New Constructive Aspects of the Lovász Local Lemma, and their Applications

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Collaborators: Bernhard Haeupler (MIT) & Barna Saha (UMD)
\( \mathcal{A} = \{ A_1, A_2, \ldots, A_m \} \): “bad” events, each defined by indep. random variables \( X_1, X_2, \ldots, X_n \).

Ubiquitous version with neighborhood relation \( \Gamma \) on \( \mathcal{A} \).

Are all \( A_i \) simultaneously avoidable?

Output = assignment to all \( X_j \); output size = \( n \).
Algorithmic versions of the LLL

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Main results:

- “Any” LLL application \( \rightarrow \) poly(\( n \))-time alg. (even if \( m \gg \text{poly}(n) \)), if we give a tiny slack in the LLL-condition;

- MAX SAT–like problems: avoiding “most” \( A_i \) (algorithmically) – interpolation between linearity of expectation and LLL.
"Pr[no \ A_i] > 0": Union Bound $\sum_i \Pr[A_i] < 1$ often too weak.

LLL (symmetric version): Suppose

- $\max_i \Pr[A_i] \leq p$, and
- each $A_i$ has $\leq D$ neighbors.

Then, $e \cdot p \cdot (D + 1) \leq 1$ implies $\Pr[\text{no } A_i \text{ holds}] > 0$.

Numerous applications. Typical case: $D \ll m$. 
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Algorithmic version?

$Pr[\bigwedge_i \overline{A_i}]$ inevitably small:

- Choose indep. set $I$ of the $A_i$ with $|I| \geq m/(D + 1)$.
- $Pr[\bigwedge_i \overline{A_i}] \leq Pr[\bigwedge_{i \in I} \overline{A_i}] = (1 - p)^{m/(D+1)} \approx \exp(-mp/D)$. 
Application: Domatic Partitions


Graph $G$; $N^+(v) =$ inclusive neighborhood of vertex $v$.

Partition vertices into a max. $\#$ dominating sets: i.e., “color” vertices with max. $\#$ colors so that

$$\forall \text{ vertices } v, \text{ all colors visible in } N^+(v).$$

[Chen-Jamieson-Balakrishnan-Morris]: wireless coordination. If $(\delta, \Delta) =$ (min., max.) degrees, $\text{OPT} \leq \delta + 1$. 

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[FHKS]: apx. threshold $= \ln \Delta$. Here: $3 \ln d$–apx. for $d$-regular $G$. 
Randomly color vertices using $\ell \sim d/(3 \ln d)$ colors.  
Bad event $A_{v,c}$: “$c$ not visible at $v$”.

\[ p = \Pr[A_{v,c}] = (1 - 1/\ell)^{d+1} \sim 1/d^3. \]

Dependence of fixed $A_{v,c}$? 
Only on $A_{w,c'}$ with $\text{dist}(v, w) \leq 2$.

$\#w < d^2$; $\#c' \leq \ell$. So, $D < d^3/(3 \ln d)$.

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Correct constant ”$3$” → ”$1$”: iterated app. of LLL, a powerful methodology ([Molloy-Reed]: “Graph colouring and the probabilistic method”).
LLL, general “asymmetric” version: If \( \exists x : A \rightarrow (0, 1) \) such that

\[
\forall i : \Pr[A_i] \leq x(A_i) \prod_{A_j \in \Gamma(A_i)} (1 - x(A_j)),
\]

then \( \Pr[\bigwedge_i A_i] \geq \prod_i (1 - x(A_i)) > 0. \)

Numerous applications:

- (Hyper-)Graph Colorings and Ramsey Numbers
- Routing [Leighton-Maggs-Rao]
- LP-Integrality gaps [Feige, Leighton-Lu-Rao-S]
- Edge-disjoint paths [Andrews] ...
The Trivial Algorithm:

```
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Theorem (LLL)

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BUT: Run-time usually exponential in $m$ (let alone $n$).
Algorithmic versions of the LLL: [Beck, Alon, Molloy-Reed, Czumaj-Scheideler, S, Moser, ...] culminating in MT:

The MT Algorithm:

start with an arbitrary assignment

while $\exists$ event $A_i$ that holds do

assign new random values to the variables of $A_i$
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The MT Algorithm:

- start with an arbitrary assignment
- while $\exists$ event $A_i$ that holds do
  - assign new random values to the variables of $A_i$

Theorem (MT)

If the LLL-conditions hold, then the above algorithm finds a satisfying assignment within an expected $\sum_i \frac{x(A_i)}{1-x(A_i)}$ iterations.
LLL-distribution and the MT-Algorithm

The trivial algorithm outputs a random sample from the conditional LLL-distribution $\mathcal{D}$, the distribution that conditions on avoiding all $A_i$. 

Well-known Bound

For any event $B = B(X_1, X_2, \ldots, X_n)$, 

$$\Pr(\mathcal{D}(B)) := \Pr(B | \bigwedge_i A_i) \leq \Pr(B) \cdot \left( \prod_{A_j \in \Gamma(B)} (1 - x(A_j)) \right)^{-1}$$

Theorem

The output distribution of the MT-algorithm satisfies (1).
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For any event $B = B(X_1, X_2, \ldots, X_n)$,

$$\Pr_{\mathcal{D}}(B) := \Pr \left( B \mid \bigwedge_i \overline{A_i} \right) \leq \Pr(B) \cdot \left( \prod_{A_j \in \Gamma(B)} (1 - x(A_j)) \right)^{-1}$$

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Examples:

- Acyclic edge coloring
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Problems with running MT:

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Problems with running MT:

1. $\mathbb{E}[\# \text{ resamplings}]: \sum_i \frac{x(A_i)}{1-x(A_i)}$
2. representation of the bad events
3. verifying a solution / finding some $A_i$ that holds currently
Let $\delta = \min_i \Pr[A_i]$. Then,

$$E[\# \text{ iterations of MT}] \leq \sum_i \frac{x(A_i)}{1 - x(A_i)}$$

$$\leq \left( \sum_i x(A_i) \right) \cdot \max_i \frac{1}{1 - x(A_i)}$$

$$\leq O(n \log(1/\delta)) \cdot \max_i \frac{1}{1 - x(A_i)}.$$ 

In all apps known to us, $\log \frac{1}{\delta} \leq O(n \log n)$. 

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New Constructive Aspects of the Lovász Local Lemma, and the
Solving Problem 2+3

How do we represent the events (implicitly) s.t.
- checking a solution or
- finding some $A_i$ that holds currently

can be done in $\text{poly}(n)$ time?

Hopeless:
In most applications this is (NP-)hard
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Algorithm: Run MT on a core-subset of the $A$, of $\text{poly}(n)$ size.
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Analysis:
  - Bound the probabilities of non-core events using (1)
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Hopeless:
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Algorithm: Run MT on a core-subset of the $A$, of poly($n$) size.

Analysis:
- Bound the probabilities of non-core events using (1)
- Use a union bound over these probabilities to prove that with high probability all the $A_i$ are avoided.
Theorem

If \( \exists \epsilon \in (0, 1) \) such that for all \( A_i \),

\[
\Pr[A]^{1-\epsilon} \leq x(A_i) \cdot \prod_{A_j \in \Gamma(A_i)} (1 - x(A_i)),
\]

then:

- for any \( p \geq \frac{1}{\text{poly}(n)} \), \( |\{A_i : \Pr[A_i] \geq p\}| \leq \text{poly}(n) \);
- if \( \log \frac{1}{\delta} \leq \text{poly}(n) \) and the above core is “checkable”, then for any desired constant \( c > 0 \), \( \exists \) Monte Carlo alg. (with \( p \sim n^{-c/\epsilon} \)) that terminates within \( O\left(\frac{n}{\epsilon} \log \frac{n}{\epsilon}\right) \) resamplings and returns a good assignment with probability at least \( 1 - n^{-c} \).
The first/best-known efficient algorithms for:

- $O(1)$-apx. for the Santa Claus problem (Feige’s proof made constructive)
- non-repetitive coloring (proof of Alon-Grytczuk-Hauszczak-Riordan made constructive)
- acyclic edge coloring
- edge-disjoint paths (Andrews [2010])
Allowing some $A_i$ to hold

Interpolating between the LLL and linearity of expectation:

**Theorem**

\[
\text{In the symmetric LLL with } p \text{ and } D, \text{ if } D \leq \alpha \cdot \left(\frac{1}{ep} - 1\right) \\
(1 < \alpha < e) \text{ then we can make at most } \sim \left(\frac{e \ln(\alpha)}{\alpha}\right) \cdot mp \text{ of the } \text{A}_i \text{ to hold, in randomized poly(m) time.}
\]
Is \( e \ln(\alpha)/\alpha \) tight?
Derandomization
Further analysis of dependencies among non-core events
How much slack is really needed?
Lopsided Local Lemma?
Full understanding of [Kolipaka-Szegedy] setting
Thank you!
Questions?