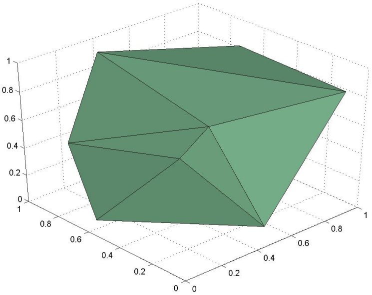
Introduction to LP and SDP Hierarchies



Madhur Tulsiani
Princeton University

Toy Problem

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Convex Relaxations for Combinatorial Optimization

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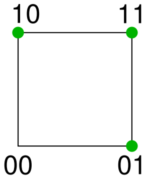

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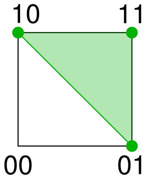

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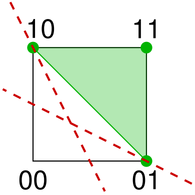

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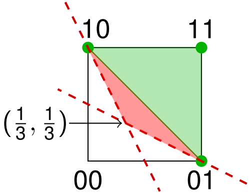

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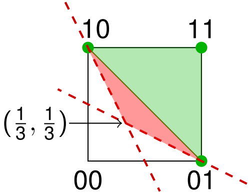

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Convex Relaxations for Combinatorial Optimization

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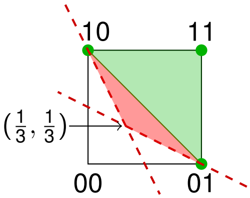
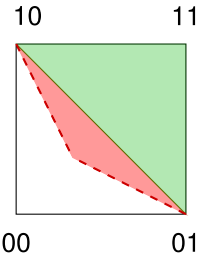


- Large number of approximation algorithms derived precisely as above.
- Analysis consists of understanding extra solutions introduced by the relaxation.
- Integrality Gap $= \dfrac{\text{Combinatorial Optimum}}{\text{Optimum of Relaxation}} = \dfrac{1}{2/3} = \dfrac{3}{2}$

Generating “tighter” relaxations

- Would like to make our relaxations less relaxed.
- Various hierarchies give increasingly powerful programs at different levels (rounds), starting from a basic relaxation.



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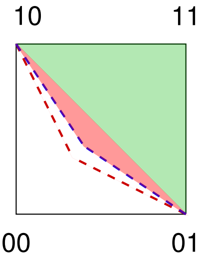

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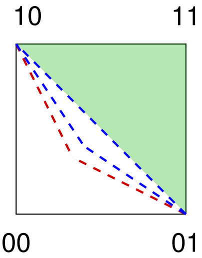
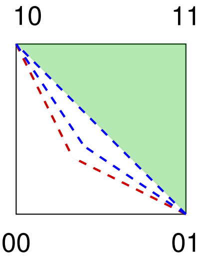


Generating "tighter" relaxations

- Would like to make our relaxations less relaxed.
- Various hierarchies give increasingly powerful programs at different levels (rounds), starting from a basic relaxation.
- Powerful computational model capturing most known LP/SDP algorithms within constant number of levels.
- Does approximation get better a higher levels?



LP/SDP Hierarchies

- Various hierarchies studied in the Operations Research literature:
  - Lovász-Schrijver (LS, LS_+)
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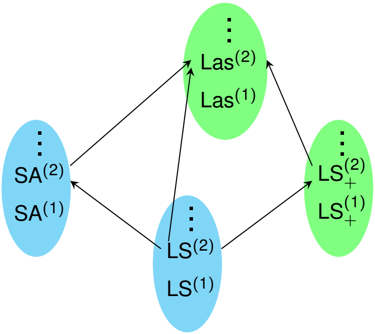

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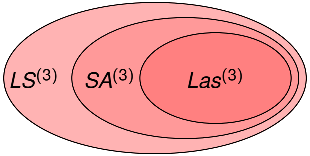

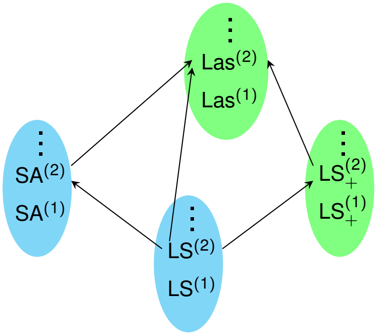

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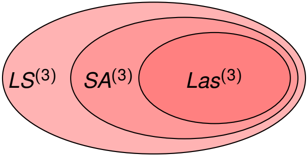
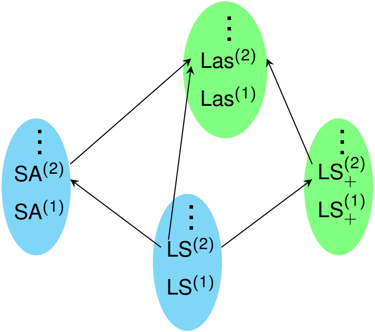
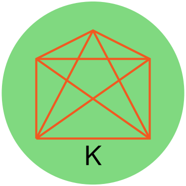




- Can optimize over r^{th} level in time $n^{O(r)}$. n^{th} level is tight.

Example: Souping up the Independent Set relaxation

$$\begin{array}{lll} \text{maximize:} & \sum_u x_u & \\ \text{subject to:} & x_u + x_v \le 1 & \forall\,(u,v) \in E \\ & x_u \in [0,1] & \end{array}$$



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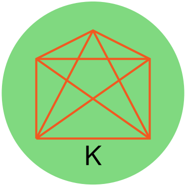

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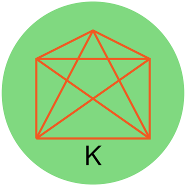

- Implied by one level of LS_+ hierarchy.
- Polytime algorithm for Independent Set on perfect graphs [GLS 81].

What Hierarchies want

Example: Maximum Independent Set for graph $G = (V, E)$

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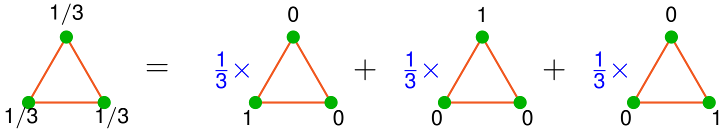

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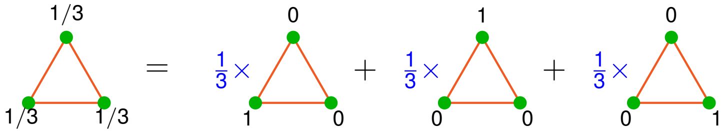

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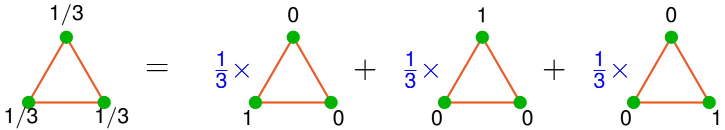

- Hierarchies add variables for conditional/joint probabilities.

The Sherali-Adams Hierarchy

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- Start with a 0/1 integer linear program.
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$$\sum_i a_i z_i \leq b$$

$$\mathbb{E}\left[\left(\sum_i a_i z_i\right) \cdot z_5 z_7 (1 - z_9)\right] \leq \mathbb{E}\left[b \cdot z_5 z_7 (1 - z_9)\right]$$

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\sum_i a_i \cdot \left(X_{\{i,5,7\}} - X_{\{i,5,7,9\}}\right) &\leq b \cdot \left(X_{\{5,7\}} - X_{\{5,7,9\}}\right)
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- LP on n^r variables.

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- $D(\{1,2,3\})$ and $D(\{1,2,4\})$ must agree with $D(\{1,2\})$.
- $SA^{(r)} \implies LCD^{(r)}$. If each constraint has at most k vars, $LCD^{(r+k)} \implies SA^{(r)}$

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- $(Y \succeq 0)$ + original constraints + consistency constraints.

The Lasserre hierarchy (constraints)

- Y is psd. (i.e. find vectors $\mathbf{U}_S$ satisfying $Y_{S_1,S_2} = \langle \mathbf{U}_{S_1}, \mathbf{U}_{S_2} \rangle$)

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- Original quadratic constraints as inner products.

SDP for Independent Set

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{i \in V} \left|\mathbf{U}_{\{i\}}\right|^2 & \\
\text{subject to} & \langle \mathbf{U}_{\{i\}}, \mathbf{U}_{\{j\}} \rangle = 0 & \forall\, (i,j) \in E \\
 & \langle \mathbf{U}_{S_1}, \mathbf{U}_{S_2} \rangle = \langle \mathbf{U}_{S_3}, \mathbf{U}_{S_4} \rangle & \forall\, S_1 \cup S_2 = S_3 \cup S_4 \\
 & \langle \mathbf{U}_{S_1}, \mathbf{U}_{S_2} \rangle \in [0,1] & \forall S_1, S_2
\end{array}
$$

The "Mixed" hierarchy

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- Level r has
  - Variables X_S for $|S| \leq r$ and all Sherali-Adams constraints.
  - Vectors $\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_n$ satisfying

$$\langle \mathbf{U}_i, \mathbf{U}_j \rangle = X_{\{i,j\}}, \langle \mathbf{U}_0, \mathbf{U}_i \rangle = X_{\{i\}} \text{ and } |\mathbf{U}_0| = 1.$$

Hands-on: Deriving some constraints

The triangle inequality

- $|\mathbf{U}_i - \mathbf{U}_j|^2 + |\mathbf{U}_j - \mathbf{U}_k|^2 \; \geq \; |\mathbf{U}_i - \mathbf{U}_k|^2$ is equivalent to

$$\langle \mathbf{U}_i - \mathbf{U}_j, \mathbf{U}_k - \mathbf{U}_j \rangle \; \geq \; 0$$

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- $\text{Mix}^{(3)} \implies \exists$ distribution on z_i, z_j, z_k such that $\mathbb{E}[z_i \cdot z_j] = \langle \mathbf{U}_i, \mathbf{U}_j \rangle$ (and so on).

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- For all integer solutions $(z_i - z_j) \cdot (z_k - z_j) \geq 0$.

The triangle inequality

- $|\mathbf{U}_i - \mathbf{U}_j|^2 + |\mathbf{U}_j - \mathbf{U}_k|^2 \ \geq \ |\mathbf{U}_i - \mathbf{U}_k|^2$ is equivalent to

$$\langle \mathbf{U}_i - \mathbf{U}_j, \mathbf{U}_k - \mathbf{U}_j \rangle \ \geq \ 0$$

- Mix$^{(3)}$ $\implies$ $\exists$ distribution on z_i, z_j, z_k such that $\mathbb{E}[z_i \cdot z_j] = \langle \mathbf{U}_i, \mathbf{U}_j \rangle$ (and so on).

- For all integer solutions $(z_i - z_j) \cdot (z_k - z_j) \ \geq \ 0$.

$$\therefore \ \langle \mathbf{U}_i - \mathbf{U}_j, \mathbf{U}_k - \mathbf{U}_j \rangle \ = \ \mathbb{E}\left[(z_i - z_j) \cdot (z_k - z_j)\right] \ \geq \ 0$$

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- For every clique K in a graph, adding the constraint

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- For $i, j \in K$, $\langle \mathbf{U}_i, \mathbf{U}_j \rangle = 0$. Also, $\forall i \, \langle \mathbf{U}_0, \mathbf{U}_i \rangle = |\mathbf{U}_i|^2 = x_i$. By Pythagoras,
$$\sum_{i \in K} \left\langle \mathbf{U}_0, \frac{\mathbf{U}_i}{|\mathbf{U}_i|} \right\rangle^2 \leq |\mathbf{U}_0|^2 = 1 \implies \sum_{i \in B} \frac{x_i^2}{x_i} \leq 1.$$

- Derived by Lovász using the ϑ-function.

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  (think $Y_{ij} = \mathbb{E}[z_i z_j] = \mathbb{P}[z_i \wedge z_j]$)
  - $Y = Y^T$
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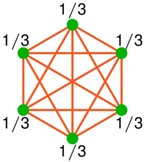
- Above is an LP (SDP) in $n^2 + n$ variables.

Lovász-Schrijver in action

- r^{th} level optimizes over distributions conditioned on r variables.



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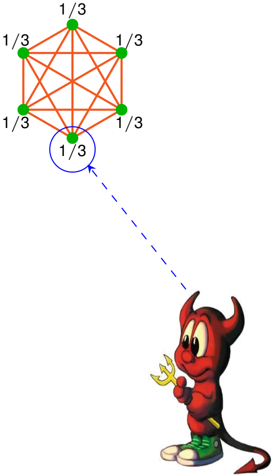

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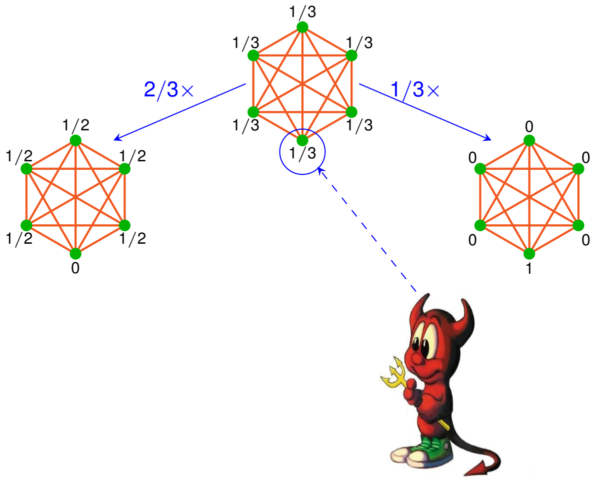

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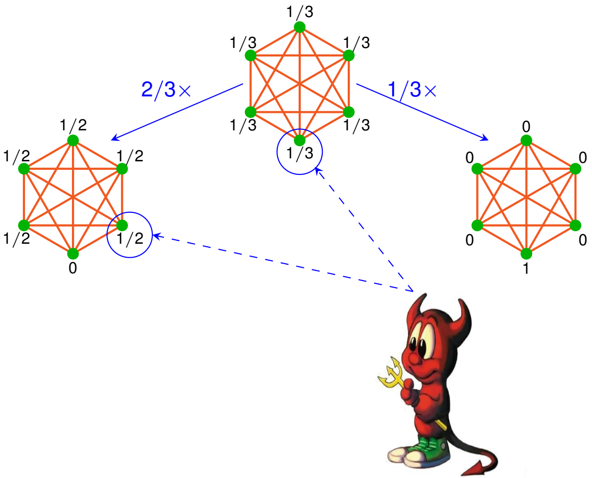

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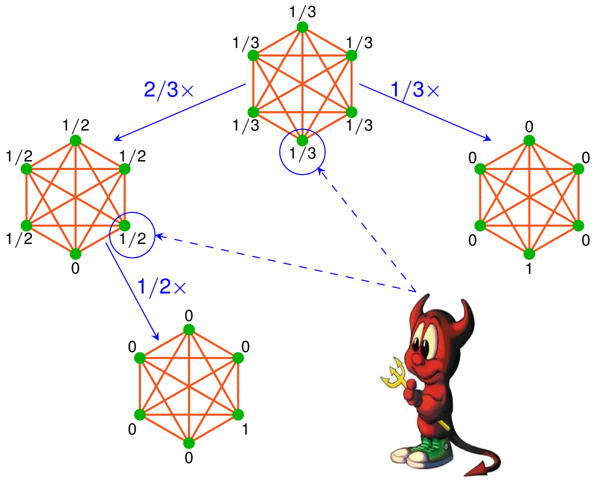

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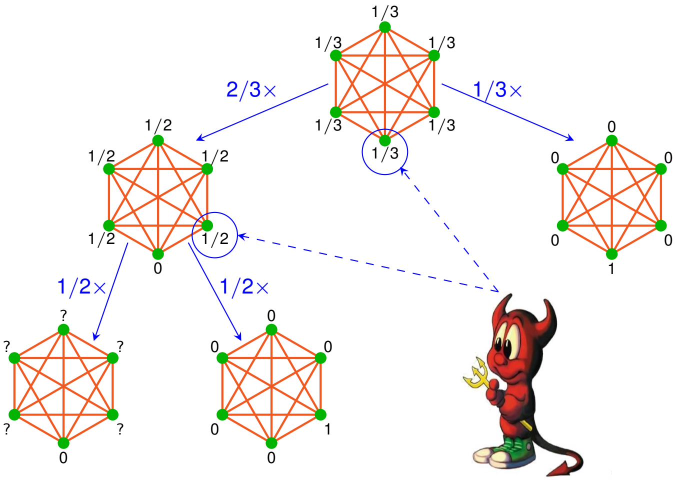

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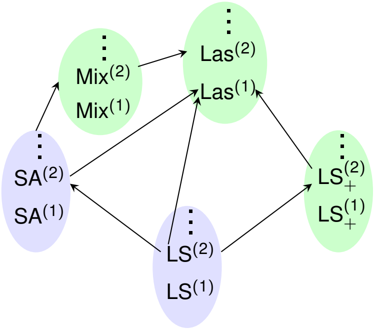

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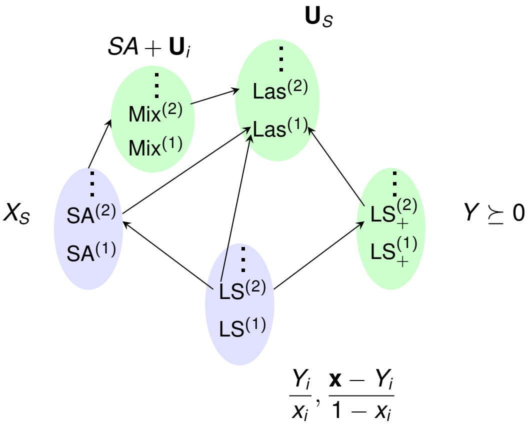

Algorithmic Applications

- Many known LP/SDP relaxations captured by 2-3 levels.
- [Chlamtac 07]: Explicitly used level-3 Lasserre SDP for graph coloring.
- [CS 08]: Algorithms using Mixed and Lasserre hierarchies for hypergraph independent set (guarantee improves with more levels).
- [KKMN 10]: Hierarchies yield a PTAS for Knapsack.
- [BRS 11, GS 11]: Algorithms for Unique Games using n^{ϵ} levels of Lassere.

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- [ABLT 06, STT 07, dlVKM 07, CMM 09]: Distributions from local probabilistic processes.
- [Charikar 02, GMPT 07, BCGM 10]: Polynomial tensoring.
- [RS 09, KS 09]: Higher level distributions from level-1 vectors.

Integrality Gaps for Expanding CSPs

CSP Expansion

- MAX k-CSP: m constraints on k-tuples of (n) boolean variables. Satisfy maximum. e.g. MAX 3-XOR (linear equations mod 2)

$$z_1 + z_2 + z_3 = 0 \qquad z_3 + z_4 + z_5 = 1 \qquad \cdots$$

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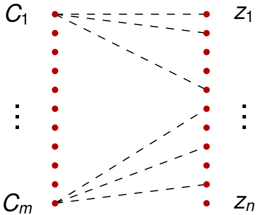

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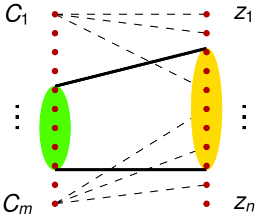

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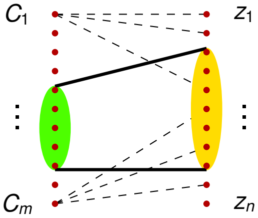

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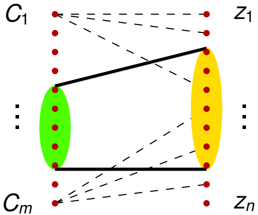

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- Used extensively in proof complexity e.g. [BW01], [BGHMP03]. For LS_+ by [AAT04].

Sherali-Adams LP for CSPs

Variables: $X_{(S,\alpha)}$ for $|S| \leq t$, partial assignments $\alpha \in \{0,1\}^S$

$$
\begin{array}{llcll}
\text{maximize} & \displaystyle\sum_{i=1}^{m} \sum_{\alpha \in \{0,1\}^{T_i}} C_i(\alpha) \cdot X_{(T_i,\alpha)} & & & \\
\text{subject to} & X_{(S\cup\{i\},\alpha\circ 0)} + X_{(S\cup\{i\},\alpha\circ 1)} & = & X_{(S,\alpha)} & \forall i \notin S \\
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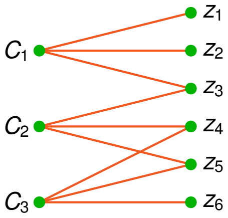
- Distributions should “locally look like" supported on satisfying assignments.

Local Satisfiability



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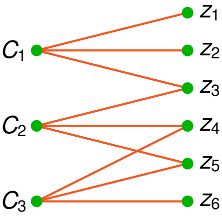

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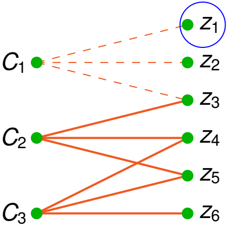

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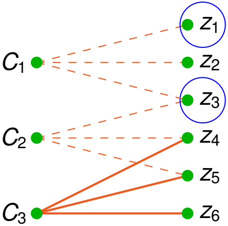

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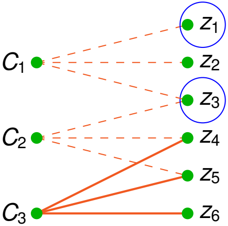

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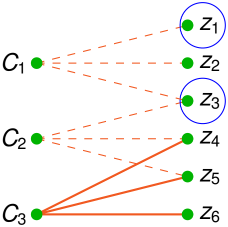

- Take $\gamma = 0.9$
- Can show any three 3-XOR constraints are simultaneously satisfiable.
- Can take $\gamma \approx (k-2)$ and any αn constraints.
- Just require $\mathbb{E}[C(z_1, \ldots, z_k)]$ over any $k-2$ vars to be constant.

$$
\begin{aligned}
& \mathbb{E}_{z_1 \ldots z_6}\left[C_1(z_1, z_2, z_3) \cdot C_2(z_3, z_4, z_5) \cdot C_3(z_4, z_5, z_6)\right] \\
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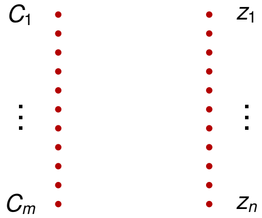
Obtaining integrality gaps for CSPs [BGMT 09]



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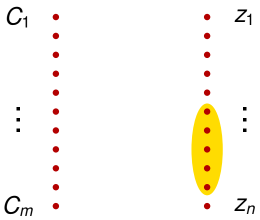

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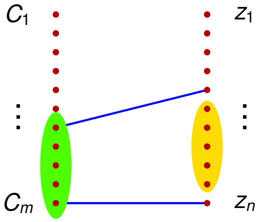

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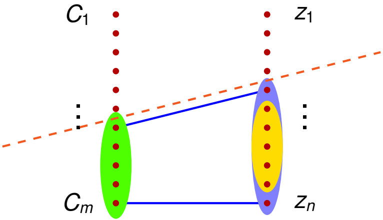

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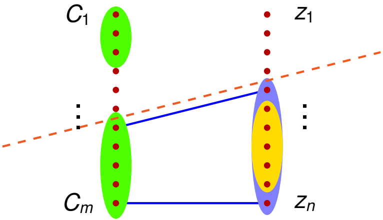

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- Remaining constraints "independent" of this assignment.
- Gives optimal integrality gaps for $\Omega(n)$ levels in the mixed hierarchy.

Vectors for Linear CSPs

A “new look” Lasserre

- Start with a $\{-1,1\}$ quadratic integer program.
  $(z_1,\ldots,z_n) \to ((-1)^{z_1},\ldots,(-1)^{z_n})$

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- Consider the psd matrix $\tilde{Y}$

$$\tilde{Y}_{S_1,S_2} = \mathbb{E}\left[\tilde{Z}_{S_1} \cdot \tilde{Z}_{S_2}\right] = \mathbb{E}\left[\prod_{i \in S_1 \Delta S_2} (-1)^{z_i}\right]$$

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- Write program for inner products of vectors $\mathbf{W}_S$ s.t.
  $\tilde{Y}_{S_1,S_2} = \langle \mathbf{W}_{S_1}, \mathbf{W}_{S_2}\rangle$

Gaps for 3-XOR

SDP for MAX 3-XOR

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{C_i \equiv (z_{i_1} + z_{i_2} + z_{i_3} = b_i)} \frac{1 + (-1)^{b_i} \left\langle \mathbf{W}_{\{i_1, i_2, i_3\}}, \mathbf{W}_{\emptyset} \right\rangle}{2} & \\
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SDP for MAX 3-XOR

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- Used by [FO 06], [STT 07] for LS_+ hierarchy.

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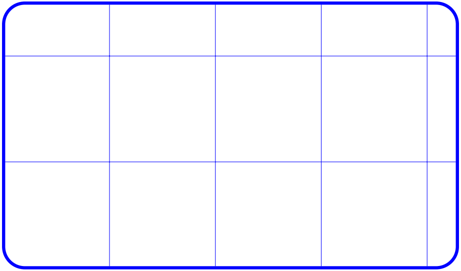

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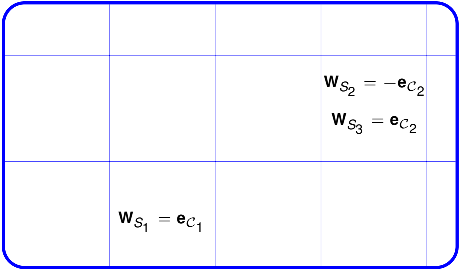

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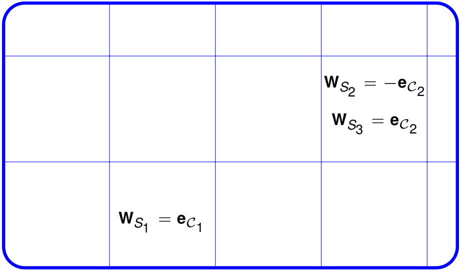
- Relies heavily on constraints being linear equations.



Reductions

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- Question posed in [AAT 04]. First done by [KV 05] from Unique Games to Sparsest Cut.

Integrality Gaps for Independent Set

- FGLSS: Reduction from MAX k-CSP to Independent Set in graph G_Φ.

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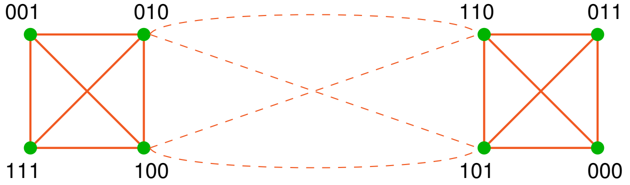

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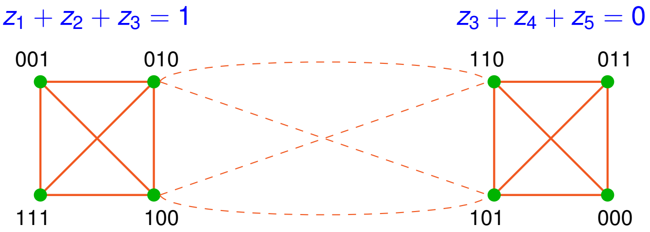

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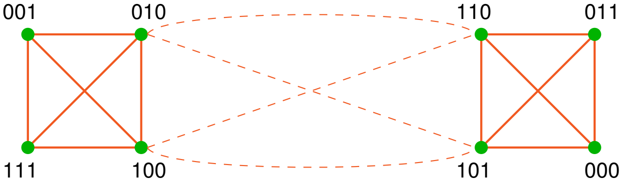

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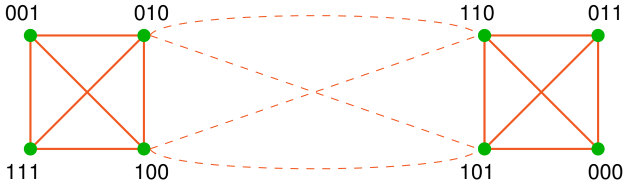

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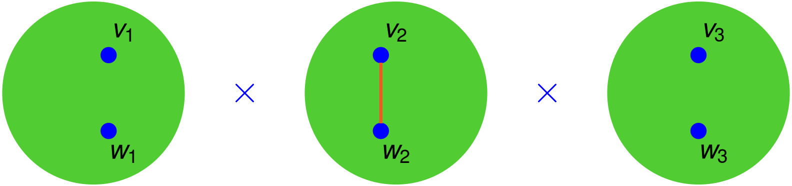
$$\mathbf{U}_{\{(z_1,z_2,z_3)=(0,0,1)\}} = \frac{1}{8}(\mathbf{W}_\emptyset + \mathbf{W}_{\{1\}} + \mathbf{W}_{\{2\}} - \mathbf{W}_{\{3\}} + \mathbf{W}_{\{1,2\}} - \mathbf{W}_{\{2,3\}} - \mathbf{W}_{\{1,3\}} - \mathbf{W}_{\{1,2,3\}})$$

Graph Products



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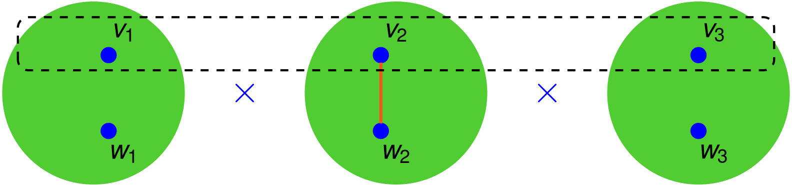

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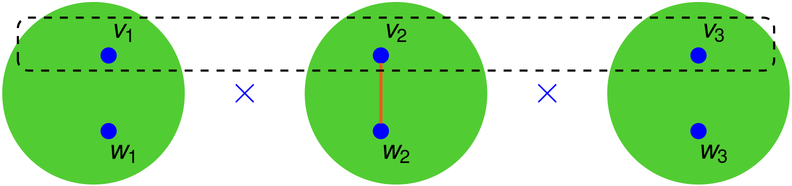

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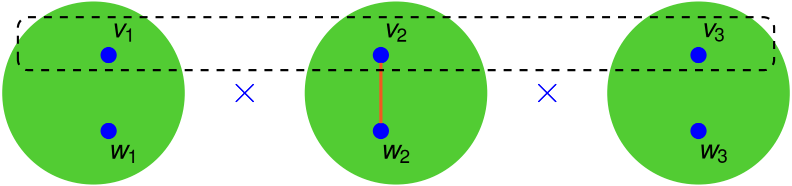

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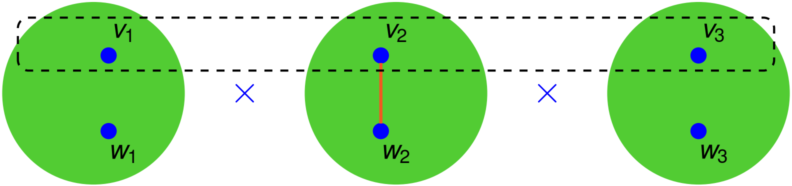

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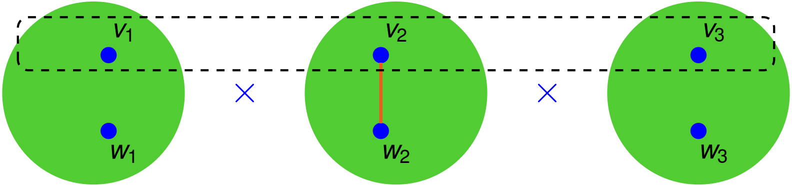

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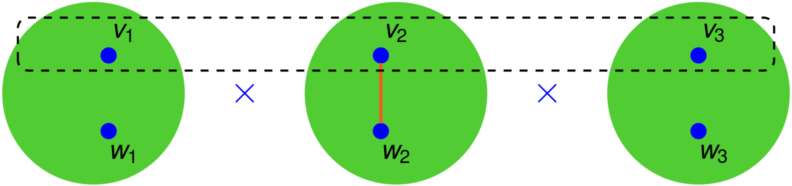

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- Together give a gap of $\frac{n}{2^{O(\sqrt{\log n \log\log n})}}$.

A few problems

Problem 1: Lasserre Gaps

- Show an integrality gap of $2 - \epsilon$ for Vertex Cover, even for $O(1)$ levels of the Lasserre hierarchy.
- Obtain integrality gaps Unique Games (and Small-Set Expansion)
    - Gaps for $O((\log \log n)^{1/4})$ levels of mixed hierarchy were obtained by [RS 09] and [KS 09].
    - Extension to Lasserre?

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- What extra constraints do vectors capture?

Thank You

Questions?