Eigenvalues and Expansion of Regular Graphs

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Abstract. The spectral method is the best currently known technique to prove lower bounds on expansion. Ramanujan graphs, which have asymptotically optimal second eigenvalue, are the best-known explicit expanders. The spectral method yielded a lower bound of $k/4$ on the expansion of linear-sized subsets of k-regular Ramanujan graphs. We improve the lower bound on the expansion of Ramanujan graphs to approximately $k/2$. Moreover, we construct a family of k-regular graphs with asymptotically optimal second eigenvalue and linear expansion equal to $k/2$. This shows that $k/2$ is the best bound one can obtain using the second eigenvalue method. We also show an upper bound of roughly $1 + \sqrt{k-1}$ on the average degree of linear-sized induced subgraphs of Ramanujan graphs. This compares positively with the classical bound $2\sqrt{k-1}$. As a byproduct, we obtain improved results on random walks on expanders and construct selection networks (respectively, extrovert graphs) of smaller size (respectively, degree) than was previously known.

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1. *Introduction*

Expander graphs are widely used in Theoretical Computer Science, in areas ranging from parallel computation[1] to complexity theory and cryptography.[2] Given an undirected k-regular graph $G = (V, E)$ and a subset X of V, the expansion of X is defined to be the ratio $|N(X)|/|X|$, where $N(X) = \{w \in V: \exists v \in X, (v, w) \in E\}$ is the set of neighbors of X. An (α, β, k, n)-expander is a k-regular graph on n nodes such that every subset of size at most αn has expansion at least β.

It is known that random regular graphs are good expanders. In particular, for any $\beta < k - 1$, there exists a constant α such that, with high probability, all the subsets of a random k-regular graph of size at most αn have expansion at least β. The explicit construction of expander graphs is much more difficult, however. The first explicit construction of an infinite family of expanders was discovered by Margulis [1973], and improved in Gabber and Galil [1981], Alon et al. [1987], and Jimbo and Maruoka [1987].

The best currently known method to calculate lower bounds on the expansion in polynomial time relies on analyzing the second eigenvalue of the graph. Since the adjacency matrix A is symmetric, all its eigenvalues are real and will be denoted by $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$. We have $\lambda_0 = k$, and $\lambda = \max(\lambda_1, |\lambda_{n-1}|) \leq k$. Tanner [1984] proved that for any subset X of V,

$$|N(X)| \geq \frac{k^2|X|}{\lambda^2 + (k^2 - \lambda^2)|X|/n}. \tag{1}$$

Therefore, in order to get high expansion, we need λ to be as small as possible. However, for any sequence $G_{n,k}$ of k-regular graphs on n vertices, $\liminf \lambda(G_{n,k}) \geq 2\sqrt{k-1}$ as n goes to infinity [Alon 1986; Lubotzky et al. 1988; Nilli 1991]. Therefore, the best expansion coefficient we can obtain by applying Tanner's result is approximately $k/4$. This bound is achieved by Ramanujan graphs, which have been explicitly constructed [Lubotzky et al. 1988; Margulis 1988] for many pairs (k, n). By definition, a Ramanujan graph is a connected k-regular graph whose eigenvalues $\neq \pm k$ are at most $2\sqrt{k-1}$ in absolute value. The relationship between the eigenvalues of the adjacency matrix and the expansion coefficient has also been investigated in Alon [1986], Alon et al. [1987], Alon and Milman [1985], and Buck [1986], but the bound they get, when applied to nonbipartite Ramanujan graphs and for sufficiently large k, is no better than Tanner's bound. Other results about expanders are contained in Bien [1989], Lubotzky [to appear], and Sarnak [1990].

Some applications, such as the construction of nonblocking networks in Arora et al. [1990], required an expansion greater than $k/2$ for linear-sized subsets. Indeed, if the expansion of a subset X is greater than $k/2$, a constant fraction of its nodes have *unique neighbors*, that is, neighbors adjacent to only one node in X. This allows the construction of a matching between X and $N(X)$ in a logarithmic number of steps and using only *local computations*. Recently, Pippenger [1993] showed that weak expanders are sufficient in applications where an expansion greater than $k/2$ was required.

[1] See Ajtai et al. [1983], Arora et al. [1990], Pippenger [1993], and Upfal [1989].
[2] See Ajtai et al. [1987], Bellare et al. [1990], Goldreich et al. [1990], and Valiant [1976].

We define the linear expansion of a family of graphs G_n on n vertices to be the best lower bound on the expansion of subsets of size up to αn, where α is an arbitrary small positive constant, that is,

$$\sup_{\alpha>0} \inf_{n} \inf_{X} \frac{|N(X)|}{|X|},$$

where X ranges over the subsets of G_n of size at most αn. We show that if (G_n) is a family of k-regular graphs whose second largest eigenvalue is upper bounded by $\tilde{\lambda}$, the linear expansion of (G_n) is at least $(k/2)(1 - \sqrt{1 - (4k-4)/\tilde{\lambda}^2})$. In particular, the expansion of linear-sized subsets of Ramanujan graphs is lower bounded by a factor arbitrary close to $k/2$. On the other hand, for any integer k such that $k-1$ is a prime congruent to 1 modulo 4, and for any function m of n such that $m = o(n)$, we explicitly construct an infinite family of k-regular graphs G_n on n vertices such that $\lambda(G_n) \leq (2 + o(1))\sqrt{k-1}$ and G_n contains a subset of size $2m$ with expansion $k/2$. Since such a family has asymptotically optimal second eigenvalue, this shows that $k/2$ is essentially the best lower bound on the linear expansion one can obtain by the second eigenvalue method. The techniques used in this construction can be applied to prove tightness of relationships between eigenvalues and diameter [Kahale 1993]. Our results provide an efficient way to test that the expansion of linear-sized subsets of random graphs is at least $k/2 - O(k^{3/4}\log^{1/2}k)$. We also show that the average degree of the induced subgraphs on linear-sized subsets of a k-regular graph G is upper bounded by a factor arbitrary close to $1 + \tilde{\lambda}/2 + \sqrt{\tilde{\lambda}^2/4 - (k-1)}$, where $\tilde{\lambda} = max(\lambda, 2\sqrt{k-1})$. This bound is equal to $1 + \sqrt{k-1}$ in the case of Ramanujan graphs, improving upon the previous known bound [Alon and Chung 1988] of $2\sqrt{k-1}$.

Sections 3–5 contain our main results. In Section 6, we apply our techniques to obtain improved results on random walks on expanders. Random walks are often used in complexity theory and cryptography, and our bound improves upon previous results in Ajtai et al. [1987] and Goldreich et al. [1990]. Applications to selection networks and extrovert graphs are described in Section 7. We conclude with some remarks in Section 8.

Some of the results in this paper have appeared in an extended abstract form in Kahale [1991; 1993a], and in a more detailed form in Kahale [1993b].

2. *Notation, Definitions, and Background*

Throughout the paper, $G = (V, E)$ will denote an undirected graph on a set V of vertices. Let $L^2(V)$ denote the set of real-valued functions on V and $L_0^2(V) = \{f \in L^2(V); \sum_{v \in V} f(v) = 0\}$. As usual, we define the scalar product of two vectors f and g of $L^2(V)$ by

$$f \cdot g = \sum_{v \in V} f(v)g(v),$$

and the euclidean norm of a vector f by $\|f\| = \sqrt{f \cdot f}$. We denote the adjacency matrix of G by A_G, or simply by A if there is no risk of confusion. The matrix A is the 0-1 $n \times n$ matrix whose (i, j) entry is equal to 1 if and only if $(i, j) \in E$. It defines a linear operator in $L^2(V)$ that maps every vector $f \in L^2(V)$ to the vector Af defined by

$$(Af)(v) = \sum_{(v,w) \in E} f(w). \tag{2}$$

This operator is selfadjoint since $\forall f, g \in L^2(V)$,

$$(Af) \cdot g = f \cdot (Ag) = \sum_{(v,w) \in E} f(v) g(w). \tag{3}$$

For any matrix or operator M with real eigenvalues, we denote by $\lambda_i(M)$ the $(i+1)$st largest eigenvalue of M, $\lambda_i(A_G)$ by $\lambda_i(G)$, and $\max(\lambda_1(G), |\lambda_{n-1}(G)|)$ by $\lambda(G)$. For any subset W of V, we denote by χ_W the characteristic vector of W: $\chi_W(v) = 1$, if $v \in W$, and 0 otherwise. We denote the adjacency matrix of the graph induced on W by A_W, the real number $\lambda_i(A_W)$ by $\lambda_i(W)$, and the set of nodes at distance at most l from W by $B_l(W)$. For the rest of this section, we assume that G is k-regular.

Fact 2.1 [*Strang* 1988]. If B is a selfadjoint operator in a vector space L, then

$$\lambda_0(B) = \max_{g \in L - \{0\}} \frac{g \cdot Bg}{\|g\|^2}.$$

Clearly, the vector χ_V is an eigenvector of A with eigenvalue k. The vector space $L_0^2(V)$ is invariant under A, and the eigenvalues of the restriction of A to $L_0^2(V)$ are $\lambda_1(G), \ldots, \lambda_{n-1}(G)$. Therefore,

Fact 2.2. For any $g \in L_0^2(V)$, we have $g \cdot Ag \leq \lambda_1(G) \|g\|^2$.

For two column vectors g and h, we say that $g \leq h$ if every coordinate of g is at most its corresponding coordinate in h.

Fact 2.3 [*Seneta* 1981, *page* 28, *ex.* 1.12]. If a real symmetric matrix has only nonnegative entries, its largest eigenvalue is nonnegative and has a corresponding eigenvector with nonnegative components. This eigenvalue is largest in absolute value.

Fact 2.4. If a real symmetric matrix B has only nonnegative entries, and $\mathbf{s}$ is a vector with positive components such that $B\mathbf{s} \leq \gamma \mathbf{s}$, then the largest eigenvalue of B is at most γ. This property still holds if only the off-diagonal entries of B are assumed to be nonnegative [Friedman 1991].

Given a graph H, the *cover graph* H' of H is the graph defined on $V' = V \times \{0, 1\}$ and where $((u, l), (v, m)) \in V' \times V'$ is an edge if and only if $(u, v) \in E$ and $l \neq m$.

Fact 2.5. The eigenvalues of H' consist of the eigenvalues of H and their negated values.

3. *Main Lemma*

In this section, we prove the main lemma (Theorem 3.6) that we will use later to derive lower bounds on the expansion.

LEMMA 3.1. *If $G = (V, E)$ is k-regular on n vertices, then for any $f \in L^2(V)$, we have*

$$f \cdot Af \leq \lambda_1(G)\|f\|^2 + \frac{k - \lambda_1(G)}{n}\Big(\sum_{v \in V} f(v)\Big)^2.$$

PROOF. We decompose f as the sum of a constant vector and a vector in $L_0^2(V)$. Let $\bar{f} = (f \cdot \chi_V/n)\chi_V$ be the orthogonal projection of f on the subspace spanned by the constant vector χ_V. Then $f_0 = f - \bar{f}$ is the orthogonal projection of f on $L_0^2(V)$. By linearity, $Af = A\bar{f} + Af_0 = k\bar{f} + Af_0$, and so $f \cdot Af = k\|\bar{f}\|^2 + f_0 \cdot Af_0$ since $\bar{f} \cdot Af_0 = A\bar{f} \cdot f_0 = k\bar{f} \cdot f_0 = 0$. By Fact 2.2 and the Pythagorean theorem, we have

$$f_0 \cdot Af_0 \leq \lambda_1(G)\|f_0\|^2 = \lambda_1(G)\big(\|f\|^2 - \|\bar{f}\|^2\big).$$

Therefore, $f \cdot Af \leq \lambda_1(G)\|f\|^2 + (k - \lambda_1(G))\|\bar{f}\|^2$. We conclude the proof by noting that

$$\|\bar{f}\|^2 = \frac{(\sum_{v \in V} f(v))^2}{n}. \qquad \square$$

LEMMA 3.2. *For any subset W of a k-regular graph G, we have $\lambda_0(W) \leq \lambda_1(G) + (k - \lambda_1(G))|W|/n$.*

PROOF. Let g be any element of $L^2(W)$. Consider the vector $f \in L^2(V)$ that coincides with g on $L^2(W)$ and is null on $V - W$. By eq. (3), we see that $g \cdot A_W g = f \cdot A_G f$. By applying Lemma 3.1 to f, we have

$$\begin{aligned} g \cdot A_W g &\leq \lambda_1(G)\|g\|^2 + \frac{k - \lambda_1(G)}{n}\Big(\sum_{v \in W} g(v)\Big)^2 \\ &\leq \left(\lambda_1(G) + \frac{k - \lambda_1(G)}{n}|W|\right)\|g\|^2. \end{aligned}$$

The second equation follows from the Cauchy–Schwartz inequality. We conclude using Fact 2.1. $\square$

A similar relation [Kahale 1993b; Sect. 4.1] holds between $\lambda_i(W)$ and $\lambda_{i+1}(G)$, for $1 \leq i \leq |W| - 1$. Lemma 3.2 already gives a restriction on the structure of induced subgraphs. For example, since the average degree of a graph is upper bounded by its largest eigenvalue, it implies that the average degree of the induced subgraph on W is at most $\lambda_1(G) + (k - \lambda_1(G))|W|/n$, which is roughly $\lambda_1(G)$ for small linear-sized subsets. To obtain a stronger restriction on the induced subgraph on a linear-sized subset X, we will apply Lemma 3.2 to the set $B_l(X)$. We start by comparing the largest eigenvalue of the subgraph induced on $B_l(X)$ to the matrix of a weighted graph associated with X.

Let Δ_X be the diagonal matrix indexed by the vertices of X and whose entry (v, v) is equal to the degree of v in the subgraph induced on X. We will simply denote by $\Delta_X(v)$ the diagonal entry $\Delta_X(v, v)$. For $\theta' > 0$ and integer l, let

$$M_X^{\theta', l} = A_X + \frac{1}{\sqrt{k-1}} \frac{\sinh(l\theta')}{\sinh((l+1)\theta')}(kI - \Delta_X).$$

The matrix $M_X^{\theta',l}$ can be regarded as the matrix of the weighted graph that is induced on X and has in addition a loop of weight $(k-1)^{-1/2}(\sinh(l\theta')/\sinh((l+1)\theta'))(k-\Delta_X(v))$ on each node v of X.

LEMMA 3.3. *Suppose $G=(V,E)$ is k-regular. Let l be a positive integer, X a nonempty subset of V, θ' a positive real number, and $\lambda' = 2\sqrt{k-1}\cosh\theta'$. If $\lambda_0(B_l(X)) \le \lambda'$, then $\lambda_0(M_X^{\theta',l}) \le \lambda'$.*

PROOF. Let $W = B_l(X)$. The idea behind the proof is as follows: Let $f \in L^2(X)$ be an eigenvector of $M_X^{\theta',l}$ corresponding to its largest eigenvalue. We will extend f to W so that it becomes roughly an eigenvector of A_W. If the largest eigenvalue of $M_X^{\theta',l}$ were too big, we would get a large eigenvalue for A_W, contradicting the fact that $\lambda_0(W) \le \lambda'$.

We define the sequence r_i as follows:

$$r_i = \frac{\sinh((l+1-i)\theta')(k-1)^{-i/2}}{\sinh((l+1)\theta')}.$$

The sequence r_i is strictly positive and decreasing for $0 \le i \le l$, and

$$\lambda' r_i = r_{i-1} + (k-1)r_{i+1} \tag{4}$$

$$\lambda' r_l = r_{l-1}. \tag{5}$$

By Fact 2.3, we can assume that all the entries of f are nonnegative. We extend the vector f to W by setting

$$f(v) = \max_{u \in X}\left(f(u) r_{d(v,u)}\right),$$

for $v \in W - X$, where $d(v,u)$ denote the distance in G between v and u. Note that f is nonnegative on W since every node in W is at distance at most l from some node in X.

CLAIM 3.4. *For any $v \in W - X$, we have $(A_W f)(v) \ge \lambda' f(v)$.*

PROOF. Let $u \in X$ be such that $f(v) = f(u) r_{d(v,u)}$. Note that $i = d(v,u)$ can be assumed to be at most l since $r_j \le 0$ for $j \ge l+1$. Let v_1 be the first node on a shortest path from v to u. Since the distance between v_1 and u is $i-1$, we have $v_1 \in W$ and

$$f(v_1) \ge f(u) r_{i-1}. \tag{6}$$

We now distinguish two cases:

Case 1. $i = l$. Combining eqs. (6) and (5), we get in this case

$$(A_W f)(v) \ge f(v_1) \ge f(u) r_{l-1} = \lambda' f(u) r_l = \lambda' f(v),$$

as required.

Case 2. $i < l$. In this case, the k neighbors of v are at distance at most $i+1 \le l$ from u. By monotonicity of the sequence r, it follows that the value of f on each of these neighbors is at least $f(u) r_{i+1}$. Using again eq. (6), we have

$$\begin{aligned}(A_W f)(v) &\ge f(v_1) + (k-1) f(u) r_{i+1} \\ &\ge f(u)(r_{i-1} + (k-1) r_{i+1}) \\ &= \lambda' f(u) r_i \\ &= \lambda' f(v).\end{aligned}$$

CLAIM 3.5. *For any $v \in X$, we have $(A_W f)(v) \geq \lambda_0(M_X^{\theta',l})f(v)$.*

PROOF. For any $v \in X$, the value of f on each of the $k - \Delta_X(v)$ neighbors of v in $W - X$ is at least $f(v)r(1)$. Therefore,

$$\begin{aligned}(A_W f)(v) &\geq (A_X f)(v) + (k - \Delta_X(v))f(v)r(1)\\ &= (M_X^{\theta',l}f)(v) = \lambda_0(M_X^{\theta',l})f(v).\end{aligned}$$ □

Since $\lambda_0(W) \leq \lambda'$, we have $(A_W f)\cdot f \leq \lambda' \|f\|^2$, by Fact 2.1. Assume for contradiction that $\lambda' < \lambda_0(M_X^{\theta',l})$. By Claim 3.4 and Claim 3.5, this implies that $(A_W f)\cdot f > \lambda'\|f\|^2$, leading to a contradiction. □

For $\theta \geq 0$, define

$$M_X^\theta = A_X + \frac{1}{\sqrt{k-1}\,e^\theta}(kI - \Delta_X).$$

The matrix M_X^θ can be regarded as the matrix of the weighted graph that is induced on X and has in addition a loop of weight $(k-1)^{-1/2}\exp(-\theta)(k - \Delta_X(v))$ on each node v of X.

THEOREM 3.6. *Suppose $G = (V, E)$ is k-regular, and let $\tilde{\lambda} = max(\lambda_1(G), 2\sqrt{k-1}) = 2\sqrt{k-1}\,cosh\ \theta$, where $\theta \geq 0$. For any nonempty subset X of V of size at most $k^{-1/\epsilon}|V|$, we have*

$$\lambda_0(M_X^\theta) \leq \tilde{\lambda}(1 + O(\epsilon)),$$

where the constant behind the O is a small absolute constant.

PROOF. Let $l = \lfloor 1/2\epsilon \rfloor$ and let W be the set of nodes at distance at most l from X. A simple calculation shows that $|W| \leq 3k^l|X| \leq 3k^{-1/(2\epsilon)}n$. It follows from Lemma 3.2 that $\lambda_0(W) \leq \lambda'$, where $\lambda' = \tilde{\lambda} + 3k^{1-1/(2\epsilon)} = 2\sqrt{k-1}$ cosh θ', with $\theta' > \theta \geq 0$. A straightforward calculation shows that $\lambda' = \tilde{\lambda}(1 + O(\epsilon))$ and cosh θ' − cosh $\theta = O(k^{1/2-1/(2\epsilon)}) = O(\epsilon^2)$. We will use the following inequalities to show that the matrix $M_X^{\theta',l}$ is approximated by M_X^θ.

CLAIM 3.7. *For $x \geq y \geq 0$, we have $(x-y)^2 \leq 2(cosh\,x - cosh\,y)$.*

PROOF. This follows immediately from Taylor's expansion formula. □

CLAIM 3.8.

$$\frac{l}{l+1}exp(-\theta') \leq \frac{sinh(l\theta')}{sinh((l+1)\theta')} \leq exp(-\theta').$$

PROOF. Since $\sinh((l+1)\theta') \geq (l+1)\sinh\theta'$, we have

$$\begin{aligned}&\exp(-\theta') - \frac{\sinh(l\theta')}{\sinh((l+1)\theta')}\\ &\quad= \frac{\exp(-l\theta') - \exp(-(l+2)\theta')}{\exp((l+1)\theta') - \exp(-(l+1)\theta')}\\ &\quad= \exp(-(l+1)\theta')\frac{\sinh\theta'}{\sinh((l+1)\theta')} \leq \frac{\exp(-\theta')}{l+1}.\end{aligned}$$

It follows from Claim 3.7 that $\theta' - \theta = O(\epsilon)$. On the other hand, Claim 3.8 implies that

$$\frac{\sinh(l\theta')}{\sinh((l+1)\theta')} = \exp(-\theta')(1 + O(1/l)) = \exp(-\theta)(1 + O(\epsilon)).$$

Therefore, all entries of the diagonal matrix $M_X^{\theta} - M_X^{\theta',l}$ are $O(\sqrt{k-1}\,\epsilon)$, and so its largest eigenvalue is $O(\sqrt{k-1}\,\epsilon)$. But, as a consequence of Fact 2.1, the function that associates to a symmetric matrix its largest eigenvalue is subadditive. Therefore,

$$\begin{aligned}\lambda_0(M_X^{\theta}) &\le \lambda_0(M_X^{\theta',l}) + O(\sqrt{k-1}\,\epsilon)\\ &\le \lambda' + O(\tilde{\lambda}\epsilon)\\ &= \tilde{\lambda}(1 + O(\epsilon)),\end{aligned}$$

where the second inequality follows from Lemma 3.3. □

Remark 3.9. The only place where we used in the proof the fact that $\lambda_1(G)$ is the second eigenvalue of G was in conjunction with Lemma 3.2 to the upper bound $\lambda_0(W)$. In particular, if λ^* is a real number such that for any subset W of V, we have $\lambda_0(W) \le \lambda^* + 2k|W|/|V|$, then Theorem 3.6 remains valid if $\lambda_1(G)$ is replaced by λ^*.

4. *Lower Bounds on the Expansion and on the Average Degree*

We will derive lower bounds on the expansion by applying Theorem 3.6 to the union of X and $N(X)$, after reducing to the case where the graph is bipartite and X is on one side of the partition. The idea is that if the expansion of X is small, a node in $N(X)$ will be adjacent to many nodes in X, in average. This implies that the largest eigenvalue of the weighted matrix associated to the subgraph induced on $X \cup N(X)$ is large, contradicting with Theorem 3.6. We also use Theorem 3.6 to derive upper bounds on the average degree of induced subgraphs.

THEOREM 4.1. *If $G = (V, E)$ is k-regular and $\tilde{\lambda} = max(\lambda_1(G), 2\sqrt{k-1})$, then for any nonempty subset X of V of size at most $k^{-1/\epsilon}|V|$,*

$$\frac{|N(X)|}{|X|} \ge \frac{k}{2}\left(1 - \sqrt{1 - \frac{4k-4}{\tilde{\lambda}^2}}\right)(1 - O(\epsilon)),$$

where the constant behind the O is a small absolute constant.

PROOF. We first show how to reduce the problem to the case where the graph G is bipartite and X is on one side of the partition. Consider the cover graph G' of G, as defined in Section 2. We show that Remark 3.9 applies to the graph G' and $\lambda^* = \lambda_1(G)$. Indeed, let W be a subset of V', $W_p \subseteq V$ the set of nodes u of V such that $(u, 0) \in W$ or $(u, 1) \in W$, and $W^* = W_p \times \{0, 1\}$. Since the largest eigenvalue of a graph is no less than the largest eigenvalue of any induced subgraph [Bollobás 1990, page 156], $\lambda_0(W) \le \lambda_0(W^*)$. (We remind the reader that $\lambda_0(W)$ (respectively, $\lambda_0(W^*)$) is the largest eigenvalue of the subgraph of G' induced on W (respectively, W^*), and $\lambda_0(W_p)$ is the largest eigenvalue of the subgraph of G induced on W_p.) On the other hand, it follows from Fact 2.5 and Fact 2.3 that $\lambda_0(W^*) = \lambda_0(W_p)$. Using Lemma 3.2, we get

$$\lambda_0(W) \le \lambda_0(W_p) \le \lambda_1(G) + k\frac{|W_p|}{|V|} = \lambda_1(G) + 2k\frac{|W|}{|V'|},$$

as required by Remark 3.9. Note that we cannot apply directly Theorem 3.6 because, by Fact 2.5, we have $\lambda_1(G') = \lambda(G)$, which may be different from $\lambda_1(G)$.

Let Y be the subset of V' equal to $X \times \{0\}$. Denote the adjacency matrix of G' by A', and the set of neighbors of Y in G' by $N'(Y)$. Let $\tilde{\lambda} = 2\sqrt{k-1}\cosh\theta$, with $\theta \ge 0$. By applying Remark 3.9, to the graph G', the set of vertices $Y \cup N'(Y)$, and $\epsilon' = 2\epsilon$, we see that the largest eigenvalue of the matrix $M' = M^{\theta}_{Y \cup N'(Y)}$ is at most $\tilde{\lambda}(1 + O(\epsilon))$. Now, consider the function $f \in L^2(Y \cup N'(Y))$ defined by $f = k\chi_Y + \tilde{\lambda}\chi_{N'(Y)}$. By Fact 2.1,

$$M'f \cdot f \le \tilde{\lambda}(1 + O(\epsilon))\|f\|^2. \tag{7}$$

The left-hand side is the sum of two terms. The first is equal to $A'f \cdot f$, and the second corresponds to the weighted self-loops. By eq. (3), we have $A'f \cdot f = 2\tilde{\lambda}k^2|Y|$. On the other hand, since the loops have no weight on Y and have average weight $\exp(-\theta)(k-1)^{-1/2}(k - k|Y|/|N'(Y)|)$ on $N'(Y)$, the second term is equal to $\exp(-\theta)(k-1)^{-1/2}(k|N'(Y)| - k|Y|)\tilde{\lambda}^2$. Thus, eq. (7) reduces to

$$2\tilde{\lambda}k^2|Y| + \frac{\tilde{\lambda}^2}{\sqrt{k-1}\exp(\theta)}k(|N'(Y)| - |Y|)$$
$$\le \tilde{\lambda}(l + O(\epsilon))(k^2|Y| + \tilde{\lambda}^2|N'(Y)|).$$

By replacing $|Y|$ by $|X|$, $|N'(Y)|$ by $|N(X)|$, we get after simplifications

$$k|X|(k - 2\exp(-\theta)\cosh\theta)$$
$$\le |N(X)|\big(\tilde{\lambda}^2 - 2k\exp(-\theta)\cosh\theta\big)(1 + O(\epsilon)). \tag{8}$$

Noting that $\tilde{\lambda}^2 - 2k\exp(-\theta)\cosh\theta = 2(k\exp(\theta) - 2\cosh\theta)\cosh\theta$, eq. 8 reduces to:

$$\frac{|N(X)|}{|X|} \ge \frac{k}{2\exp(\theta)\cosh\theta}(1 - O(\epsilon)).$$

We conclude the proof using the formula

$$\frac{1}{\exp(\theta)\cosh\theta} = 1 - \sqrt{1 - \frac{1}{\cosh^2\theta}} = 1 - \sqrt{1 - \frac{4k-4}{\tilde{\lambda}^2}}. \qquad \square$$

THEOREM 4.2. *If $G = (V, E)$ is k-regular and $\tilde{\lambda} = max(\lambda_1(G), 2\sqrt{k-1})$, then for any nonempty subset X of V of size at most $k^{-1/\epsilon}|V|$, the average degree σ of the subgraph of G induced on X is at most*

$$\left(1 + \frac{\tilde{\lambda}}{2} + \sqrt{\frac{\tilde{\lambda}^2}{4} - (k-1)}\right)(1 + O(\epsilon)),$$

where the constant behind the O is a small absolute constant.

PROOF. We use the same notations as in Theorem 3.6. As noted before, the matrix M_X^θ can be regarded as the matrix of the weighted graph on X that is induced on X and has in addition a loop of weight $(k-1)^{-1/2}\exp(-\theta)(k - \Delta_X(v))$ on each node v of X. By Fact 2.1, we have $\chi_X \cdot M_X^\theta \chi_X \le \lambda_0(M_X^\theta)|X|$, which translates into

$$\sigma + \frac{k-\sigma}{\sqrt{k-1}\exp(\theta)} \le 2\sqrt{k-1}(1+O(\epsilon))\cosh\theta.$$

This implies

$$\begin{aligned}\sigma &\le \frac{2(k-1)\exp(\theta)\cosh\theta - k}{\sqrt{k-1}\exp(\theta)-1}(1+O(\epsilon))\\ &= \left(\sqrt{k-1}\exp(\theta)+1\right)(1+O(\epsilon)).\end{aligned}$$

We conclude by noting that $\exp(\theta) = \cosh\theta + \sqrt{\cosh^2\theta - 1}$. □

5. *A Family of Almost Ramanujan Graphs with Expansion $k/2$*

In this section, we construct explicitly a family of k-regular graphs G_n containing subsets of sublinear size having expansion $k/2$, and such that $\lambda(G_n) = (2+o(1))\sqrt{k-1}$. For this, we need the following lemma.

LEMMA 5.1. *Consider a graph on a vertex set W, a subset X of W, a positive integer h, and $s \in L^2(W)$. Let X_i be the set of nodes at distance i from X. Assume the following conditions hold*:

(1) *For $h-1 \le i, j \le h$, all nodes in X_i have the same number of neighbors in X_j.*
(2) *The vector s is constant on X_{h-1} and on X_h.*
(3) *s has positive components and $As \le \mu s$ on $B_{h-1}(X)$, where μ is a positive real number.*

Then for any $g \in L^2(W)$ such that $|Ag(u)| = \mu|g(u)|$ for $u \in B_{h-1}(X)$, we have

$$\frac{\sum_{v\in X_h} g(v)^2}{\sum_{v \in X_h} s(v)^2} \ge \frac{\sum_{v\in X_{h-1}} g(v)^2}{\sum_{v\in X_{h-1}} s(v)^2}.$$

PROOF. Let P, P_{h-1} and P_h be the projections on the sets $B_{h-1}(X)$, X_{h-1} and X_h, respectively. We need to show that $\|P_h g\|^2/\|P_h s\|^2 \ge \|P_{h-1}g\|^2/\|P_{h-1}s\|^2$. Let $A_h = (P+P_h)A(P+P_h)$. The operator A_h corresponds in some sense to the adjacency matrix of the subgraph induced on $B_h(X)$, but it acts on $L^2(W)$. By the conditions of the lemma, there exist positive coefficients α, β, and γ such that $P_h A_h s = \gamma P_h s$ and $A_h P_h s = \alpha P_h s + \beta P_{h-1} s$. By hypothesis, we have $A_h s \le \mu Ps + \gamma P_h s$. Premultiplying both sides of this equation by P and A_h yields successively

$$\begin{aligned}PA_h s &\le \mu Ps\\ &= \mu s - \mu P_h s,\\ A_h P A_h s &\le \mu A_h s - \mu(\alpha P_h s + \beta P_{h-1} s)\\ &\le \mu^2 Ps + \mu(\gamma - \alpha)P_h s - \mu\beta P_{h-1} s.\end{aligned}$$

But the matrix $A_h P A_n - \mu^2 P - \mu(\gamma - \alpha) P_h + \mu\beta P_{h-1}$ has only nonnegative entries off its diagonal, and so its largest eigenvalue is 0 since s is positive (Fact 2.4). The quadratic form associated to this matrix is therefore negative semidefinite (Fact 2.1), and so

$$A_h P A_h g \cdot g \leq \mu^2 P g \cdot g + \mu(\gamma - \alpha) P_h g \cdot g - \mu\beta P_{h-1} g \cdot g. \tag{9}$$

Since both A_h and P are selfadjoint and since $P^2 = P$, the left-hand side of eq. (9) is equal to $\|PA_h g\|^2$. We can rewrite eq. (9) as follows:

$$\|PA_h g\|^2 \leq \mu^2 \|Pg\|^2 + \mu(\gamma - \alpha)\|P_h g\|^2 - \mu\beta \|P_{h-1} g\|^2.$$

But $\|PA_h g\| = \mu\|Pg\|$ by hypothesis, and so

$$(\gamma - \alpha)\|P_h g\|^2 \geq \beta\|P_{h-1} g\|^2. \tag{10}$$

On the other hand, since A_h and P_h are selfadjoint, we have $A_h P_h s \cdot s = P_h A_h s \cdot s$, and so $\alpha\|P_h s\|^2 + \beta\|P_{h-1} s\|^2 = \gamma\|P_h s\|^2$. Comparing this with eq. (10) concludes the proof. $\square$

THEOREM 5.2. *For any integer k such that $k - 1$ is prime, we can explicitly construct an infinite family of k-regular graphs G_n on n vertices whose linear expansion is $k/2$ and such that $\lambda_1(G_n) \leq 2\sqrt{k-1}(1 + 2\log^2 \log n / \log_k^2 n)$.*

PROOF. We construct the family (G_n) by altering the known constructions of explicit Ramanujan graphs, so that the expansion of (G_n) is $k/2$. From Lubotzky et al. [1988] and Margulis [1988], we know that we can explicitly construct an infinite family of bipartite Ramanujan graphs (F_n) on n vertices whose girth $c(F_n)$ is $(4/3 + o(1))\log_{k-1} n$. Let $F_n = (V, E)$ be an element of the family, $u \in V$ a vertex of F_n and $l = \lfloor c(F_n)/2 \rfloor - 2$. Let $u_1, \ldots, u_k$ be the neighbors of u and let $v_1, \ldots, v_k$ be k vertices at distance two from u such that $(u_i, v_i) \in E$. The subgraph of F_n induced on $B_{l+1}(\{u\})$ is a tree since it contains no cycles. Let u' and v' be two elements not belonging to V. Consider the k-regular graph $G_{n+2} = (V', E')$, where $V' = V \cup \{u', v'\}$ and $E' = E \cup \bigcup_{i=1}^{k} \{(u', u_i), (u_i, u'), (v', v_i), (v_i, v')\} - \bigcup_{i=1}^{k} \{(u_i, v_i), (v_i, u_i)\}$. Figure 1 shows the graph G_{n+2} in the neighborhood of u in the case $k = 3$. For shorthand, we denote A_{F_n} by A, $A_{G_{n+2}}$ by A' and $\lambda_1(A')$ by λ'. We need to show that $\lambda' \leq (2 + o(1))\sqrt{k-1}$. Assume that $\lambda' > 2\sqrt{k-1}$ (otherwise, we are done), and let $\lambda' = 2\sqrt{k-1}\cosh\theta'$, with $\theta' > 0$. Let $g \in L_0^2(V')$ be an eigenvector corresponding to λ'.

We outline informally the basic ideas of the proof. Roughly speaking, we will show that the values that g takes on the nodes u', v', u_i, v_i are small compared to $\|g\|$. This implies that g is close in l_2-norm to its restriction f on V. Lemma 3.1 then implies that $f \cdot Af \leq (2 + o(1))\sqrt{k-1}\|g\|^2$. But since $g(u')$, $g(v')$, $g(u_i)$, and $g(v_i)$ are small, the scalar product $f \cdot Af$ is close to $g \cdot A'g = \lambda'\|g\|^2$, and so $\lambda' \leq (2 + o(1))\sqrt{k-1}$.

Since u and u' have the same neighbors in G_{n+2} and $\lambda' \neq 0$, we have $g(u) = g(u')$. By eq. (3), we have

$$\begin{aligned}\lambda'\|g\|^2 &= g \cdot A'g \\ &= f \cdot Af - 2\sum_{i=1}^{k} g(u_i)g(v_i) + 2\sum_{i=1}^{k} g(u')g(u_i) + 2\sum_{i=1}^{k} g(v')g(v_i).\end{aligned} \tag{11}$$

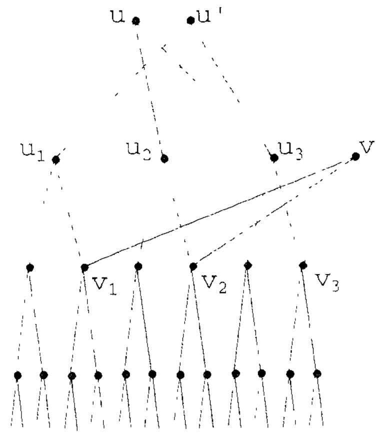

FIG. 1. The graph G_{n+2} in the neighborhood of u in the case $k = 3$. The dotted edges are those belonging to $E - E'$.

We upper bound $-2g(u_i)g(v_i)$ by $g(u_i)^2 + g(v_i)^2$. On the other hand, the equality $(A'g)(u') = \lambda' g(u')$ implies that

$$\sum_{i=1}^{k} g(u')g(u_i) = \frac{1}{\lambda'}\left(\sum_{i=1}^{k} g(u_i)\right)^2 \le \frac{k}{\lambda'}\sum_{i=1}^{k} g(u_i)^2.$$

A similar relation holds for v'. Combining this with eq. (11), we get

$$\lambda'\|g\|^2 \le f \cdot Af + \left(1 + \frac{2k}{\lambda'}\right)\sum_{i=1}^{k}\left(g(u_i)^2 + g(v_i)^2\right). \tag{12}$$

We use Lemma 3.1 to bound the term $f \cdot Af$. Note that $\sum_{w \in V} f(w) = -g(u') - g(v')$ since $g \in L_0^2(V')$, and so

$$\begin{aligned} f \cdot Af &\le \lambda_1(A)\|f\|^2 + \frac{k}{n}(g(u') + g(v'))^2 \\ &\le 2\sqrt{k-1}\left(\|g\|^2 - g(u')^2 - g(v')^2\right) + \frac{2k}{n}\left(g(u')^2 + g(v')^2\right) \\ &\le 2\sqrt{k-1}\,\|g\|^2, \end{aligned}$$

for sufficiently large n. Combining this with eq. (12) and noting that $1 + 2k/\lambda' \le 4\sqrt{k-1}$, we obtain

$$\lambda'\|g\|^2 \le 2\sqrt{k-1}\left(\|g\|^2 + 2\sum_{i=1}^{k}\left(g(u_i)^2 + g(v_i)^2\right)\right). \tag{13}$$

We now compare $g(u_i)$ and $g(v_i)$ to $\|g\|$. Let s be the function on V' defined by $s(u) = s(u') = k$ and, for $v \in V' - \{u, u'\}$ at distance i from u, $s(v) = 2(k-1)^{1-i/2}\cosh i\theta'$. An easy calculation shows that the function s verifies the conditions of Lemma 5.1 for the graph G', with $X = \{u, u'\}$, $\mu = \lambda'$, and for any integer h between 1 and $l+1$. Hence

$$\begin{aligned}\|g\|^2 &\geq \sum_{v \in X_{l+1}} g(v)^2 \\ &\geq \frac{\sum_{v \in X_{l+1}} s(v)^2}{\sum_{v \in X_1} s(v)^2} \sum_{v \in X_1} g(v)^2 \\ &= \frac{k-2}{k-1} \frac{\cosh^2(l+1)\theta'}{\cosh^2\theta'} \sum_{i=1}^{k} g(u_i)^2 \\ &\geq \frac{1}{2}\cosh^2(l\theta') \sum_{i=1}^{k} g(u_i)^2.\end{aligned}$$

Similarly, $\|g\|^2 \geq 1/2\cosh^2(l\theta')\sum_{i=1}^{k} g(v_i)^2$. Combining this with eq. (13), we get

$$\cosh\theta' \leq 1 + \frac{8}{\cosh^2 l\theta'}. \tag{14}$$

Solving eq. (14) yields $\theta' \leq (\log l)/l$ for sufficiently large n, and so

$$\lambda' = 2\sqrt{k-1}\left(1 + \frac{\theta'^2}{2}(1+o(1))\right) \leq 2\sqrt{k-1}\left(1 + 2\frac{\log^2\log n}{\log_k^2 n}\right).$$

Theorem 4.1 implies that the linear expansion of the family (G_n) is at least $k/2$. Since the subset $\{u, u'\}$ has k neighbors, this bound is tight. □

If $k-1$ is a prime congruent to 1 modulo 4, we know from Lubotzky et al. [1988] that there exists an infinite family of nonbipartite k-regular Ramanujan graphs with girth at least $(2/3 + o(1))\log_{k-1} n$. By repeating the construction in Theorem 5.2, we obtain k-regular graphs whose second largest eigenvalue *in absolute value* is $(2 + o(1))\sqrt{k-1}$ and linear expansion $k/2$. Moreover, by adjoining nodes at regions of the graph at sufficiently large distance from each other, we can construct for any $m = m(n) = o(n)$ a family of k-regular graphs whose second largest eigenvalue in absolute value is $(2 + o(1))\sqrt{k-1}$ and containing a subset of size $2m$ with expansion $k/2$. This can be shown by a proof similar to Theorem 5.2.

6. *Random Walks*

We show that the probability that a walk stays inside a given set has an exponential decay in the length of the walk. Our bound improves upon previous results in Ajtia et al. [1987] and Goldreich et al. [1990] and is shown to be optimal for many values of the parameters.

COROLLARY 6.1. *If $G = (V, E)$ is k-regular and W a subset of V, the fraction of walks in G of length l whose all vertices belong to W is at most $\mu(\alpha + \mu - \alpha\mu)^l$, where $\alpha = \lambda_1/k$ and $\mu = |W|/n$.*

PROOF. The number of walks of length l in W is equal to the sum of entries of A_W^l, and so the fraction of walks of length l in W is equal to $((A_W)^l\chi_W \cdot \chi_W)/(k^l n)$. On the other hand, since $\lambda_0(A_W)$ is the largest eigenvalue of A_W

in absolute value, the largest eigenvalue of A_W^l is $(\lambda_0(A_W))^l$. Using Lemma 3.2 and Fact 2.1, we conclude that

$$(A_W^l \chi_W) \cdot \chi_W \le (\lambda_0(A_W))^l \|\chi_W\|^2 \le \mu n(\lambda_1 + k\mu - \lambda_1 \mu)^l,$$

as desired. □

Define the density of a subset of vertices to be the ratio of its size to the total number of vertices. Corollary 6.1 is optimal in the following sense: for any rational $\alpha \in [1/2, 1]$ and any rational $\mu \in [0, 1]$, there exists an arbitrary large k-regular graph G and a subset W of G of density μ such that $\lambda_1(G) = \alpha k$ and, for any integer l, the fraction of walks in W of length l is equal to $\mu(\alpha + \mu - \alpha\mu)^l$. Indeed, let $G = K_{a+2} \times K_b$, where $b = a(1 - \alpha)/\alpha$ and $K_{a+2} \times K_b$ is the graph on $V = \{1, \dots, a + 2\} \times \{1, \dots, b\}$, with $((i, j), (i', j')) \in E$ if and only if $i = i'$ XOR $j = j'$. The graph G is regular of degree $a + b$; its eigenvalues are $a + b$, $a, b - 2$ and -2, and $\lambda_1(G) = a = \alpha(a + b)$. Let $W = \{1, \dots, a + 2\} \times \{1, \dots, \mu b\}$. The set W has density μ in V, and the fraction of walks in W of length l is $\mu(a + \mu b)^l/(a + b)^l$, which is equal to the value given by Corollary 6.1.

7. *Other Applications*

We list three applications of Theorems 4.1 and 4.2.

(1) *Random Regular Graphs*. It was shown in Friedman [1991] that, if k is even, then for a random k-regular graphs G, we have $\lambda_1(G) \le 2\sqrt{k-1} + O(\log k)$ with high probability. Using Theorem 4.1, we deduce that for a random regular graph, we can prove with high probability *in polynomial time* that linear-sized subsets (of density at most $k^{-1/\epsilon}$, where $\epsilon = k^{-1/4}$) have expansion at least

$$\frac{k}{2}\left(1 - \sqrt{1 - \frac{4k - 4}{(2\sqrt{k-1} + O(\log k))^2}}\right)(1 + O(k^{-1/4}))$$
$$= \frac{k}{2} - O(k^{3/4}\log^{1/2} k).$$

(2) *Selection Networks*. We can use Theorem 4.1 to build explicit selection networks of small size. A selection network is a network of comparators that classifies a set of n numbers, where n is even, into two subsets of $n/2$ numbers such that any element in the first subset is smaller than any element in the second subset. In Pippenger [1991], a probabilistic construction of a selection network is given using an asymptotic upper bound of $2n \log_2 n$ comparators. Also, an upper bound slightly less than $6n \log_2 n$ is shown by a deterministic construction. Using Theorem 4.1, we can construct selection networks of asymptotic size $(3 + o(1))n \log_2 n$. Indeed, it is shown in Pippenger [1991] how to construct selection networks of size $(2 + o(1))n \log_2 n$ from expanders of degree 4 having linear expansion at least 3. The construction can be easily generalized to build selection networks of size $k(1/2 + o(1))n \log_2 n$ from expanders of degree k having linear expansion at least 3. Theorem 4.1 then shows that we can build

explicit selection networks of size $(3 + o(1))n \log_2 n$ using 6-regular Ramanujan graphs.

(3) *Extrovert Graphs*. Given a graph $G = (V, E)$ and a subset X of V, an element of X is said to be extrovert if at least half of its neighbors are outside X. A family of graphs is called extrovert if all linear-sized subsets contain a constant fraction of extrovert nodes. Such graphs have been used [Broder et al. 1992] to solve the token distribution problem. Theorem 4.2 shows that the average degree of the nodes of a linear-sized induced subgraph of a k-regular Ramanujan graph is upper bounded by roughly $1 + \sqrt{k-1}$, which is less than $k/2$ for $k \geq 7$. This shows that Ramanujan graphs of degree at least 7 are extrovert graphs. Classical results [Alon and Chung 1989] require the degree to be at least 15.

8. *Concluding Remarks and Further Work*

(1) Let H be a graph of maximum degree at most k, and $\bar{\lambda}$ a real number no smaller than $2\sqrt{k-1}$. Theorem 3.6 implies that, if there exists an infinite family G_n of k-regular graphs containing H as an induced subgraph and such that $\lambda(G_n) \leq (1 + o(1))\bar{\lambda}$, then $\lambda_0(M_H^\theta) \leq \bar{\lambda}$. ($M_H^\theta$ can be defined similarly to M_X^θ.) If $k - 1$ is a prime congruent to 1 modulo 4, this condition can be shown to be sufficient [Kahale 1993b].

(2) It is still an open question whether there exists a family of Ramanujan graphs with linear expansion at most $k/2$.

(3) It would be interesting to calculate the exact value of the linear expansion of the known explicit constructions of Ramanujan graphs [Lubotzky et al. 1988; Margulis 1988]. Theorem 4.1 shows that it is at least $k/2$. On the other hand, an easy combinatorial argument shows that the linear expansion of any family of k-regular graphs is at most $k - 1$. Besides being Ramanujan, the graphs constructed in Lubotzky et al. [1988] and Margulis [1988] have other interesting combinatorial properties. For example, they are Cayley graphs and have high girth, unlike the graphs that we constructed in Section 5. This leads us to conjecture that their linear expansion is strictly greater than $k/2$. Any explicit construction of k-regular graphs with provable linear expansion strictly greater than $k/2$ would also be interesting.

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