

The Gaber-Galil expander

1 The graph

We define 4 maps on $V = \mathbb{Z}_n^2$: $A(x_1, x_2) = (x_1 + x_2, x_2)$, $B(x_1, x_2) = (x_1, x_1 + x_2)$, $E_1(x_1, x_2) = (x_1 + 1, x_2)$ and $E_2(x_1, x_2) = (x_1, x_2 + 1)$ (all additions are modulo n). We also think of the maps A and B as 2×2 matrices so that $A^t = B$ and vice versa. We wish to show that these maps (with their inverses) give an expander graph on V . Let $S = \{A, A^{-1}, B, B^{-1}, E_1, E_1^{-1}, E_2, E_2^{-1}\}$. For each map $T \in S$ we associate an $n^2 \times n^2$ permutation matrix M_T so that $M_T(x, y) = 1$ if $y = T(x)$ and is zero otherwise.

Let $\Omega = L^2(\mathbb{Z}_n^2)$. We think of functions in Ω as column vectors of length n^2 with real (or complex) entries. For a function $f \in \Omega$ and a map $T : V \mapsto V$ we have $M_T f \in \Omega$ with $(M_T f)(x) = f(Tx)$. The adjacency matrix of our 8 regular graph is $M = \sum_{T \in S} M_T$. For $f \in \Omega$, we denote $R(f) = \langle f, Mf \rangle = \sum_{x \in V} f(x)(Mf)(x)$. By the spectral definition of expanders, we need to show:

Theorem 1.1. *Let $f \in \Omega$ be so that $\sum_x f(x) = 0$. Then, there exists $\lambda > 0$ (independent of n) so that*

$$R(f) \leq (8 - \lambda)\|f\|^2.$$

The idea of the proof is as follows. We split the matrix M into two matrices:

$$M_1 = M_{E_1} + M_{E_1^{-1}} + M_{E_2} + M_{E_2^{-1}},$$

$$M_2 = M_A + M_{A^{-1}} + M_B + M_{B^{-1}}$$

and define $R_1(f) = \langle f, M_1 f \rangle$, $R_2(f) = \langle f, M_2 f \rangle$. We will show that, for each f (with entries summing to zero) either $|R_1(f)| \leq (4 - \lambda)\|f\|^2$ or $|R_2(f)| \leq (4 - \lambda)\|f\|^2$. In other words, either the maps E_1, E_2 ‘expand’ f (call this case I) or the maps A, B ‘expand’ f (case II). To characterize those functions f in each case, we need to define the Fourier transform.

2 Fourier transform

For $x, y \in \mathbb{Z}_n^2$ we let $x \cdot y = x_1 y_1 + x_2 y_2 \pmod{n}$ to avoid confusion with the inner product notation over Ω . Let $\omega = e^{2\pi i/n}$. We define the Fourier coefficients of $f \in \Omega$ by $\hat{f}(y) =$

$\sum_a f(a)\omega^{-a \cdot y}$. We thus have:

$$f(x) = \frac{1}{n^2} \sum_{y \in V} \hat{f}(y) \omega^{y \cdot x},$$

$$\langle f, g \rangle = \frac{1}{n^2} \langle \hat{f}, \hat{g} \rangle.$$

It is also easy to check that, if $g(x) = f(Tx + b)$, with T an invertible 2×2 matrix and $b \in \mathbb{Z}_n^2$, then

$$\hat{g}(y) = \omega^{(T^{-1}b) \cdot y} \hat{f}((T^{-1})^t y).$$

For our 4 maps A, B, E_1, E_2 we have

$$\widehat{M_A f}(y) = \hat{f}(B^{-1}y),$$

$$\widehat{M_B f}(y) = \hat{f}(A^{-1}y),$$

$$\widehat{M_{E_1} f}(y) = \omega^{y_1} \hat{f}(y),$$

$$\widehat{M_{E_2} f}(y) = \omega^{y_2} \hat{f}(y),$$

and similarly for the inverses. Notice the first two identities imply that for all $f \in \Omega$, $\widehat{M_2 f} = M_2 \hat{f}$ and so

$$R_2(f) = \langle f, M_2 f \rangle = n^{-2} \langle \hat{f}, \widehat{M_2 f} \rangle = n^{-2} \langle \hat{f}, M_2 \hat{f} \rangle = n^{-2} R_2(\hat{f}). \quad (1)$$

In other words, the expansion of M_2 can be analyzed for f or for \hat{f} interchangeably.

3 The two cases

Roughly speaking, if M_1 does not expand f (i.e., $R_1(f)$ is large) then it means that the Fourier transform of f is concentrated in ‘low frequencies’: meaning most of the Fourier mass is at points $y \in V$ with both y_1 and y_2 ‘close’ to zero (or to n). This makes sense, as the shift operators E_1 and E_2 should not expand ‘smooth’ functions (this is similar to taking derivatives). Thus, it is enough to show that M_2 expands functions whose Fourier mass is concentrated around the origin (but is zero at $(0, 0)$). By (1) it is enough to bound $R_2(\hat{f})$ or, equivalently, to show that M_2 expands sets that are around the origin (the transition from functions to sets is via Cheeger). But analyzing the action of M_2 on sets around the origin is the same as analyzing its action on \mathbb{Z}^2 , which is easily seen to be expanding.

We now set up the necessary notations: For $t \in \mathbb{Z}_n$ we let $|t| = \min\{t, n - t\}$. We define the set $W = \{(x_1, x_2) \in V \mid |x_1|, |x_2| \leq n/4\}$. For a function $f \in \Omega$ and a set $C \subset \mathbb{Z}_n^2$ we define the restriction of f to C as $(f|_C)(x) = f(x)$ for $x \in C$ and zero otherwise.

The first Lemma takes care of functions that have ‘enough’ high frequencies.

Lemma 3.1. *[Expansion in high frequencies] Let $f \in \Omega$ be such that $\|\hat{f}|_W\|^2 \leq (1 - \epsilon)\|\hat{f}\|^2$. Then*

$$|R_1(f)| \leq (4 - 2\epsilon)\|f\|^2.$$

The second lemma deals with the other case:

Lemma 3.2. *[Expansion in low frequencies] Let $f \in \Omega$ be such that $\sum_x f(x) = 0$ and $\|\hat{f}|_W\|^2 \geq (1 - \epsilon)\|\hat{f}\|^2$. Then*

$$R_2(f) \leq (4 - 1/64 + \epsilon)\|f\|^2.$$

The theorem clearly follows from these two lemmas (taking ϵ to be sufficiently small and separating into the two cases).

4 Proof of Lemma 3.1

We have $\langle \hat{f}, \widehat{M_{E_i} f} \rangle = \langle \hat{f}, \omega^{y_i} \hat{f} \rangle$ and similarly for E_i^{-1} (with ω^{-y_i} instead of ω^{y_i}). Therefore

$$\begin{aligned} \langle \hat{f}, \widehat{M_1 f} \rangle &= \langle \hat{f}, (\omega^{y_1} + \omega^{-y_1}) \hat{f} \rangle + \langle \hat{f}, (\omega^{y_2} + \omega^{-y_2}) \hat{f} \rangle \\ &= \sum_{y \in V} |\hat{f}(y)|^2 \phi(y), \end{aligned}$$

with $\phi(y) = 2 \cos(2\pi y_1/n) + 2 \cos(2\pi y_2/n)$. If $y \notin W$ then one of the cosines is negative and so $\phi(y) \leq 2$. Since $\phi(y) \leq 4$ everywhere else, we get

$$\begin{aligned} \langle \hat{f}, \widehat{M_1 f} \rangle &\leq 4 \sum_{y \in W} |\hat{f}(y)|^2 + 2 \sum_{y \notin W} |\hat{f}(y)|^2 \\ &\leq 4(1 - \epsilon)\|\hat{f}\|^2 + 2\epsilon\|\hat{f}\|^2 = (4 - 2\epsilon)\|\hat{f}\|^2. \end{aligned}$$

Thus,

$$R_1(f) = \langle f, M_1 f \rangle = n^{-2} \langle \hat{f}, \widehat{M_1 f} \rangle \leq n^{-2} (4 - 2\epsilon) \|\hat{f}\|^2 = (4 - 2\epsilon) \|f\|^2.$$

This completes the proof of the lemma. □

5 Proof of Lemma 3.2

We first argue about the case where the support of \hat{f} is completely contained in W . Then the Lemma will follow by loosing an additional $O(\epsilon)$ coming from the mass outside W . So, suppose that $\hat{f}(y) = 0$ for all $y \notin W$. We define $h(y) = |\hat{f}(y)|$ so that h is a non negative function with support contained in W .

A *level-set* of a non negative function $h : V \mapsto \mathbb{R}$ is a set of the form $L = \{x \in V \mid h(x) \geq \ell\}$ for some ℓ . We call the level set ‘trivial’ if $\ell = 0$. For a graph $G = (V, E)$, the edge-expansion $\phi_G(S)$ for a set $S \subset V$ is defined as

$$\phi_G(S) = \frac{|E(S, \bar{S})|}{|S|}.$$

The following statement is the main part in the proof of the Cheeger inequality.

Theorem 5.1 (Cheeger). *Let $G = (V, E)$ be a d -regular graph with adjacency matrix M . Let $h : V \mapsto \mathbb{R}$ be a non-negative function and suppose that $\phi_G(S) \geq \tau$ for any non trivial level set S of h . Then,*

$$\langle h, Mh \rangle \leq (d - \lambda) \|h\|^2,$$

with $\lambda \geq \tau^2/4d$.

To apply this theorem on our $h = |\hat{f}|$ we have to observe that all level sets of h are contained in $W \setminus \{(0, 0)\}$ and show that all such sets expand by A, B, A^{-1}, B^{-1} . We first show that this set of 4 maps is an expander in \mathbb{Z}^2 and then argue that this implies expansion also for sets in $W \subset \mathbb{Z}_n^2$ (we could argue directly on \mathbb{Z}_n^2 but moving to \mathbb{Z} makes the expansion more transparent). For this purpose we let $G_{\mathbb{Z}}$ denote the graph on \mathbb{Z}^2 and edges given by A, B, A^{-1}, B^{-1} and $G_{\mathbb{Z}_n}$ the same graph on vertex set \mathbb{Z}_n^2 . We denote by $\Gamma_G(S)$ the set of vertices in G that have at least one neighbor in S (in the graph G).

Claim 5.2 (Expansion in \mathbb{Z}^2). *Let $S \subset \mathbb{Z}^2 \setminus \{(0, 0)\}$ be a finite set. Then, $\phi_{G_{\mathbb{Z}}}(S) \geq \frac{1}{2}$.*

Proof. Let \mathbb{Z}_0 be the union of the two axis (i.e., all points with one zero coordinate). We break S into 5 sets: $S_0 = S \cap \mathbb{Z}_0$ and S_1, S_2, S_3, S_4 the intersections of S with the 4 quadrants. It is easy to see that for each $i \in [4]$ we have $|\Gamma(S_i)| = 2|S_i|$ (e.g., A, B map S_1 into two disjoint sets) and the 4 sets $\Gamma(S_i)$ are disjoint from each other. This means that

$$|E(S - S_0, \bar{S})| \geq |S| - |S_0|.$$

Now, every edge leaving S_0 (there are $4|S_0|$ edges) has its other endpoint outside S_0 . Some of these might be in $S - S_0$ but the number of these is at most $3(|S| - |S_0|)$ (since we showed that at least $|S| - |S_0|$ leave $S - S_0$ into the complement of S). Thus, we have

$$|E(S_0, \bar{S})| \geq 4|S_0| - 3(|S| - |S_0|) = 7|S_0| - 3|S|.$$

Considering the two cases $|S_0| > |S|/2$ and $|S_0| \leq |S|/2$ we see that in both we have $|E(S, \bar{S})| \geq \frac{1}{2}|S|$. \square

Corollary 5.3 (Expansion in $W \subset \mathbb{Z}_n^2$). *Let $S \subset W \setminus \{(0, 0)\}$. Then, $\phi_{G_{\mathbb{Z}_n}}(S) \geq \frac{1}{2}$.*

Proof. The set W is naturally divided into 4 disjoint parts, corresponding to the 4 quadrants of \mathbb{Z}^2 . We ‘re-arrange’ these into a new set $S' \subset \mathbb{Z}^2$ as follows: Let $\theta : \mathbb{Z}_n \mapsto \mathbb{Z}$ be defined as $\theta(t) = \text{sign}(t) \cdot |t|$ with $\text{sign}(t) = 1$ for $t \leq n/2$ and -1 for $t > n/2$. Then $S' = \theta(S)$ (applied coordinate-wise). So each vertex in S' has a single vertex in S associated with it. Observe that, if an edge of $G_{\mathbb{Z}}$ goes from S' to its complement, then the same edge in $G_{\mathbb{Z}_n}$ (leaving the corresponding vertex) will go from S to its complement (this is true since edges are ‘too short’ to reach from one of the four parts of W to another). This completes the proof. \square

Using the corollary, and Theorem 5.1 (Cheeger), we get that

$$\langle h, M_2 h \rangle \leq (4 - 1/64) \|h\|^2.$$

Using (1), and the triangle inequality (moving from \hat{f} to h), this gives:

$$\begin{aligned} R_2(f) &= n^{-2} \langle \hat{f}, M_2 \hat{f} \rangle \leq n^{-2} \langle h, M_2 h \rangle \\ &\leq (4 - 1/64) n^{-2} \|h\|^2 = (4 - 1/64) n^{-2} \|\hat{f}\|^2 = (4 - 1/64) \|f\|^2. \end{aligned}$$

To prove the lemma, we need to argue about functions f so that $\|\hat{f}|_W\| \geq (1 - \epsilon) \|f\|$. For such an f we can write f as a sum $f = f_1 + f_2$ with $\hat{f}_1 = \hat{f}|_W$ and $\hat{f}_2 = \hat{f}|_{\bar{W}}$. By the properties of the Fourier transform, and since $\langle \hat{f}_1, \hat{f}_2 \rangle = 0$ we have $\|f\|^2 = \|f_1\|^2 + \|f_2\|^2$. Notice also that $\sum_x f_1(x) = 0$ since $\hat{f}_1(0,0) = \hat{f}(0,0) = 0$. For f_1 we have established the inequality $R_2(f_1) = \langle f_1, M_2 f_1 \rangle \leq (4 - 1/64) \|f_1\|^2$. We also know that $\|f_2\|^2 \leq \epsilon \|f\|^2$. Putting things together, and using $\|M_2 v\| \leq 4 \|v\|$ for any v , we get:

$$\begin{aligned} R_2(f) &= \langle f, M_2 f \rangle = \langle f_1 + f_2, M_2(f_1 + f_2) \rangle \\ &= R_2(f_1) + \langle f_2, M_2 f_1 \rangle + \langle f_1, M_2 f_2 \rangle + \langle f_2, M_2 f_2 \rangle \\ &\leq (4 - 1/64) \|f_1\|^2 + 4\epsilon \|f\|^2 + 4\epsilon \|f\|^2 + 4\epsilon^2 \|f\|^2 \\ &\leq (4 - 1/64 + 3\epsilon) \|f\|^2. \end{aligned}$$

This completes the proof of the lemma. \square