}\left(\Delta^+\big|_{Z_{d-1}(X_n)}\right)$ converge to zero.

Assume therefore that there are arbitrarily small $\|z_n\|$, and by passing to a subsequence that $\|z_n\| \to 0$. Then $\|b_n\| \to \|f\| > 0$, giving $\frac{\|\Delta^- b_n\|}{\|b_n\|} = \frac{\|\Delta^- f_n\|}{\|b_n\|} \to 0$. This implies that $\lambda_n' = \min \operatorname{Spec}\left(\Delta^-\big|_{B^{d-1}(X_n)}\right)$ converge to zero. However,

$$\begin{aligned}\lambda_n' &= \min \operatorname{Spec}\left(\Delta^-\big|_{B^{d-1}(X_n)}\right) = \min \operatorname{Spec}\left(\partial_{d-1}^* \partial_{d-1}\big|_{B^{d-1}(X_n)}\right) \\ &\overset{\star}{=} \min \operatorname{Spec}\left(\partial_{d-1} \partial_{d-1}^*\big|_{B_{d-2}(X_n)}\right) = \min \operatorname{Spec}\left(\Delta_{d-2}^+\big|_{B_{d-2}(X_n)}\right) \\ &\ge \min \operatorname{Spec}\left(\Delta_{d-2}^+\big|_{Z_{d-2}(X_n)}\right) = \lambda_{d-2}(X_n)\end{aligned}$$

where $\star$ is due to the fact that B^{d-1} and B_{d-1} are the orthogonal complements of $\ker \partial_{d-1}$ and $\ker \partial_{d-1}^*$ respectively. This is a contradiction, since $\lambda_{d-2}(X_n)$ are bounded away from zero. □

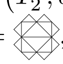
Remarks.

(1) If $X^{(j)}$ denote the j-skeleton of a complex X, i.e. the subcomplex consisting of all cells of dimension $\le j$, then one can look at $\lambda\left(X_n^{(d-1)}\right)$ instead of at $\lambda_{d-2}(X_n)$. Since we have different weight functions in codimension one, these are not equal. However, since we assumed that all

$(d-1)$ and $(d-2)$ degrees are globally bounded (and nonzero), the norms induced by these choices of weight functions are equivalent, and thus $\lambda\left(X_n^{(d-1)}\right)$ are bounded away from zero iff $\lambda_{d-2}\left(X_n\right)$ are.

(2) The Alon-Boppana theorem for graphs follows from condition *(2)* in this Proposition (as done in [GŻ99]), since zero is never an isolated point in the spectrum of the Laplacian of an infinite connected graph. Otherwise, it would correspond to an eigenfunction, which is some multiple of the degree function, hence not in L^2.

7.6 Analysis of balls in T_2^2

In this section we analyze the spectrum of balls in the 2-regular triangle complex T_2^2, proving in particular that they constitute a counterexample for the higher-dimensional analogue of Alon-Boppana (Theorem 7.10). We denote here $X_r = B_r\left(T_2^2, e_0\right)$, the ball of radius r around an edge e_0 in T_2^2: X_0 is a single edge, $X_1 =$ , $X_2 =$ , $X_3 =$ , and so on. For $r \geq 1$ we define three $r \times r$ matrices denoted $M_{++}^{(r)}, M_{+-}^{(r)}, M_{--}^{(r)}$, and for $r \geq 0$ a $(r+1)\times(r+1)$ matrix $M_{-+}^{(r)}$, as follows:

$$M_{-+}^{(0)} = \begin{pmatrix} 1 \end{pmatrix}, \qquad M_{++}^{(1)} = M_{+-}^{(1)} = M_{--}^{(1)} = \begin{pmatrix} 0 \end{pmatrix}$$

$$M_{-+}^{(1)} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}, \qquad M_{++}^{(2)} = M_{+-}^{(2)} = \begin{pmatrix} \frac{1}{2} & -1 \\ -1 & 2 \end{pmatrix}, \qquad M_{--}^{(2)} = \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & 2 \end{pmatrix}$$

$$M_{++}^{(r)} = M_{+-}^{(r)} = \left.\begin{pmatrix} \frac{1}{2} & -1 & & & \\ -\frac{1}{2} & \frac{3}{2} & -1 & & \\ & & \ddots & & \\ & & -\frac{1}{2} & \frac{3}{2} & -1 \\ & & & -1 & 2 \end{pmatrix}\right\} r$$

$$M_{--}^{(r)} = \left.\begin{pmatrix} \frac{3}{2} & -1 & & & \\ -\frac{1}{2} & \frac{3}{2} & -1 & & \\ & & \ddots & & \\ & & -\frac{1}{2} & \frac{3}{2} & -1 \\ & & & -1 & 2 \end{pmatrix}\right\} r$$

$$M_{-+}^{(r)} = \left.\begin{pmatrix} 1 & -2 & & & \\ -\frac{1}{2} & \frac{3}{2} & -1 & & \\ & & \ddots & & \\ & & -\frac{1}{2} & \frac{3}{2} & -1 \\ & & & -1 & 2 \end{pmatrix}\right\} r+1$$

Theorem 7.11. *The spectrum of $X_r = B_r\left(T_2^2\right)$ is given (including multiplicities) by*

$$\operatorname{Spec}\Delta^+\left(X_r\right) = \operatorname{Spec} M_{++}^{(r)} \cup \operatorname{Spec} M_{+-}^{(r)} \cup \operatorname{Spec} M_{--}^{(r)} \cup \operatorname{Spec} M_{-+}^{(r)} \cup \bigcup_{j=1}^{r-1}\left[\operatorname{Spec} M_{++}^{(j)}\right]^{2^{r-j+1}}$$

where $[X]^i$ means that X is repeated i times.

To make this clear, this gives

$$\left|\operatorname{Spec}\Delta^+\left(X_r\right)\right| = 4r + 1 + \sum_{j=1}^{r-1} 2^{r-j+1}\cdot j = 2^{r+2} - 3 = \left|X_r^1\right| = \dim\Omega^1\left(X_r\right),$$

as ought to be.

Proof. The symmetry group of X_r (for $r \geq 1$) is $G = \{id, \tau_h, \tau_v, \sigma\}$, where τ_h is the horizontal reflection, τ_v is the vertical reflection (around the middle edge e_0), and $\sigma = \tau_h \circ \tau_v = \tau_v \circ \tau_h$ is a rotation by π. The irreducible representations of G are given in Table 7.1.

	e	τ_h	τ_v	σ
V_{++}	1	1	1	1
V_{+-}	1	1	-1	-1
V_{-+}	1	-1	1	-1
V_{--}	1	-1	-1	1

Table 7.1: The irreducible representations of $G = \mathrm{Sym}(X_r)$.

We define four orientations for X_r, denoted $X_r^{\pm\pm}$, demonstrated in Figure 7.3. In all of them e_0 is oriented from left to right, and the first (top right) quadrant is oriented clockwise. Each of the other quadrants is then oriented according to the corresponding representation, e.g. X_r^{+-} satisfies the following: for every oriented edge e, if $e \in X_r^{+-}$ then $\tau_h e \in X_r^{+-}$, while $\tau_v e, \sigma e \notin X_r^{+-}$ (so that $\overline{\tau_v e}, \overline{\sigma e} \in X_r^{+-}$).

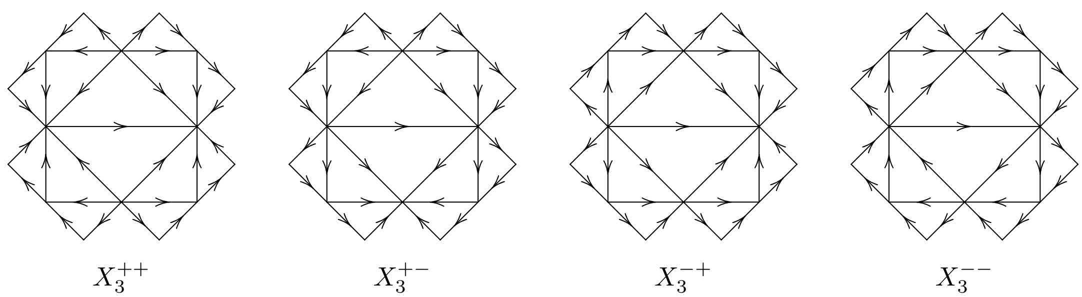


Figure 7.3: The four choices of orientations for X_r, depicted for $r = 3$.

The space of 1-forms $\Omega^1(X_r)$ is naturally a representation of $G = \mathrm{Sym}(X_r)$, by $(\gamma f)(e) = f(\gamma^{-1}e)$ (where $\gamma \in G$, $f \in \Omega^1(X_r)$, $e \in X_r^1$). We denote by $\Omega_{\pm\pm}^{(r)} = \Omega_{\pm\pm}^1(X^r)$ its $V_{\pm\pm}$-isotypic components. For example, $f \in \Omega_{+-}^{(r)}$ if and only if it satisfies $\tau_h f = f$ and $\tau_v f = -f$ (which implies that $\sigma f = \tau_v \tau_h f = -f$).

We say that a 1-form on X_r is $++$-*spherical*, denoted $f \in \mathcal{S}_{++}^{(r)}$, if it is

(1) spherical in absolute value (i.e. $|f(e)| = |f(e')|$ whenever $\mathrm{dist}(e_0, e) = \mathrm{dist}(e_0, e')$), and

(2) V_{++}-isotypic (namely $f \in \Omega_{++}^{(r)}$, or equivalently, f is of constant sign on X_r^{++}).

The definition of $\mathcal{S}_{+-}^{(r)}, \mathcal{S}_{-+}^{(r)}, \mathcal{S}_{--}^{(r)}$ are analogue.

Let $e_1, \ldots, e_r$ be edges in the first quadrant of X_r oriented as in $X_r^{\pm\pm}$, and with $\mathrm{dist}(e_i, e_0) = i$. Let f be an eigenform of Δ^+ with eigenvalue λ, which is in one of the $\mathcal{S}_{\pm\pm}^{(r)}$. Then for $2 \leq i \leq r-1$

$$\lambda f(e_i) = \left(\Delta^+ f\right)(e_i) = f(e_i) - \frac{1}{2}\left[f(e_{i-1}) - f(e_i) + 2f(e_{i+1})\right]$$

and

$$\lambda f(e_r) = \left(\Delta^+ f\right)(e_r) = f(e_r) - \left[f(e_{r-1}) - f(e_r)\right].$$

The behavior of f around e_0, e_1 depends on the isotypic component. We assume $r \geq 2$, and leave it to the reader to verify the cases $r = 0, 1$. Every form in $\Omega_{++}^{(r)}, \Omega_{+-}^{(r)}, \Omega_{--}^{(r)}$ must vanish on the middle edge e_0: for the first two, since

$$f(e_0) = (\tau_h f)(e_0) = f(\tau_h e_0) = f(\overline{e_0}) = -f(e_0),$$

and for the last one since $f(e_0) = (-\tau_v f)(e_0) = -f(\tau_v e_0) = -f(e_0)$. For a spherical $(-+)$-functions we have

$$\lambda f(e_0) = \left(\Delta^+ f\right)(e_0) = f(e_0) - \frac{1}{2}\left[4 \cdot f(e_1)\right],$$

and at e_1 we have (using the fact that $f(e_0) = 0$ for $f \in \Omega_{++}^{(r)}, \Omega_{+-}^{(r)}, \Omega_{--}^{(r)}$)

$$\lambda f(e_1) = \left(\Delta^+ f\right)(e_1) = \begin{cases} f(e_1) - \frac{1}{2}\left[f(e_1) + 2f(e_2)\right] & f \in \Omega_{++}^{(r)}, \Omega_{+-}^{(r)} \\ f(e_1) - \frac{1}{2}\left[f(e_0) - f(e_1) + 2f(e_2)\right] & f \in \Omega_{-+}^{(r)} \\ f(e_1) - \frac{1}{2}\left[-f(e_1) + 2f(e_2)\right] & f \in \Omega_{--}^{(r)}. \end{cases}$$

The matrices $M_{\pm\pm}^{(r)}$ represent these equations, and thus the $++$-spherical spectrum of X^r is $\operatorname{Spec} \Delta^+\big|_{\mathcal{S}_{++}^{(r)}} = \operatorname{Spec} M_{++}^{(r)}$, and likewise for the other $\mathcal{S}_{\pm\pm}^{(r)}$.

Until now we have only accounted for the spherical part of $\Omega^1(X)$, finding in total $4r+1$ eigenvalues. The other eigenvalues are obtained by using spherical eigenforms of X^i with $i < r$.

Denote by $X_r^{\mathfrak{h}}$ the upper half of X_r, including e_0, which is a fundamental domain for $\{id, \tau_v\}$. Observe that $X_r \backslash \mathring{X}_1$ (by which we mean X_r after deleting e_0 and the two triangles adjacent to it, but not the other four edges), is comprised of four copies of $X_{r-1}^{\mathfrak{h}}$, which intersect only in vertices. Denote these four copies of $X_{r-1}^{\mathfrak{h}}$ by $Y_1, \ldots, Y_4$. Let $f \in \mathcal{S}_{++}^{(r-1)}$ be a $(++)$-spherical λ-eigenform on X_{r-1}, and define $g \in \Omega^1(X_r)$ by $g\big|_{Y_1} = f\big|_{X_{r-1}^{\mathfrak{h}}}$ and $g\big|_{Y_2} = g\big|_{Y_3} = g\big|_{Y_4} = 0$. We show now that g is a λ-eigenform of X^r. Since $f \in \Omega_{++}^{(r-1)}$, $g(e_1) = f(e_0) = 0$, where e_1 is the edge incident to e_0 in Y_1. Therefore, $\Delta^+ g = \lambda g$ holds everywhere outside Y_1. It also holds at e_1, since if e_2, e_2' are the two edges incident to e_1 in Y_1, then $g(e_2) = -g(e_2')$ since f is symmetric with respect to τ_h. Obviously, $\Delta^+ g = \lambda g$ holds in $Y_1 \backslash \{e_1\}$, and we are done. We could have taken $g\big|_{Y_i} = f\big|_{X_{r-1}^{\mathfrak{h}}}$ for any $i \in \{1, 2, 3, 4\}$, and the resulting eigenforms are independent. We remark that taking $f \in \Omega_{+-}^{(r-1)}$ would also work, but would give again the same eigenforms, while $f \in \Omega_{-+}^{(r)}, \Omega_{--}^{(r)}$ would not define an eigenform on X_r.

More generally, $X_r \backslash \mathring{X}_j$ is comprised of 2^{j+1} copies of $X_{r-j}^{\mathfrak{h}}$, and in a similar way every eigenform of $\Delta^+\big|_{\mathcal{S}_{++}^{(r-j)}}$ contributes 2^{j+1} eigenforms to X^r. We recall that for $f \in \mathcal{S}_{++}^{(r-j)}$ we always have $f(e_0) = 0$, and observe that due to the recursion relations if $f \neq 0$ then $f(e_1) \neq 0$. Therefore, the eigenforms obtained from copies of $X_{r-j}^{\mathfrak{h}}$ for various j are all linearly independent, as they are supported outside different balls in X^r. Together with the $4r+1$ spherical eigenforms, this accounts for

$$4r + 1 + \sum_{j=1}^{r} 2^{j+1} \cdot \left|\operatorname{Spec} \Delta^+\big|_{\mathcal{S}_{++}^{(r-j)}}\right| = 4r + 1 + \sum_{j=1}^{r-1} 2^{j+1}(r-j) = 2^{r+2} - 3$$

independent eigenforms, and since this is the dimension of $\Omega^1(X_r)$ we are done. $\square$

Proposition 7.12. *For every $r \in \mathbb{N}$ and $\lambda \in \operatorname{Spec} M_{\pm\pm}^{(r)}$, either $\lambda = 0$ or $\frac{3}{2} - \sqrt{2} < \lambda$.*

Proof. Let $p_{++}^{[r]}(\lambda) = \det\left(M_{++}^{(r)} - \lambda I\right)$, and similarly for the other $\pm\pm$. Expanding $M_{--}^{(r)} - \lambda I$ by minors gives

$$p_{--}^{[1]}(\lambda) = 1 - \lambda, \quad p_{--}^{[2]}(\lambda) = \lambda^2 - \frac{7}{2}\lambda + 2, \quad p_{--}^{[3]} = -\lambda^3 + 5\lambda^2 - \frac{27}{4}\lambda + 2$$
$$p_{--}^{[r]}(\lambda) = \left(\frac{3}{2} - \lambda\right) p_{--}^{[r-1]}(\lambda) - \frac{1}{2} p_{--}^{[r-2]}(\lambda) \qquad (r \geq 4).$$

This yields a quadratic recurrence formula in $\mathbb{Q}[\lambda]$ whose solution (for $r \geq 2$) is $p_{--}^{[r]}(\lambda) = \alpha(\lambda)\mu_+(\lambda)^r + \beta(\lambda)\mu_-(\lambda)^r$, where

$$\alpha(\lambda) = 2 - \beta(\lambda) = \frac{(2\lambda - 2)\sqrt{4\lambda^2 - 12\lambda + 1} + 4\lambda^2 - 10\lambda - 2}{(2\lambda - 3)\sqrt{4\lambda^2 - 12\lambda + 1} + 4\lambda^2 - 12\lambda + 1},$$
$$\mu_\pm(\lambda) = \frac{3}{4} - \frac{\lambda}{2} \pm \frac{1}{4}\sqrt{4\lambda^2 - 12\lambda + 1}.$$

For $0 < \lambda < \frac{3}{2} - \sqrt{2}$ one can verify that $\beta(\lambda) < 0 < \alpha(\lambda)$ and $0 < \mu_-(\lambda) < \mu_+(\lambda)$, and for $r \geq 2$

$$p_{--}^{[r]}(\lambda) = \mu_+(\lambda)^r \left(\alpha(\lambda) + \beta(\lambda)\left(\frac{\mu_-(\lambda)}{\mu_+(\lambda)}\right)^r\right) \geq \mu_+(\lambda)^r \left(\alpha(\lambda) + \beta(\lambda)\left(\frac{\mu_-(\lambda)}{\mu_+(\lambda)}\right)^2\right)$$
$$= \mu_+(\lambda)^{r-2}\left(\alpha(\lambda)\mu_+(\lambda)^2 + \beta(\lambda)\mu_-(\lambda)^2\right) = \mu_+(\lambda)^{r-2} p_{--}^{[2]}(\lambda) > 0.$$

Using the solution for $p_{--}^{[r]}$ one can write $p_{+-}^{[r]}$, for $r \geq 4$, as

$$p_{+-}^{[r]}(\lambda) = \left(\frac{1}{2} - \lambda\right) p_{--}^{[r-1]}(\lambda) - \frac{1}{2} p_{--}^{[r-2]}(\lambda)$$
$$= \alpha(\lambda)\left(\left(\frac{1}{2} - \lambda\right)\mu_+(\lambda) - \frac{1}{2}\right)\mu_+(\lambda)^{r-2} + \beta(\lambda)\left(\left(\frac{1}{2} - \lambda\right)\mu_-(\lambda) - \frac{1}{2}\right)\mu_-(\lambda)^{r-2}$$

Now $\alpha(\lambda)\left(\left(\frac{1}{2} - \lambda\right)\mu_+(\lambda) - \frac{1}{2}\right) < 0 < \beta(\lambda)\left(\left(\frac{1}{2} - \lambda\right)\mu_-(\lambda) - \frac{1}{2}\right)$ for $0 < \lambda < \frac{3}{2} - \sqrt{2}$, and it follows that $p_{+-}^{[r]}(\lambda)$ does not vanish in this interval. This takes care of $p_{++}^{[r]}(\lambda)$ as well, since $M_{++}^{[r]} = M_{+-}^{[r]}$. The considerations for $p_{-+}^{[r]}(\lambda)$ are similar, and we leave them to the reader. $\square$

We can conclude now that $\{X_r\}_{r \in \mathbb{N}}$ constitute a counterexample for high-dimensional Alon-Boppana:

Proof of Theorem 7.10. By the results in this section, the spectrum of $\Delta_{X_r}^+$ is contained in $\{0\} \cup \left(\frac{3}{2} - \sqrt{2}, 3\right]$. Since X_r is contractible, its first homology is trivial and thus the zeros in the spectrum all belong to coboundaries, i.e., $\operatorname{Spec} \Delta_{X_r}^+\big|_{Z_1} \subseteq \left(\frac{3}{2} - \sqrt{2}, 3\right]$. Therefore, $\liminf_{r\to\infty} \lambda(X_r) \geq \frac{3}{2} - \sqrt{2}$. In fact, $\liminf_{r\to\infty} \lambda(X_r) = \frac{3}{2} - \sqrt{2}$. This follows from $\frac{3}{2} - \sqrt{2} \in \operatorname{Spec} T_2^2$ (by Theorem 7.3), together with Theorem 7.8, which asserts that there exist $\lambda_r \in \operatorname{Spec} \Delta_{X_r}^+$ such that $\lambda_r \to \frac{3}{2} - \sqrt{2}$. As λ_r can be assumed to be nonzero, they are in fact in $\operatorname{Spec} \Delta^+_{X_r}\big|_{Z_1}$, so that $\liminf_{r\to\infty} \lambda(X_r) \le \lim_{r\to\infty} \lambda_r = \frac{3}{2} - \sqrt{2}$. Finally, by Lemma 7.1 and Theorem 7.3 we have $\lambda\left(T_2^2\right) = 0$. $\square$

7.7 Spectral radius and random walk

The spectral radius of an operator T is $\rho(T) = \max\{|\lambda| \mid \lambda \in \operatorname{Spec} T\}$. If T is a self-adjoint operator on a Hilbert space then $\rho(T) = \|T\|$. In this section we observe the transition operator $A = A(X,p)$ acting on $\Omega^{d-1}_{L^2}$, and relate it to the asymptotic behavior of the expectation process on X. Under additional conditions, this can be translated to a result on the spectral gap of the complex.

Proposition 7.13. *Let $\mathcal{E}^\sigma_n$ be the expectation process associated with the p-lazy $(d-1)$-walk on a finite or countable d-complex X with bounded $(d-1)$-degrees.*

(1) For all values of p

$$\sup_{\sigma\in X^{d-1}_\pm} \limsup_{n\to\infty} \sqrt[n]{|\mathcal{E}^\sigma_n(\sigma)|} = \|A\| = \rho(A).$$

(2) If $\frac{1}{2} \le p \le 1$ then

$$\sup_{\sigma\in X^{d-1}_\pm} \limsup_{n\to\infty} \sqrt[n]{\mathcal{E}^\sigma_n(\sigma)} = \|A\| = \max \operatorname{Spec} A = \frac{p(d-1)+1}{d} - \frac{1-p}{d}\cdot \min \operatorname{Spec} \Delta^+.$$

(3) If $\frac{1}{2} \le p \le 1$ and all $(d-2)$-cells in X are of infinite degree, then

$$\sup_{\sigma\in X^{d-1}_\pm} \limsup_{n\to\infty} \sqrt[n]{\mathcal{E}^\sigma_n(\sigma)} = \frac{p(d-1)+1}{d} - \frac{1-p}{d}\cdot \lambda(X).$$

Proof. For an oriented $(d-1)$-cell σ,

$$\mathcal{E}^\sigma_n(\sigma) = A^n \mathcal{E}^\sigma_0(\sigma) = \deg\sigma \langle A^n \mathbb{1}_\sigma, \mathbb{1}_\sigma\rangle = \deg\sigma \int_{\mathbb{C}} z^n d\mu_\sigma(z) = \deg\sigma \int_{\operatorname{Spec} A} z^n d\mu_\sigma(z),$$

where μ_σ is the spectral measure of A with respect to $\mathbb{1}_\sigma$. It follows that

$$\limsup_{n\to\infty} \sqrt[n]{|\mathcal{E}^\sigma_n(\sigma)|} = \limsup_{n\to\infty} \sqrt[n]{\deg\sigma \left| \int_{\operatorname{supp}\mu_\sigma} z^n d\mu_\sigma(z)\right|} = \max\{|\lambda| \mid \lambda \in \operatorname{supp}\mu_\sigma\},$$

and by $\operatorname{Spec}(A) = \bigcup_{\sigma\in X^{d-1}_\pm} \operatorname{supp}(\mu_\sigma)$ (see (7.10))

$$\sup_{\sigma\in X^{d-1}_\pm} \limsup_{n\to\infty} \sqrt[n]{|\mathcal{E}^\sigma_n(\sigma)|} = \sup_{\sigma\in X^{d-1}_\pm} \max\{|\lambda| \mid \lambda\in \operatorname{supp}\mu_\sigma\} = \rho(A),$$

settling *(1)*. Since $\operatorname{Spec}(A) \subseteq \left[2p-1, \frac{p(d-1)+1}{d}\right]$, in the case $p \ge \frac{1}{2}$ the spectrum of A is nonnegative. Therefore,

$$\mathcal{E}^\sigma_n(\sigma) = A^n \mathcal{E}^\sigma_0(\sigma) = \deg\sigma \langle A^n \mathbb{1}_\sigma, \mathbb{1}_\sigma\rangle \ge 0$$

so that $|\mathcal{E}_n^\sigma(\sigma)| = \mathcal{E}_n^\sigma(\sigma)$, and in addition $\rho(A) = \max \operatorname{Spec} A$. This accounts for *(2)*, and combining this with Lemma 7.1 gives *(3)*. $\square$

This proposition is a generalization of the classic connection between return probability and spectral radius in an infinite connected graph. Namely, for any vertex v the non-lazy walk on this graph satisfies

$$\lim_{n\to\infty} \sqrt[n]{\mathbf{p}_n^v(v)} = 1 - \lambda(G) = \max \operatorname{Spec} A = \|A\| = \rho(A),$$

where A is the transition operator of the walk. There are slight differences, though: in general dimension $p \geq \frac{1}{2}$ is needed for some of these equalities, and in addition one must take the supremum over all possible starting points for the process. For graphs this is not necessary (provided the graph is connected), and we do not know whether the same is true in general dimension. One case in which this is not necessary is when the complex is $(d-1)$-transitive, in the sense that its symmetry group acts transitively on X^{d-1}. This (together with Theorem 7.3) leads to the following corollary:

Corollary 7.14. *For the k-regular arboreal d-complex T_k^d, the non-lazy random walk starting at any $(d-1)$-cell σ satisfies*

$$\limsup_{n\to\infty} \sqrt[n]{|\mathbf{p}_n^\sigma(\sigma) - \mathbf{p}_n^\sigma(\overline{\sigma})|} = \frac{d-1+2\sqrt{d(k-1)}}{kd}.$$

For $p \geq \frac{1}{2}$, the p-lazy walk satisfies

$$\limsup_{n\to\infty} \sqrt[n]{\mathbf{p}_n^\sigma(\sigma) - \mathbf{p}_n^\sigma(\overline{\sigma})} = \begin{cases} p + \frac{1-p}{d} & 2 \leq k \leq d+1 \\ p + (1-p)\frac{1-d+2\sqrt{d(k-1)}}{kd} & d+1 \leq k \end{cases}.$$

Another corollary of Proposition 7.13 is the following:

Corollary 7.15. *If $\dim X = d$ and there exists some $\tau \in X^{d-2}$ of finite degree (in particular, if X is finite), then the $p \geq \frac{1}{2}$ lazy random walk satisfies*

$$\sup_{\sigma \in X_\pm^{d-1}} \limsup_{n\to\infty} \sqrt[n]{\mathbf{p}_n^\sigma(\sigma) - \mathbf{p}_n^\sigma(\overline{\sigma})} = p + \frac{1-p}{d}.$$

Proof. The form $\delta_{d-1}\mathbb{1}_\tau$ is in $\Omega_{L^2}^{d-1}$ and in $\ker \delta_d$, so that $0 \in \operatorname{Spec} \Delta^+$. $\square$

7.8 Amenability, transience and recurrence

An infinite connected graph with finite degrees is said to be *amenable* if its Cheeger constant

$$h(X) = \min_{\substack{A \subseteq V \\ 0 < |A| < \infty}} \frac{|E(A, V\backslash A)|}{|A|}$$

is zero. It is called *recurrent* if with probability one the random walk on it returns to its starting point, and *transient* otherwise. A nonamenable graph is always transient.

All three notions have many equivalent characterizations. Among these are the following, which relate to the Laplacian of the graph:

(1) If X has bounded degrees, then it is amenable if and only if $\lambda(X) = \min \operatorname{Spec} \Delta^{+} = 0$. This follows from the discrete Cheeger inequalities for infinite graphs (see [Dod84, Tan84, AM85, Alo86]).

(2) X is transient if and only if $\mathbb{E}^{v}\left[\begin{smallmatrix}\text{number of}\\ \text{visits to } v\end{smallmatrix}\right] = \sum_{n=0}^{\infty} \mathbf{p}_{n}^{v}(v) < \infty$ for some v, or equivalently for all v.

(3) X is transient if and only if there exists $f \in \Omega_{L^2}^{1}(X)$ such that $\partial f = \mathbb{1}_{v}$ for some v, or equivalently for all v [Lyo83].

This suggests observing the following generalizations of these notions for a simplicial complex of dimension d:

(**A**) $\lambda(X) = 0$.

(**A′**) $\min \operatorname{Spec} \Delta^{+} = 0$.

(**T**) $\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n}^{\sigma}(\sigma) < \infty$ for every $\sigma \in X^{d-1}$, where $\widetilde{\mathcal{E}}$ is the normalized expectation process of laziness p on X, for some $\frac{1}{2} \leq p < 1$ (equivalently, all p - see Proposition 7.16*(5)*).

(**T′**) For every $\sigma \in X^{d-1}$ there exists $f \in \Omega_{L^2}^{d}(X)$ such that $\partial_{d} f = \mathbb{1}_{\sigma}$.

For infinite graphs, (**A**) and (**A′**) are the same and are equivalent to amenability, and (**T**) (for any p) and (**T′**) are equivalent to transience. These definitions suggests many questions, some of which are presented in §9. The next proposition points out some observations regarding them. Let us also define the property:

(**S**) All $(d-2)$-cells in X have infinite degrees,

which holds in any infinite graph.

Proposition 7.16. *Let X be a complex of dimension d with bounded $(d-1)$-degrees. Then*

(1) (**A**) $\Rightarrow$ (**A′**).

(2) (**A′**) + (**S**) $\Rightarrow$ (**A**).

(3) $\neg$ (**A′**) $\Rightarrow$ (**T′**) $\Rightarrow$ (**S**).

(4) $\neg$ (**A′**) $\Rightarrow$ (**T**).

(5) If (**T**) *holds for some $\frac{1}{2} \leq p < 1$, then it holds for any such p.*

(6) If zero is an isolated point in $\operatorname{Spec} \Delta^{+}$ *then* $\neg$ (**T**).

Proof. *(1)* is trivial and *(2)* follows from Lemma 7.1.

(3) If $(\mathbf{A}')$ fails then $0 \notin \operatorname{Spec}\Delta^{+}$, which means that Δ^{+} is invertible on $\Omega_{L^2}^{d-1}(X)$. Thus, for every $\sigma \in X^{d-1}$ there exists $\psi \in \Omega_{L^2}^{d-1}$ s.t. $\Delta^{+}\psi = \mathbb{1}_{\sigma}$, and taking $f = \delta_d \psi$ gives $(\mathbf{T}')$. If $(\mathbf{S})$ fails, then some $\tau \in X^{d-2}$ has finite degree. In this case for any $f \in \Omega^d$ one has

$$(\partial_{d-1}\partial_d f)(\tau) = \sum_{v \lhd \tau} (\partial_d f)(v\tau) = \sum_{v \lhd \tau} \sum_{w \lhd v\tau} f(wv\tau) = 0,$$

since this sums over every d-cell containing τ exactly twice, with opposite orientations. If σ is any $(d-1)$-cell containing τ, then $\partial_d f = \mathbb{1}_{\sigma}$ would give $0 = (\partial_{d-1}\partial_d f)(\tau) = (\partial_{d-1}\mathbb{1}_{\sigma})(\tau) = 1$, so that $(\mathbf{T}')$ fails.

(4) If $\min \operatorname{Spec}\Delta^{+} > 0$ then by Proposition 7.13*(2)*

$$\sup_{\sigma \in X_{\pm}^{d-1}} \limsup_{n\to\infty} \sqrt[n]{\widetilde{\mathcal{E}}_n^{\sigma}(\sigma)} = 1 - \frac{1-p}{p(d-1)+1} \cdot \min \operatorname{Spec}\Delta^{+} < 1$$

which gives $\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{\sigma}(\sigma) < \infty$ for every σ.

(5) Let $\frac{1}{2} \leq p$. Denote by $\left\{\widetilde{\mathcal{E}}_n^{p,\sigma}\right\}_{n=0}^{\infty}$ the p-lazy normalized expectation process starting from σ, and let $\widetilde{A}_p = \frac{p(d-1)+1}{d} \cdot A_p$. Recall that $\widetilde{\mathcal{E}}_n^{p,\sigma} = \widetilde{A}_p^n \widetilde{\mathcal{E}}_0^{p,\sigma} = \widetilde{A}_p^n \mathbb{1}_{\sigma}$, and let $\mu_p = \mu_{\sigma}^{\widetilde{A}_p}$ be the spectral measure of $\widetilde{A}_p$ w.r.t. σ. Then

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{p,\sigma}(\sigma) = \sum_{n=0}^{\infty} \widetilde{A}_p^n \mathbb{1}_{\sigma}(\sigma) = \deg\sigma \sum_{n=0}^{\infty} \left\langle \widetilde{A}_p^n \mathbb{1}_{\sigma}, \mathbb{1}_{\sigma} \right\rangle = \deg\sigma \sum_{n=0}^{\infty} \int_{\operatorname{Spec}\widetilde{A}_p} \lambda^n d\mu_p(\lambda).$$

Since $\operatorname{Spec}\widetilde{A}_p \subseteq \left[\frac{d(2p-1)}{p(d-1)+1}, 1\right] \subseteq [0,1]$, by monotone convergence

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{p,\sigma}(\sigma) = \deg\sigma \sum_{n=0}^{\infty} \int_{\operatorname{Spec}\widetilde{A}_p} \lambda^n d\mu_p(\lambda) = \deg\sigma \int_{\operatorname{Spec}\widetilde{A}_p} \frac{d\mu_p(\lambda)}{1-\lambda} \tag{7.12}$$

which is to be understood as ∞ if μ_p has an atom at $\lambda = 1$. Given $p < q < 1$ one has $\widetilde{A}_q = \pi\widetilde{A}_p + (1-\pi)I$, where $\pi = \pi(p,q,d) = \frac{1-q}{1-p} \cdot \frac{p(d-1)+1}{q(d-1)+1} \in (0,1)$. The spectral measure of $\widetilde{A}_q$ w.r.t. σ is thus given by $\mu_q = \mu_{\sigma}^{\widetilde{A}_q} = \mu_{\sigma}^{\widetilde{A}_p} \circ g^{-1}$ where $g(\lambda) = \pi\lambda + 1 - \pi$, so that

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{q,\sigma}(\sigma) = \deg\sigma \int_{\operatorname{Spec}(\widetilde{A}_q)} \frac{d\mu_q(\lambda)}{1-\lambda} = \deg\sigma \int_{\operatorname{Spec}\widetilde{A}_p} \frac{d\mu_p(\lambda)}{1-(\pi\lambda+1-\pi)} = \frac{1}{\pi} \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{p,\sigma}(\sigma)$$

which completes the proof. Finally, *(6)* follows from (7.12) as an isolated point in the spectrum implies an atom at 1.

□

8 Isospectrality

This section, which is based on the paper [Par13b], treats the results on isospectrality which are introduced in §1.6.

8.1 G-sets

Recall that our isospectral construction relies on the condition (1.7):

$$\forall g \in G: \quad \sum_{i=1}^{r} \frac{|[g] \cap H_i|}{|H_i|} = \sum_{i=1}^{r} \frac{|[g] \cap K_i|}{|K_i|}.$$

To explain where this condition comes from, we invoke the theory of G-sets. We start by recalling the basic notions and facts.

For a group G, a (left) *G-set* X is a set equipped with a (left) action of G, i.e. a multiplication rule $G \times X \to X$. Such an action partitions X into *orbits*, the subsets of the form $Gx = \{gx \,|\, g \in G\}$ for $x \in X$. A G-set with one orbit is said to be *transitive*, and every G-set decomposes uniquely as a disjoint union of transitive ones, its orbits. For every subgroup H of G, the set of left cosets ${}^{G}\!/\!_{H}$ is a transitive (left) G-set.

We denote by $\mathrm{Hom}_G(X, Y)$ the set of G-set homomorphisms from X to Y, which are the functions $f : X \to Y$ which commute with the actions, i.e. satisfy $f(gx) = gf(x)$ for all $g \in G$, $x \in X$. An isomorphism is, as usual, an invertible homomorphism.

Every transitive G-set is isomorphic to ${}^{G}\!/\!_{H}$, for some subgroup H of G, and ${}^{G}\!/\!_{H}$ and ${}^{G}\!/\!_{K}$ are isomorphic if and only if H and K are conjugate subgroups of G. More generally, every G-set is isomorphic to $\bigcup_{i \in I} {}^{G}\!/\!_{H_i}$ for some collection (possibly with repetitions) of subgroups H_i $(i \in I)$ in G, and these are determined uniquely up to order and conjugacy. Namely, $X = \bigcup {}^{G}\!/\!_{H_i}$ and $Y = \bigcup {}^{G}\!/\!_{K_i}$ are isomorphic if and only if after some reordering H_i is conjugate to K_i for every i.

A *right G-set* is a set equipped with a right action of G, i.e. a multiplication rule $X \times G \to X$ (satisfying $x(gg') = (xg)g'$). The classification of right G-sets by right cosets is analogous to that of left G-sets by left ones.

8.1.1 Linearly equivalent G-sets

Henceforth G is a finite group, and all G-sets are finite, so that every G-set is isomorphic to a finite disjoint union of the form $\bigcup {}^{G}\!/\!_{H_i}$. For a G-set X, $\mathbb{C}[X]$ denotes the $\mathbb{C}G$-module (i.e. complex representation of G) having X as a basis, with G acting on $\mathbb{C}[X]$ by the linear extension of its action on X, i.e. $g \sum a_i x_i = \sum a_i g x_i$ $(g \in G,\ a_i \in \mathbb{C},\ x_i \in X)$.

If $X \cong Y$ (as G-sets), then $\mathbb{C}[X] \cong \mathbb{C}[Y]$ (as $\mathbb{C}G$-modules), but not vice versa. In fact, this is precisely where (1.6) and (1.7) come from:

Proposition 8.1. *For two (finite) G-sets X, Y the following are equivalent:*

(1) $\mathbb{C}[X] \cong \mathbb{C}[Y]$ *as complex representations of G.*

(2) Every $g \in G$ fixes the same number of elements in X and in Y.

(3) $X \cong \bigcup {}^{G}/_{H_i}$ *and* $Y \cong \bigcup {}^{G}/_{K_i}$ *for* $H_i, K_i \leq G$ *satisfying* (1.7).

Proof. The character of $\mathbb{C}[X]$ is $\chi_{\mathbb{C}[X]}(g) = |\mathrm{fix}_X(g)|$, hence by character theory (*1*) is equivalent to (*2*). It is a simple exercise to show that $\left|\mathrm{fix}_{G/H}(g)\right| = \frac{|[g]\cap H||C_G(g)|}{|H|}$, so that for H_i such that $X \cong \bigcup {}^{G}/_{H_i}$ we obtain

$$|\mathrm{fix}_X(g)| = \sum_i \left|\mathrm{fix}_{G/H_i}(g)\right| = |C_G(g)| \cdot \sum_i \frac{|[g]\cap H_i|}{|H_i|},$$

showing that (*2*) is equivalent to (*3*). □

Definition 8.2. G-sets X and Y as in Proposition 8.1 are said to be *linearly equivalent.*

Remark. In the literature one encounters also the terms *arithmetically equivalent*, *almost equivalent*, *Gassman pair*, or *Sunada pair*. Also, sometimes the "trivial case", namely when $X \cong Y$ as G-sets, is excluded.

8.1.2 Back to the example

In (1.8) we presented subgroups H_i, K_i of $G = \{e, \sigma, \tau, \sigma\tau\} \cong {}^{\mathbb{Z}}/_{2\mathbb{Z}} \times {}^{\mathbb{Z}}/_{2\mathbb{Z}}$, which satisfied condition (1.7). Figure 8.1 shows the corresponding G-sets $X = \bigcup {}^{G}/_{H_i}$ and $Y = \bigcup {}^{G}/_{K_i}$, and one indeed sees that

$$|\mathrm{fix}_X(g)| = |\mathrm{fix}_Y(g)| = \begin{cases} 6 & g = e \\ 2 & g = \sigma, \tau, \sigma\tau \end{cases}$$

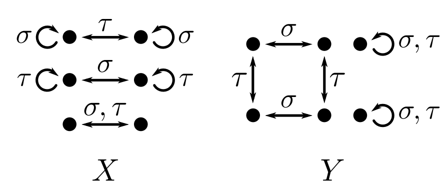


Figure 8.1: X and Y are linearly equivalent G-sets for $G = \{e, \sigma, \tau, \sigma\tau\}$, corresponding to the subgroups in (1.8).

We note that X and Y are not isomorphic as G-sets, as the sizes of their orbits are different: X has three orbits of size two, whereas Y has one orbit of size four and two orbits of size one.

8.1.3 The transitive case - Gassman-Sunada pairs

When restricting to transitive G-sets, X and Y are linearly equivalent exactly when $X \cong {}^{G}/_{H}$, $Y \cong {}^{G}/_{K}$ for $H, K \leq G$ satisfying the Sunada condition (1.6). In the literature H, K are known as *almost conjugate, locally conjugate, arithmetically equivalent, linearly equivalent, Gassman pair*, or *Sunada pair*, and again one usually excludes the trivial case, which is when H and K are conjugate. For a group to have a Sunada pair its order must be a product of at least five primes [DiP09], but there exist such n (the smallest being 80), for which no group of size n has one. The smallest group which admits a Sunada pair is ${}^{\mathbb{Z}}/_{8\mathbb{Z}} \rtimes \mathrm{Aut}\left({}^{\mathbb{Z}}/_{8\mathbb{Z}}\right)$ (of size 32).

8.1.4 Tensor product of G-sets

The theory of G-sets is parallel in many aspects to that of R-modules (where R stands for a non-commutative ring). This section describes in some details the G-set analogue of the tensor product of modules. Except for Definition 8.3, and the universal property (8.1), this section may be skipped by abstract nonsence haters.

If M is a right R-module, for every abelian group A the group of homomorphisms $\mathrm{Hom}_{Ab}(M, A)$ has a structure of a (left) R-module, by $(rf)(m) = f(mr)$. In fact, $\mathrm{Hom}_{Ab}(M, \_)$ is a functor from Ab to $Rmod$, the category of left R-modules. This functor has a celebrated left adjoint, the tensor product $M \otimes_R \_ : Rmod \to Ab$. This means that for every R-module N there is an isomorphism

$$\mathrm{Hom}_{Ab}(M \otimes_R N, A) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_{Ab}(M, A))$$

which is natural in N and A.

The analogue for G-sets is this: If X is a right G-set, then for every set S the set $\mathrm{Hom}_{Set}(X, S)$ has a structure of a (left) G-set, by $(gf)(x) = f(xg)$. Here $\mathrm{Hom}_{Set}(X, \_)$ is a functor from Set to $Gset$ (the category of left G-sets), and again it has a left adjoint:

Definition 8.3. The tensor product over G of a right G-set X and a left G-set Y, denoted $X \times_G Y$, is the set ${}^{X\times Y}/_{(xg,y)\sim(x,gy)}$, i.e. the quotient set of the Cartesian product $X \times Y$ by the relations $(xg, y) \sim (x, gy)$ (for all $x \in X$, $g \in G$, $y \in Y$).

The functor $X \times_G \_ : Gset \to Set$ is indeed the left adjoint of $\mathrm{Hom}_{Set}(X, \_)$: For every G-set Y there is an isomorphism (natural in Y and S)

$$\mathrm{Hom}_{Set}(X \times_G Y, S) \cong \mathrm{Hom}_G(Y, \mathrm{Hom}_{Set}(X, S)).$$

As it is custom to write B^A for $\mathrm{Hom}_{Set}(A, B)$, this can be written as

$$S^{X\times_G Y} \cong \mathrm{Hom}_G\left(Y, S^X\right) \tag{8.1}$$

which for $G = 1$ is the familiar isomorphism of sets $S^{X\times Y} \cong \left(S^X\right)^Y$.

The tensor product of G-sets behaves much like that of modules, e.g., there are natural isomorphisms as follows:

- Distributivity: $(\bigcup X_i) \times_G Y \cong \bigcup (X_i \times_G Y)$.
- Associativity: $(X \times_G Y) \times_H Z \cong X \times_G (Y \times_H Z)$ (where Y is a (G, H)-biset, i.e. $(gy)h = g(yh)$ holds for all $g \in G$, $y \in Y$, $h \in H$).
- Neutral element: $G \times_G X \cong X$.
- Extension of scalars: if $H \leq G$, G is a (G, H)-biset. For an H-set X, this gives $G \times_H X$ a G-set structure (by $g'(g, x) = (g'g, x)$). This construction is adjoint to the restriction of scalars: for any G-set Y one has
$$\mathrm{Hom}_G(G \times_H X, Y) \cong \mathrm{Hom}_H(X, Y). \tag{8.2}$$

Remark. A point in which groups and rings differ is the following: A left G-set can be regarded as a right one, by defining the right action to be $xg = g^{-1}x$. Thus, we shall allow ourselves to regard left G-sets as a right ones, and vice versa[(†)]. Going back to Definition 8.3, if we choose to regard X as a left G-set, we obtain

$$X \times_G Y = \frac{X \times Y}{(xg, y) \sim (x, gy)} = \frac{X \times Y}{(g^{-1}x, y) \sim (x, gy)} = \frac{X \times Y}{(x, y) \sim (gx, gy)} = {}^{X\times Y}\!/_{G}$$

i.e. the tensor product is the orbit set of the normal (Cartesian) product of the left G-sets X and Y. A word of caution: the process of turning a left G-set into a right one does not give it, in general, a (G, G)-biset structure.

8.2 Action and spectrum

8.2.1 Tensor product of G-manifolds

Assume we have an action of G by isometries on a Riemannian manifold M and on a finite G-set X. Our purpose is to study $M \times_G X$, which has a Riemannian orbifold structure as a quotient of $M \times X$ (where X is given the discrete topology)[(‡)]. In §1.6 we discussed unions of the form $\bigcup {}^{M}\!/_{H_i}$ for subgroups $H_i \leq G$, and this is still our object of study: we can choose subgroups H_i of G such that $X \cong \bigcup {}^{G}\!/_{H_i}$, and for any such choice we have an isometry $M \times_G X \cong \bigcup {}^{M}\!/_{H_i}$. This can be verified directly, or by the tensor properties:

$$\begin{aligned} M \times_G X &\cong M \times_G \left(\bigcup {}^{G}\!/_{H_i} \right) \cong \bigcup \left(M \times_G {}^{G}\!/_{H_i} \right) \cong \bigcup \left(M \times_G \left(G \times_{H_i} \mathbf{1} \right) \right) \\ &\cong \bigcup \left(\left(M \times_G G \right) \times_{H_i} \mathbf{1} \right) \cong \bigcup \left(M \times_{H_i} \mathbf{1} \right) \cong \bigcup {}^{M}\!/_{H_i} \end{aligned}$$

where $\mathbf{1}$ denotes a one-element set. In this light, the tensor product generalizes the notion of quotients, since quotients by subgroups of G correspond to tensoring with transitive G-sets: ${}^{M}\!/_{H} \cong M \times_G {}^{G}\!/_{H}$. The advantage of studying $M \times_G X$ rather than $\bigcup {}^{M}\!/_{H_i}$ is that the former is free of choices, and thus more suitable for functorial constructions, and yields more elegant proofs. On the other hand, $\bigcup {}^{M}\!/_{H_i}$ is much more familiar, and the reader is encouraged to envision $M \times_G X$ as a union of quotients of M.

The next theorem, which describes the space of functions on $M \times_G X$, is the heart of our isospectrality technique.

Theorem 8.4. *If a finite group G acts by isometries on a Riemannian manifold M then for every finite G-set X there is an isomorphism*

$$L^2 \left(M \times_G X \right) \cong \operatorname{Hom}_{\mathbb{C}G} \left(\mathbb{C} \left[X \right], L^2 \left(M \right) \right)$$

(where $L^2 (M)$ is a representation of G by $(gf)(m) = f\left(g^{-1}m\right)$.)

[(†)] For rings, a left R-module can only be regarded as a right R^{opp}-module, and in general $R \ncong R^{opp}$. In groups, $G \cong G^{\mathrm{opp}}$ canonically by the inverse map.

[(‡)] More generally, if M and M' are G-manifolds, $M \times_G M'$ is an orbifold (manifold, if G acts freely on $M \times M'$), but here we shall only consider the tensor product of a G-manifold and a finite G-set (which can be regarded as a compact manifold of dimension 0).

Remark. In the language of [BPBS09, PB10], this means that $M \times_G X$ is an ${}^{M}\!/\!{}_{\mathbb{C}[X]}$-manifold, and since $M \times_G X \cong \bigcup {}^{M}\!/\!{}_{H_i}$, this is implied in [BPBS09, §9.3]. However, the perspective of tensor product gives a direct proof.

Proof. We have isomorphisms of vector spaces:

$$\mathbb{C}^{M \times_G X} \cong \mathrm{Hom}_G\left(X, \mathbb{C}^M\right) \cong \mathrm{Hom}_{\mathbb{C}G}\left(\mathbb{C}\left[X\right], \mathbb{C}^M\right). \tag{8.3}$$

The left one is by adjointness of tensor and hom (8.1), and it is given explicitly by sending $f \in \mathbb{C}^{M \times_G X}$ to $F \in \mathrm{Hom}_G\left(X, \mathbb{C}^M\right)$ defined by $F\left(x\right)\left(m\right) = f\left(m, x\right)$. The next isomorphism is by adjointness of the free construction $X \mapsto \mathbb{C}\left[X\right]$ and the forgetful functor $\mathbb{C}Gmod \to Gset$, i.e.

$$\mathrm{Hom}_G\left(X, \_\right) \cong \mathrm{Hom}_{\mathbb{C}G}\left(\mathbb{C}\left[X\right], \_\right), \tag{8.4}$$

and is given explicitly by linear extension, namely, defining $F\left(\sum a_i x_i\right) = \sum a_i F\left(x_i\right)$. The correspondence of the L^2 conditions then follows from the finiteness of G and X, and the fact that $\int_{M \times X} \left|f\right|^2 = \sum_{x \in X} \int_M \left|f\left(\,\cdot\,, x\right)\right|^2$. $\square$

Definition 8.5. The *spectrum* of a Riemannian manifold M is the function $\mathrm{Spec}_M : \mathbb{R} \to \mathbb{N}$ which prescribes to every number its multiplicity as an eigenvalue of the Laplace operator on M, i.e. $\mathrm{Spec}_M\left(\lambda\right) = \dim L^2_\lambda\left(M\right)$ where $L^2_\lambda\left(M\right) = \left\{f \in L^2\left(M\right) \,\middle|\, \Delta f = \lambda f\right\}$.

Corollary 8.6. *If G acts on M, and X and Y are linearly equivalent G-sets, then $M \times_G X$ and $M \times_G Y$ are isospectral.*

Remark. For transitive X and Y, this is equivalent to Sunada's theorem.

Proof. By Theorem 8.4, we have $L^2\left(M \times_G X\right) \cong L^2\left(M \times_G Y\right)$, but we must verify that this isomorphism respects the Laplace operator. If $y \mapsto \sum_{x \in X} a_{y,x} x$ is a $\mathbb{C}G$-module isomorphism from $\mathbb{C}\left[Y\right]$ to $\mathbb{C}\left[X\right]$, then $\mathcal{T} : L^2\left(M \times_G X\right) \xrightarrow{\cong} L^2\left(M \times_G Y\right)$ is given explicitly by $\left(\mathcal{T} f\right)\left(m, y\right) = \sum_{x \in X} a_{y,x} f\left(m, x\right)$ ($\mathcal{T}$ is a *transplantation* map, see [Bus86, Bér92, CDS94, Cha95]). This isomorphism commutes with the Laplace operators on their domains of definition, hence inducing isomorphism of eigenspaces, and in particular equality of spectra. Alternatively, one can replace L^2 throughout Theorem 8.4 with L^2_λ, obtaining directly $L^2_\lambda\left(M \times_G X\right) \cong \mathrm{Hom}_{\mathbb{C}G}\left(\mathbb{C}\left[X\right], L^2_\lambda\left(M\right)\right)$, and thus $L^2_\lambda\left(M \times_G X\right) \cong L^2_\lambda\left(M \times_G Y\right)$. $\square$

The theorem and corollary above give us isospectral manifolds, but do not tell us whether they are isometric or not. First of all, if X and Y are isomorphic as G-sets then $M \times_G X$ and $M \times_G Y$ are certainly isometric. However, this may happen also for non-isomorphic G-sets[†]. The next section deals with this inconvenience.

[†]For example, if H and K are isomorphic subgroups of G, and the action of G on M can be extended to an action of some supergroup $\widehat{G}$ in which H and K are conjugate, then ${}^{M}\!/\!{}_{H}$ and ${}^{M}\!/\!{}_{K}$ are also isometric.

8.2.2 Unbalanced pairs

In §8.1.2 we concluded that the G-sets X and Y in Figure 8.1 were non-isomorphic by pointing out differences in the sizes of their orbits. This property is stronger than just being non-isomorphic, and we give it a name.

Definition 8.7. For a finite group G, a pair of finite G-sets X, Y is an *unbalanced pair* if they are linearly equivalent (i.e. $\mathbb{C}[X] \cong \mathbb{C}[Y]$ as $\mathbb{C}G$-modules), and if in addition they differ in the sizes of their orbits, namely, for some n the number of orbits of size n in X and the number of such orbits in Y are different.

Remark 8.8. Since the size of a G-set X equals $\dim \mathbb{C}[X]$, and the number of orbits in X equals $\dim \left(\mathbb{C}[X]^G\right)$ (†), linearly equivalent G-sets necessarily have the same size and number of orbits. Thus, there are no unbalanced pairs in which one of the sets is transitive, and in particular there are no unbalanced Sunada pairs.

Theorem 8.9. *If X, Y is an unbalanced pair of G-sets, then for any faithful action of G by isometries on a compact connected manifold M, the manifolds (or orbifolds) $M \times_G X$ and $M \times_G Y$ are isospectral and non-isometric.*

Proof. Isospectrality was obtained in Corollary 8.6. To show that $M \times_G X$ and $M \times_G Y$ cannot be isometric, we choose H_i such that $X \cong \bigcup G/H_i$, and observe that

- Since M is connected, $\{M/H_i\}$ form the connected components of $M \times_G X$.
- Since G acts faithfully and M is connected, $\operatorname{vol} M/H_i = \frac{\operatorname{vol} M}{|H_i|}$.

Thus, the sizes of orbits in X correspond to the volumes of connected components in $M \times_G X$ (‡). Therefore, if X and Y form an unbalanced pair then $M \times_G X$ and $M \times_G Y$ differ in the volumes of their connected components. To be precise, if X and Y have different numbers of orbits of size n, then $M \times_G X$ and $M \times_G Y$ have different numbers of connected components of volume $\frac{n \cdot \operatorname{vol} M}{|G|}$. □

8.2.3 The Burnside ring and the lattice of isospectral quotients

A nice point of view is attained from $\Omega(G)$, the Burnside ring of the group G. Its elements are formal differences of isomorphism classes of finite G-sets, namely $X - Y$ where X and Y are finite G-sets, with $X - Y = X' - Y'$ whenever $X \cup Y' \cong X' \cup Y$. The operations in $\Omega(G)$ are disjoint union and Cartesian product (extended to formal differences by distributivity). If we fix representatives $H_1, \ldots, H_r$ for the conjugacy classes of subgroups in G, the classification of G-sets (see §8.1) tells us that $\Omega(G) = \{\sum_{i=1}^r n_i \cdot G/H_i \mid n_i \in \mathbb{Z}\}$, so that as an abelian group $\Omega(G)^+ \cong \mathbb{Z}^r$ with $\{G/H_i\}_{i=1}^r$ being a basis.

Now, instead of looking at a pair of G-sets (X, Y), we look at the element $X - Y$ in $\Omega(G)$. First, we note that some information is lost: For any G-set Z, the pair (X, Y) and the pair $(X' = X \cup Z, Y' = Y \cup Z)$ both correspond to the same element in $\Omega(G)$, i.e. $X - Y = X' - Y'$.

(†) V^G denotes the G-invariant part of a representation V: $V^G = \{v \in V \mid gv = v \, \forall g \in G\}$.

(‡) This correspondence between sizes of orbits and volumes of components is apparent in Figures 8.1 and 1.3.

Second, we notice this is in fact desirable. In order to produce elegant isospectral pairs, one would like to "cancel out" isometric connected components shared by two isospectral manifolds (as in [Cha95]), and the pair $M \times_G X'$, $M \times_G Y'$ is just the pair $M \times_G X$, $M \times_G Y$ with each manifold added $M \times_G Z$.

Thus, we would like to look at *reduced pairs*, pairs of G-sets X, Y which share no isomorphic sub-G-sets (equivalently, no isomorphic orbits). The map $(X, Y) \mapsto X - Y$ gives a correspondence between reduced pairs and the elements of $\Omega(G)^{(\dagger)}$. Since $X \cong Y$ if and only if $X - Y = 0$, nonzero elements in $\Omega(G)$ correspond to reduced pairs of non-isomorphic G-sets, and 0 corresponds to the (reduced) pair $(\varnothing, \varnothing)$.

A second ring of interest is $R(G)$, the representation ring of G. Its elements are formal differences of isomorphism classes of complex representations of G, with the operations being direct sum and tensor product. $R(G)$ also denotes the ring of virtual characters of G, which is isomorphic to the representation ring (see e.g. [Ser77, §9.1]). There is a ring homomorphism from $\Omega(G)$ into $R(G)$, given by $X \mapsto \mathbb{C}[X]$ (or $X \mapsto \chi_{\mathbb{C}[X]}$, considering $R(G)$ as the character ring). We denote the kernel of this homomorphism by $\mathcal{L}(G)$, and say that its elements are *linearly trivial*. The formal difference $X - Y$ is in $\mathcal{L}(G)$ if and only if $\mathbb{C}[X] \cong \mathbb{C}[Y]$, so that we have a correspondence between linearly trivial elements in $\Omega(G)$ and reduced pairs of linearly equivalent G-sets.

Since $\mathcal{L}(G)$, the ideal of linearly trivial elements, is a subgroup of the free abelian group $\Omega(G)^+ \cong \mathbb{Z}^r$, it is also free abelian: $\mathcal{L}(G) \cong \mathbb{Z}^m$ for some $m \leq r$. This means that we can find a $\mathbb{Z}$-basis for $\mathcal{L}(G)$ (we demonstrate how to compute such a basis in §8.4). This gives a lattice of linearly equivalent reduced pairs, as follows: if $\{X_i - Y_i\}_{i=1..m}$ is a basis for $\mathcal{L}(G)$, and we define for $\bar{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m$

$$X_{\bar{n}} = \left(\bigcup_{i : n_i > 0} n_i X_i \right) \cup \left(\bigcup_{i : n_i < 0} |n_i| \, Y_i \right)$$
$$Y_{\bar{n}} = \left(\bigcup_{i : n_i < 0} |n_i| \, X_i \right) \cup \left(\bigcup_{i : n_i > 0} n_i Y_i \right)$$

then every reduced pair of linearly equivalent G-sets (X, Y) is obtained by canceling out common factors in $(X_{\bar{n}}, Y_{\bar{n}})$, for a unique $\bar{n} \in \mathbb{Z}^m$.

Given an action of G on a manifold M, we associate with every G-set X a manifold, namely $M \times_G X$. The lattice of linearly equivalent pairs then maps to a lattice of isospectral pairs (see the example in §8.4). For a general manifold M, this might be only a sublattice of the lattice of isospectral quotients, which can be described as follows: We pull the spectrum function backwards to $\Omega(G)$, defining $\mathrm{Spec}_{X-Y} = \mathrm{Spec}_{M \times_G X} - \mathrm{Spec}_{M \times_G Y}$ (so that we have $\mathrm{Spec} : \Omega(G) \to \mathbb{Z}^{\mathbb{R}}$). Isospectral pairs of the form $(M \times_G X, M \times_G Y)$ are exactly those for which $X - Y \in \ker \mathrm{Spec}$, and Corollary 8.6 states that this kernel (for any M) contains $\mathcal{L}(G)$.

8.3 Construction of unbalanced pairs

Our objective in this section is to find unbalanced pairs. That is, given a group G, to find two G-sets X, Y which differ in the number of orbits of some size, and such that $\mathbb{C}[X] \cong \mathbb{C}[Y]$ as $\mathbb{C}G$-modules.

$^{(\dagger)}$Just like the map $(x, y) \mapsto \frac{x}{y}$ gives a correspondence between reduced pairs of positive integers ($x, y \in \mathbb{N}$ such that $\gcd(x, y) = 1$), and positive rationals.

We shall do so by "balancing" unions of transitive G-sets, which correspond to coset spaces of the form ${}^{G}\!/_{H}$. For every subgroup $H \leq G$ we denote by $\mathscr{S}_H$ the function

$$\mathscr{S}_H(g) = \chi_{\mathbb{C}[G/H]}(g) = \left|\mathrm{fix}_{G/H}(g)\right| = \frac{|[g] \cap H|\,|C_G(g)|}{|H|} \tag{8.5}$$

$\mathbb{C}[{}^{G}\!/_{H}]$ is sometimes called the *quasiregular representation* of G on H, and $\mathscr{S}_H$ is thus the *quasiregular character*. It also bears the names $\mathbf{1}_H^G$, $\mathbf{1}\!\uparrow_H^G$, or $\mathrm{Ind}_H^G \mathbf{1}$, being the induction of the trivial character of H to G. Lastly, it is the image of ${}^{G}\!/_{H}$ under the map $\Omega(G) \to R(G)$, when the latter is regarded as the ring of virtual characters of G.

In light of Proposition 8.1, we shall seek H_i, K_i such that $\sum_i \mathscr{S}_{H_i} = \sum_i \mathscr{S}_{K_i}$, and then check that the obtained linearly equivalent pair is unbalanced. We use a few easy computations:

(1) For the trivial subgroup $1 \leq G$, we have

$$\mathscr{S}_1(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases} \tag{8.6}$$

(2) For $H = G$,

$$\mathscr{S}_G \equiv 1 \tag{8.7}$$

(3) For any H,

$$\mathscr{S}_H(e) = [G : H] \tag{8.8}$$

(4) For G abelian $[g] = \{g\}$ and $C_G(g) = G$, so that $\mathscr{S}_H = [G : H] \cdot \mathbf{1}_H$, i.e.

$$\mathscr{S}_H(g) = \begin{cases} [G : H] & g \in H \\ 0 & g \notin H \end{cases} \tag{8.9}$$

8.3.1 Cyclic groups

Finite cyclic groups have no unbalanced pairs. This follows from the following:

Proposition 8.10. *If G is finite cyclic, linearly equivalent G-sets are isomorphic.*

Proof. Let $G = {}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$, and $D = \{d \,|\, d > 0,\, d \mid n\}$. The subgroups of G are $H_d = \langle d \rangle$ for $d \in D$, and by (8.9) $\mathscr{S}_{H_d} = \frac{n}{d} \cdot \mathbf{1}_{H_d}$. A non-trivial pair of linearly equivalent G-sets corresponds to two different $\mathbb{N}$-combinations of $\{\mathscr{S}_{H_d}\}_{d \in D}$ that agree as functions. Finding such a pair is equivalent to finding a nonzero $\mathbb{Z}$-combination of $\{\mathscr{S}_{H_d}\}_{d \in D}$ which vanishes. However, the matrix $(\mathscr{S}_{H_d}(d'))_{d,d' \in D}$ is upper triangular with non-vanishing diagonal, which means that $\left\{\mathscr{S}_{H_d}\big|_D\right\}_{d \in D}$ are linearly independent over $\mathbb{Q}$, hence so are $\{\mathscr{S}_{H_d}\}_{d \in D}$. $\square$

8.3.2 $G = {}^{\mathbb{Z}}\!/_{p\mathbb{Z}} \times {}^{\mathbb{Z}}\!/_{p\mathbb{Z}}$

Here we generalize the pair which appeared in Sections 1.6 and 8.1.2. Let p be a prime. $G = {}^{\mathbb{Z}}\!/_{p\mathbb{Z}} \times {}^{\mathbb{Z}}\!/_{p\mathbb{Z}}$ has $p+1$ subgroups of size (and index) p: $H_\lambda = \left\{(x, y) \,\middle|\, \frac{x}{y} = \lambda\right\}$, where $\lambda \in P^1(\mathbb{F}_p) =$

$\{0, 1, .., p-1, \infty\}$. Every non-identity element in G appears in exactly one of these, and we obtain by (8.8) and (8.9)

$$\sum_{\lambda \in P^1(\mathbb{F}_p)} \mathscr{S}_{H_\lambda}(g) = \begin{cases} p(p+1) & g = e \\ p & g \neq e \end{cases}.$$

Consulting (8.6) and (8.7), we find that this is the same as $p \cdot \mathscr{S}_G + \mathscr{S}_1$, so there is linear equivalence between

$$X = \bigcup_{\lambda \in P^1(\mathbb{F}_p)} {}^{G}\!/\!_{H_\lambda} \qquad \text{and} \qquad Y = \underbrace{\mathbf{1} \cup \ldots \cup \mathbf{1}}_{p} \cup G\,, \tag{8.10}$$

where $\mathbf{1}$ denotes the G-set with one element (corresponding to ${}^{G}\!/\!_{G}$). Obviously, this is an unbalanced pair (X has $p+1$ orbits of size p, and Y has one orbit of size p^2 and p orbits with a single element). Figure 8.1 shows X, Y for $p = 2$ (by their Schreier graphs with respect to the standard basis of ${}^{\mathbb{Z}}\!/\!_{2\mathbb{Z}} \times {}^{\mathbb{Z}}\!/\!_{2\mathbb{Z}}$).

8.3.3 Application - Hecke pairs

Let

$$G = \langle \sigma, \tau \mid \sigma^p = \tau^p = 1, \sigma\tau = \tau\sigma \rangle \cong {}^{\mathbb{Z}}\!/\!_{p\mathbb{Z}} \times {}^{\mathbb{Z}}\!/\!_{p\mathbb{Z}}$$

act on the torus $T = {}^{\mathbb{R}^2}\!/\!_{\mathbb{Z}^2}$ by the rotations $\sigma \cdot (x, y) = \left(x, y + \frac{1}{p}\right)$ and $\tau \cdot (x, y) = \left(x + \frac{1}{p}, y\right)$. From the unbalanced pair (8.10) one obtains the isospectral pair $T \times_G X$ and $T \times_G Y$, each a union of $p+1$ tori. These examples were constructed using different techniques by Doyle and Rossetti, who baptized them "Hecke pairs" [DR11]. The cases $p = 2, 3, 5$ are illustrated in Figure 8.2. One can verify that the analogue pair for $p = 4$, for example, is not isospectral - the reason is that unlike in the prime case the subgroups

$$H_\lambda = \begin{cases} \{(x, \lambda x) \mid x \in {}^{\mathbb{Z}}\!/\!_{4\mathbb{Z}}\} & \lambda = 0..3 \\ \{(0, x) \mid x \in {}^{\mathbb{Z}}\!/\!_{4\mathbb{Z}}\} & \lambda = \infty \end{cases}$$

do not cover $({}^{\mathbb{Z}}\!/\!_{4\mathbb{Z}} \times {}^{\mathbb{Z}}\!/\!_{4\mathbb{Z}}) \setminus \{0\}$ evenly.

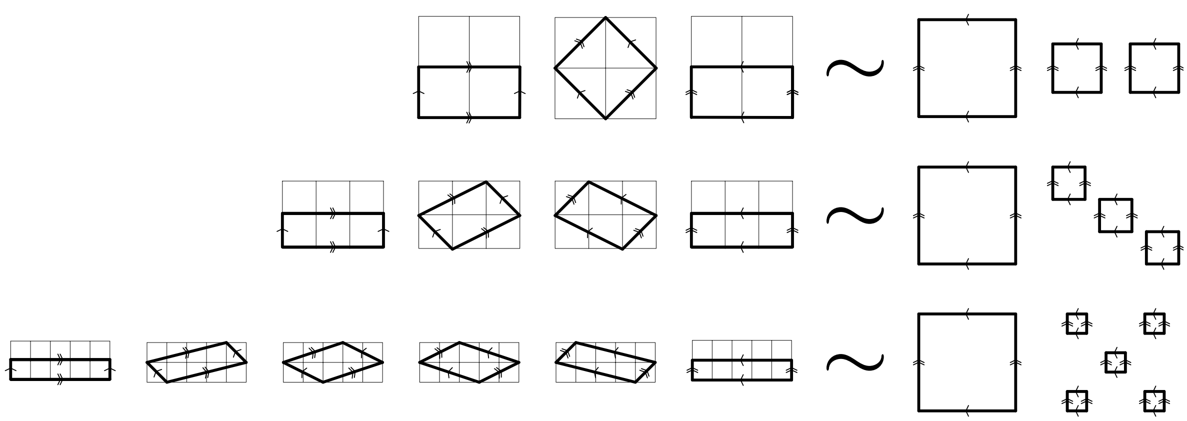


Figure 8.2: Isospectral pairs consisting of unions of tori, obtained as the tensor product over $G = {}^{\mathbb{Z}}\!/\!_{p\mathbb{Z}} \times {}^{\mathbb{Z}}\!/\!_{p\mathbb{Z}}$ of the torus ${}^{\mathbb{R}^2}\!/\!_{\mathbb{Z}^2}$ and the G-sets in (8.10), for $p = 2, 3, 5$. Grids are drawn to clarify the sizes.

Remark. Since the spectrum of a flat torus is represented by a quadratic form, isospectrality between flat tori can be interpreted as equality in the representation numbers of forms[†]. For example, isospectrality in the case $p = 2$ (Figure 8.2, top) asserts that together the quadratic forms $4m^2 + n^2$, $2m^2 + 2n^2$ and $4m^2 + n^2$ represent (over the integers) every value the same number of times as do $m^2 + n^2$, $4m^2 + 4n^2$, and $4m^2 + 4n^2$ together.

8.3.4 $G = \mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$

Now let G be the non-abelian group of size pq, where p and q are primes such that $q \equiv 1 \pmod{p}$. G has one subgroup Q of size q, and q subgroups $P_1, P_2, \ldots, P_q$ of size p. Since Q is normal we have

$$[g] \cap Q = \begin{cases} [g] & g \in Q \\ \varnothing & g \notin Q \end{cases} \qquad \Rightarrow \qquad \mathscr{S}_Q(g) = \begin{cases} p & g \in Q \\ 0 & g \notin Q \end{cases}.$$

Every non-identity element of G generates its entire centralizer, for otherwise it would be in the center. Thus for $g \neq e$

$$\sum_{i=1}^{q} \mathscr{S}_{P_i}(g) = \frac{|C_G(g)|}{p} \sum_{i=1}^{q} |[g] \cap P_i| = \frac{|C_G(g)|}{p} \cdot |[g] \cap (G \backslash Q)| = \begin{cases} 0 & g \in Q \backslash \{e\} \\ q & g \notin Q \end{cases}$$

but since P_i are all conjugate we have $\mathscr{S}_{P_i} = \mathscr{S}_{P_1}$ for all i. Denoting $P = P_1$, we have by the above and (8.8)

$$\mathscr{S}_P(g) = \begin{cases} q & g = e \\ 0 & g \in Q \backslash e \\ 1 & g \notin Q \end{cases}$$

and we find that

$$(p \cdot \mathscr{S}_P + \mathscr{S}_Q)(g) = (p \cdot \mathscr{S}_G + \mathscr{S}_1)(g) = \begin{cases} pq + p & g = e \\ p & g \neq e \end{cases}$$

which gives us the unbalanced pair

$$X = \underbrace{G/P \cup \ldots \cup G/P}_{p} \cup G/Q \qquad \text{and} \qquad Y = \underbrace{\mathbf{1} \cup \ldots \cup \mathbf{1}}_{p} \cup G.$$

This pair was discovered and used for constructing isospectral surfaces by Hillairet [Hil08].

8.3.5 Example - dihedral groups

A nice family of groups of the form $\mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$ is formed by the dihedral groups of order $2q$, where q is an odd prime. $D_q = \left\langle \sigma, \tau \,\middle|\, \sigma^q, \tau^2, (\sigma\tau)^2 \right\rangle$ acts by symmetries on the regular q-gon (say, with Neumann boundary conditions). In this case, the unbalanced pair we obtained above is $X = D_q/\langle\tau\rangle \cup D_q/\langle\tau\rangle \cup D_q/\langle\sigma\rangle$, $Y = \mathbf{1} \cup \mathbf{1} \cup D_q$, which gives for every q an isospectral pair consisting of six orbifolds, five of which are

[†] This insight (in the opposite direction) led Milnor to the first construction of isospectral manifolds [Mil64].

planar domains with Neumann boundary conditions, and the sixth (the quotient by $\langle\sigma\rangle$) a $\frac{2\pi}{q}$-cone. Figure 8.3 shows the case $q = 5$.

Let us remark that similar pairs with different boundary conditions can be constructed by observing other representations of D_q and its subgroups - see [BPBS09, §9.3] for an example.

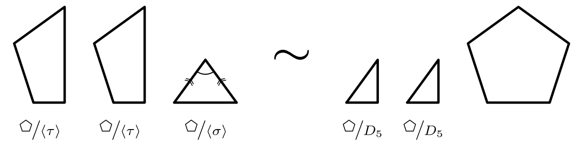


Figure 8.3: An isospectral pair obtained from the action of $D_5 \cong {}^{\mathbb{Z}}/_{5\mathbb{Z}} \rtimes {}^{\mathbb{Z}}/_{2\mathbb{Z}}$ on a regular pentagon. All boundary conditions are Neumann.

8.3.6 Non-cyclic groups

A group H is said to be *involved* in a group G if there exist some $L \trianglelefteq K \leq G$ such that ${}^{K}/_{L} \cong H$.

Proposition 8.11. *If a group H which has an unbalanced pair is involved in G, then G has an unbalanced pair.*

Proof. It is enough to assume that H is either a subgroup or a quotient of G. Assume first that $H \leq G$. If X, Y is an unbalanced pair of H-sets, the induced G-sets $G \times_H X$ and $G \times_H Y$ (see §8.1.4) form an unbalanced pair as well:

- They are linearly equivalent: we have natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathbb{C}G}\left(\mathbb{C}\left[G \times_H X\right], \_\right) &\cong \operatorname{Hom}_G\left(G \times_H X, \_\right) \\ &\cong \operatorname{Hom}_H\left(X, \_\right) \cong \operatorname{Hom}_{\mathbb{C}H}\left(\mathbb{C}\left[X\right], \_\right)\end{aligned}$$

  where the first and last isomorphisms are by (8.4), and the middle one is by (8.2). Since $\mathbb{C}[X] \cong \mathbb{C}[Y]$ as $\mathbb{C}H$-modules, we obtain that $\mathbb{C}[G \times_H X] \cong \mathbb{C}[G \times_H Y]$ as $\mathbb{C}G$-modules.

- The sizes of orbits in $G \times_H X$ are the sizes of orbits in X multiplied by $[G : H]$, since if $X \cong \bigcup {}^{H}/_{H_i}$ is a decomposition of X into H-orbits then

$$\begin{aligned}G \times_H X &\cong G \times_H \left(\bigcup {}^{H}/_{H_i}\right) \cong \bigcup G \times_H {}^{H}/_{H_i} \\ &\cong \bigcup G \times_H \times H \times_{H_i} \mathbf{1} \cong \bigcup G \times_{H_i} \mathbf{1} \cong \bigcup {}^{G}/_{H_i}\end{aligned}$$

  is a decomposition of $G \times_H X$ into G-orbits.

Assume now that $\pi : G \twoheadrightarrow H$ is an epimorphism. An H-set X has a G-set structure by $gx = \pi(g)x$, and an unbalanced pair of H-sets X, Y is also an unbalanced pair of G-sets: since G realizes the same permutations in $\operatorname{Sym}(X)$ as does H, a linear H-equivariant isomorphism $\mathbb{C}[X] \cong \mathbb{C}[Y]$ is also G-equivariant, and the orbits in X as a G-set and as an H-set are the same. $\square$

Remark. If G acts on a manifold M, and X is an H-set for some $H \leq G$, then we have

$$M \times_G (G \times_H X) \cong (M \times_G G) \times_H X \cong M \times_H X$$

i.e. the induced G-set gives the same manifold as does the original H-set.

Theorem 8.12. *Every non-cyclic finite group has an unbalanced pair.*

Proof. Let G be a non-cyclic finite group. Assume first that some p-Sylow group $P \leq G$ is not cyclic (in particular this is the case if G is abelian). Let $\Phi(P)$ be the Frattini subgroup of P, which is the intersection of all of its maximal proper subgroups. For any p-group P the quotient ${}^{P}\!/\!_{\Phi(P)}$ is an elementary p-group of the same rank as P, so that if P is non-cyclic then ${}^{P}\!/\!_{\Phi(P)}$ must contain ${}^{\mathbb{Z}}\!/\!_{p\mathbb{Z}} \times {}^{\mathbb{Z}}\!/\!_{p\mathbb{Z}}$. Therefore, ${}^{\mathbb{Z}}\!/\!_{p\mathbb{Z}} \times {}^{\mathbb{Z}}\!/\!_{p\mathbb{Z}}$ is involved in G and we are done by Proposition 8.11 and §8.3.2.

Zassenhaus classified the finite groups whose Sylow subgroups are all cyclic [Hal76, Thm. 9.4.3]. They are of the form

$$G_{m,n,r} = \left\langle a, b \,\middle|\, a^m = b^n = e, b^{-1}ab = a^r \right\rangle = {}^{\mathbb{Z}}\!/\!_{m\mathbb{Z}} \rtimes_{\vartheta_r} {}^{\mathbb{Z}}\!/\!_{n\mathbb{Z}}$$

for m, n, r satisfying $(m, n(r-1)) = 1$ (here $\vartheta_r(1)(1) = r$, and $r^n \equiv 1 \pmod{m}$ is implied to make $\vartheta_r : {}^{\mathbb{Z}}\!/\!_{n\mathbb{Z}} \to \mathrm{Aut}\left({}^{\mathbb{Z}}\!/\!_{m\mathbb{Z}}\right) \cong \left({}^{\mathbb{Z}}\!/\!_{m\mathbb{Z}}\right)^{\times}$ a homomorphism). Since $0 \times \ker\vartheta_r \leq Z(G)$, and the quotient ${}^{G}\!/\!_{Z(G)}$ is never cyclic for nonabelian G, we can assume (by Proposition 8.11) that ϑ_r is injective. We can also assume that n is prime, for otherwise for any nontrivial factor k of n we have a proper subgroup $\left\langle a, b^k \right\rangle = {}^{\mathbb{Z}}\!/\!_{m\mathbb{Z}} \rtimes_{\vartheta_{r^k}} {}^{\mathbb{Z}}\!/\!_{\frac{n}{k}\mathbb{Z}}$ which is non-cyclic by the injectivity of ϑ_r. We can further assume that m is prime. Otherwise, pick some prime q dividing m, and consider $\left\langle a^{m/q}, b \right\rangle$: it is cyclic only if ϑ_r fixes $a^{m/q}$, i.e. $a^{rm/q} = a^{m/q}$, so that $m \mid \frac{m}{q}(r-1)$, which is impossible since $(m, n(r-1)) = 1$. Thus, by §8.3.4 we are done. □

Since unbalanced G-sets are in particular non-isomorphic, this together with Proposition 8.10 give the following:

Corollary 8.13. *For a finite group G, the map $\Omega(G) \to R(G)$ which takes a G-set X to the representation $\mathbb{C}[X]$ is injective iff G is cyclic.*

Theorems 8.9 and 8.12 together imply the results announced in §1.6:

Corollary 8.14. *If a finite non-cyclic group G acts faithfully by isometries on a compact connected Riemannian manifold M, then there exist G-sets X, Y such that $M \times_G X$ and $M \times_G Y$ are isospectral and non-isometric.*

From this follows:

Corollary 8.15. *If M is a compact connected Riemannian manifold (or orbifold) such that $\pi_1(M)$ has a finite non-cyclic quotient, then M has isospectral non-isometric covers.*

Proof. Let $\widetilde{M}$ be the universal cover of M, and N a normal subgroup in $\pi_1(M)$ such that $G = {}^{\pi_1(M)}\!/\!_{N}$ is finite non-cyclic. $\widehat{M} = {}^{\widetilde{M}}\!/\!_{N}$ is a finite cover of M and thus compact, and G acts on it faithfully, with ${}^{\widehat{M}}\!/\!_{G} = M$. By the previous corollary there exist isospectral non-isometric unions of quotients of $\widehat{M}$ by subgroups of G, and these are covers of M. □

8.4 Computation

Here we show how to compute, using GAP [GAP13], a basis for $\mathcal{L}(G)$, the ideal of linearly trivial elements in the Burnside ring $\Omega(G)$, which correspond to reduced pairs of linearly equivalent G-sets. We then consider an action of G and compute the isospectral pairs which correspond to this basis and action.

Let $G = D_6$ (see §8.3.5), and let $\{H_i\}$ be a set of representatives for the conjugacy classes of subgroups of G (so that $\{{}^{G}/_{H_i}\}$ is a $\mathbb{Z}$-basis of $\Omega(G)$). In the example which follows we compute the corresponding quasiregular characters $c_i = \mathscr{S}_{H_i}$, which are the images of this basis under the map $\Omega(G) \to R(G)$. We then compute a basis for $\mathcal{L}(G)$, the kernel of this map, and apply the LLL algorithm to this basis in order to possibly obtain a sparser one.

```
gap > G := DihedralGroup(12);;
gap > H := List(ConjugacyClassesSubgroups(G), Representative);;
gap > c := List(H, h -> List(PermutationCharacter(G,h)));;
gap > LLLReducedBasis(NullspaceIntMat(c)).basis;
        [[0,1,0,−1,0,0,−1,0,1,0], [1,−1,0,−1,0,0,0,−1,0,2],
        [−1,1,0,1,1,0,−1,0,−1,0], [−1,1,1,1,0,−2,0,0,0,0]]
```

For example, the first element in the basis we obtained tells us that ${}^{G}/_{H_2} - {}^{G}/_{H_4} - {}^{G}/_{H_7} + {}^{G}/_{H_9}$ vanishes in $R(G)$, so that ${}^{G}/_{H_2} \cup {}^{G}/_{H_9}$ is linearly equivalent to ${}^{G}/_{H_4} \cup {}^{G}/_{H_7}$. One has to explore the output of `ConjugacyClassesSubgroups(G)` to find out which subgroups these exactly are, or alternatively, to construct H_i oneself (in this case, for example, H_2 belongs to the conjugacy class of $\langle\tau\rangle$). The first line in Table 8.1 presents representatives for the classes returned by `ConjugacyClassesSubgroups(G)`, and the bottom four lines of the table show the basis that was calculated for $\mathcal{L}(D_6)$ above. One may check that pairs II, III and IV are unbalanced.

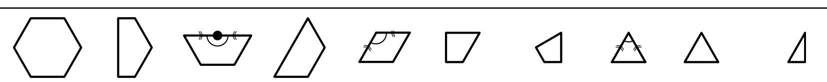
H_i	$\langle 1\rangle$	$\langle\tau\rangle$	$\langle\sigma^3\rangle$	$\langle\tau\sigma\rangle$	$\langle\sigma^2\rangle$	$\langle\tau,\tau\sigma^3\rangle$	$\langle\tau,\tau\sigma^2\rangle$	$\langle\sigma\rangle$	$\langle\tau\sigma,\tau\sigma^3\rangle$	$\langle\tau,\tau\sigma\rangle$
$⬡/H_i$										
I	0	1	0	-1	0	0	-1	0	1	0
II	1	-1	0	-1	0	0	0	-1	0	2
III	-1	1	0	1	1	0	-1	0	-1	0
VI	-1	1	1	1	0	-2	0	0	0	0

Table 8.1: Representatives for the conjugacy classes of subgroups in D_6, displayed with the corresponding quotients of the hexagon, and a basis for $\mathcal{L}(D_6) = \ker(\Omega(D_6) \to R(D_6))$.

Given an action of G on a manifold M, every difference of G-sets $X - Y \in \mathcal{L}(G)$ gives rise to an isospectral pair, namely $M \times_G X$, $M \times_G Y$. We consider the standard action of D_6 on the regular hexagon, which we denote by ⬡. The second line in Table 8.1 shows the quotients $⬡/H_i$ corresponding to the subgroups $H_i \leq D_6$ in the topmost line, and we see that in this case there are no isometric quotients arising from non-isomorphic G-sets. The isospectral pairs corresponding to the basis we obtained for $\mathcal{L}(D_6)$ are shown in Table 8.2.

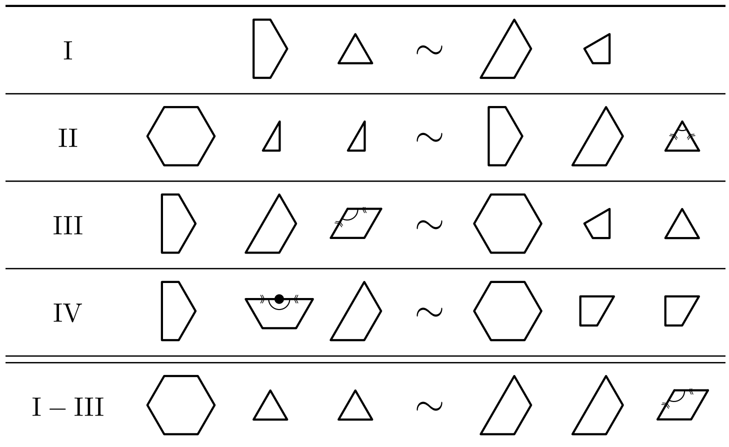


Table 8.2: The isospectral pairs corresponding to the basis for $\mathcal{L}(D_6)$ described in Table 8.1, and an example of an element obtained as a combination of these.

All isospectral pairs which arise from linear equivalences between D_6-sets are spanned by these four, as explained in §8.2.3. The bottom line in Table 8.2 demonstrates such a pair (corresponding to the element $\mathrm{I} - \mathrm{III}$). We remark that the pair corresponding to I is a hexagonal analogue of Chapman's "two piece band" [Cha95] - such analogues exist for every n (but for odd n the isospectral pair obtained is also isometric).

9 Generalizations and open questions

Isoperimetric constant

- As remarked after the statement of Theorem 1.2, one always has $h(X) = 0$ for X with a non-complete skeleton. One possible definition of a Cheeger constant for general complexes appears in 5.5. Another natural candidate is the following:

$$\widetilde{h}(X) = \min_{V=\coprod_{i=0}^{d} A_i} \frac{n \cdot |F(A_0, A_1, \ldots, A_d)|}{|F^{\partial}(A_0, A_1, \ldots, A_d)|},$$

  where $F^{\partial}(A_0, A_1, \ldots, A_d)$ denotes the set of $(d-1)$-spheres (i.e. copies of the $(d-1)$-skeleton of the d-simplex) having one vertex in each A_i. For a complex X with a complete skeleton, $\widetilde{h}(X) = h(X)$ as $F^{\partial}(A_0, \ldots, A_d) = A_0 \times \ldots \times A_d$. It is not hard to see that a lower Cheeger inequality does not hold here: consider any non-minimal triangulation of the $(d-1)$-shpere, and attach a single d-simplex to one of the $(d-1)$-cells on it. The obtained complex has $\lambda = 0$, and $\widetilde{h} = n$. However, we conjecture that the upper bound still holds, namely, that the inequality $\lambda(X) \leq \widetilde{h}(X)$ holds for every d-complex.

- In Riemannian geometry, the Cheeger constant of a Riemannian manifold M is concerned with its partitions into two submanifolds along a common boundary of codimension one. The original Cheeger inequalities, due to Cheeger [Che70] and Buser [Bus82], relate the Cheeger constant to the smallest eigenvalue of the Laplace-Beltrami operator on $C^{\infty}(M) = \Omega^0(M)$. Can one define an isoperimetric quantity which concerns partitioning of M into $d+1$ parts, and relate it to the spectrum of the Laplace-Beltrami operator on $\Omega^{d-1}(M)$, the space of smooth $(d-1)$-forms?

Random simplicial complexes

- In Lemma 5.6 it is shown that Linial-Meshulam complexes with expected degree $O(\log n)$ are expanders. Is there a similar model for general complexes, for which the skeletons are not complete? Specifically, is there one in which the expected degrees of cells are only logarithmic in the number of vertices - for example, a random triangle complex with n vertices, $O(n \log n)$ edges, and $O(n \log^2 n)$ triangles, which is expanding (in contrast, we only know this for $X\left(2, n, \frac{C \log n}{n}\right)$ which has $O\left(n^2 \log n\right)$ triangles).

- In the random graph model $G = G(n,p) = X(1,n,p)$, taking $p = \frac{k}{n}$ with a fixed k gives disconnected G a.a.s. However, random k-regular graphs are a.a.s. connected, and in fact are excellent expanders (see e.g. [Fri08, Pud12]). In higher dimension, $X = X\left(d, n, \frac{k}{n}\right)$ has a.a.s. a nontrivial $(d-1)$-homology, and also $h(X) = 0$ (by Corollary 5.7 (2)). Can one construct a model for random regular complexes, and are these complexes high-dimensional expanders? This is interesting even for a weak notion of regularity, such as having a bounded fluctuation of degrees, or having all links of vertices isomorphic.

Random walk

- In an infinite connected graph, the limit of $\sqrt[n]{\mathbf{p}_n^v(v)}$ (which describes the spectral radius of the transition operator, see §7.7) is independent of the starting point v. Is the same true in higher dimension? Namely, is $\limsup_{n\to\infty}\sqrt[n]{\mathcal{E}_n^\sigma(\sigma)}$ independent of σ for a $(d-1)$-connected complex?

- If $X \twoheadrightarrow Y$ is a covering map of graphs, then $\lambda(X) \geq \lambda(Y)$ (see e.g. [Kes59, Lemma 3.1], but beware - Kesten uses $\lambda(X)$ for what we denote by $1-\lambda(X)$). Does the same hold in higher dimension? If $\pi: X \twoheadrightarrow Y$ is a covering map of d-complexes, then the same argumentation as in graphs shows that for any $\widetilde{\sigma} \in X^{d-1}$ and $\sigma = \pi(\widetilde{\sigma}) \in Y^{d-1}$ one has $\mathbf{p}_n^{\widetilde{\sigma}}(\widetilde{\sigma}) \leq \mathbf{p}_n^\sigma(\sigma)$ and also $\mathbf{p}_n^{\widetilde{\sigma}}\left(\overline{\widetilde{\sigma}}\right) \leq \mathbf{p}_n^\sigma(\overline{\sigma})$. This, however, does not suffice to show that $\mathcal{E}_n^{\widetilde{\sigma}}(\widetilde{\sigma}) \leq \mathcal{E}_n^\sigma(\sigma)$. Showing that this hold (or even that it holds after taking n^{th}-roots and letting $n\to\infty$) would give the desired result.

- It is not hard to see that a $(d+1)$-partite d-complex is disorientable, but for $d \geq 2$ one can also construct examples of disorientable complexes which are not $(d+1)$-partite. It seems reasonable to conjecture that for simply connected complexes these properties coincide. Is this indeed the case?

- The suggestions for higher-dimensional analogues of amenability and transience raise several questions:

  - Can high amenability and transience be characterized in non-spectral terms (i.e. combinatorial expansion, or some $1-0$ event in the $(d-1)$-walk model)?
  - Are the transience properties $(\mathbf{T})$ and $(\mathbf{T}')$ equivalent under some conditions?
  - Are all d-complexes with degrees bounded by $d+1$ d-amenable?

- In classical settings, the Brownian motion on a Riemannian manifold constitutes a continuous limit of the discrete random walk. Can one define a continuous process, say, on the $(d-1)$-sphere bundle of a Riemannian manifold, which relates to its $(d-1)$-homology and to the spectrum of the Laplace-Beltrami operator on $(d-1)$-forms?

- There are surprising and useful connections between random walks on graphs and electrical networks (see e.g. [DS84, LP05]). Can a parallel theory be devised for the random $(d-1)$-walk on d-complexes?

Isospectrality

The isospectrality technique described in §8 (and thus Sunada's technique as well) has actually little to do with spectral geometry, since no property of the Laplace operator is used apart from being linear and commuting with isometries. For any linear operator F (on function spaces or other bundles, over manifolds or general spaces), these methods produce F-isospectral objects, given an action of a group which commutes with F.

However, it seems that in much more general settings, when a group action is studied, Sunada pairs are worth looking at. The most famous examples are Galois theory, giving Gassmann's construction of arithmetically equivalent number fields [Gas26], and Riemannian coverings, giving Sunada's isospectral construction; but Sunada pairs were also studied in the context of Lie groups [DGL89], ergodic systems [LTW02], dessin d'enfants [MP10], the spectrum of discrete graphs [Bro96] and metric ones [SS06], the Ihara zeta function of graphs [ST00], and the Witten zeta function of a Lie group [Lar04].

Sunada pairs in G correspond to linearly equivalent transitive G-sets, and we have seen that in the context of Riemannian coverings Sunada's technique generalizes to non-transitive G-sets as well. It is natural to ask whether other applications of Sunada pairs can be generalized in an analogous way. Of particular interest are unbalanced pairs, which do not exist in the transitive case (see Remark 8.8). In the settings of Riemannian manifolds they allowed us to deduce non-isometry, and one may hope that they play interesting roles in other situations.

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