PCPs and Inapproximability:
Recent Milestones and Continuing Challenges

Venkatesan Guruswami
Carnegie Mellon University

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Intended to be a survey* of some key developments of last ≈ 10 years

Let’s begin with a glimpse of where PCP theory was ≈ a decade back
PCPs circa 2000
PCP theorem

- PCP theorem: [AS, ALMSS] Polynomial size witnesses for NP languages that can be checked by a randomized polytime verifier by just probing 3 bits (with soundness error < 0.999)

- Equivalently, Gap3SAT$_{1,\alpha}$ is NP-hard for some $\alpha < 1$

- (Surprising?) Connection to approximating Clique (based on FGLSS graph) discovered before connection to approximating Max 3SAT
  - PCP theorem implies factor $n^\gamma$ inapproximability for Clique
PCPs: early years

- PCP theorem implied APX hardness for many problems (via their classic reductions from 3SAT).
  - Generally very weak factors.
  - Quest for better (optimal?) factors followed

- 2-prover 1-round proof systems (“bipartite 2-query PCPs”) emerged as the canonical PCP to reduce from:
  - Constraint satisfaction problem called Label Cover
  - Example early Label Cover-based success:
    \( \Omega(\log n) \) hardness for set cover [Lund-Yannakakis]
Label Cover

- Binary (arity two) CSP over large domain $[R]$ with “projection” constraints.

- Instance consists of:
  - Bipartite graph $G=(V,W,E)$
  - For each $e = (v,w) \in E$, a function $\pi_e : [R] \rightarrow [R]$.

- Assignment (labeling) $A : V \cup W \rightarrow [R]$ satisfies an edge $e = (v,w)$ if $\pi_e(A(v)) = A(w)$
  - Value of instance = maximum fraction of edges satisfied by a labeling

- Raz’s parallel repetition theorem gives a strong (value $1$ vs $1/R^\gamma$) gap hardness for Label Cover
Strong(er)/optimal PCPs

• Improvements in PCP params, aimed in part at better hardness results (using specific predicates for PCP check)
  – Label Cover used as “outer” PCP
  – Composed with “inner” PCP (trading off soundness for much smaller # queries)
    • Paradigm: Encode labels & test codewords
    • “Inner” task: Check if \( \pi(a) = b \) reading very few bits of \( Enc(a) \) and \( Enc(b) \)

• Which code to use?
  Brilliant invention of [Bellare-Goldreich-Sudan]: LONG CODE
Long Code
(aka Dictator functions)

• For \(a \in \{1,2,\ldots,R\}\), \(\text{LONG}(a) : \{0,1\}^R \to \{0,1\}\)
  \[
  \text{LONG}(a)(x) = x_a \text{ for every } x \in \{0,1\}^R
  \]

  – Very redundant (encodes \(\log R\) bits into \(2^R\) bits)
    • has the value of every function \([R] \to \{0,1\}\) at \(a\)
    • but doesn’t hurt to have around if \(R\) is constant.
  – Surprisingly useful!
The first optimal PCPs

- [Håstad'96]: zero “amortized free bit” PCP \( \Rightarrow \) factor \( n^{1-\varepsilon} \) inapproximability for Clique

- [Håstad'97]: \text{Gap3LIN}_{1-\varepsilon,\frac{1}{2}+\varepsilon} \text{ is NP-hard} (3-query PCP with completeness \( 1-\varepsilon \) and soundness \( \frac{1}{2}+\varepsilon \)). Optimal!

Also, similar result mod p, and NP-hardness of

- \text{Gap3SAT}_{1,\frac{7}{8}+\varepsilon} \text{ & Gap-4-Set-Splitting}_{1,\frac{7}{8}+\varepsilon} \text{ Optimal!}

- Several hardness results (currently best known, under NP \( \neq \) P) via gadget reductions from Håstad’s results:
  - MaxCut: \( \frac{16}{17} \)  Max2SAT: \( \frac{21}{22} \)  MaxDiCut: \( \frac{11}{12} \)
  - NAE-3SAT: \( \frac{15}{16} \)  3-Set-Splitting: \( \frac{19}{20} \)  3-coloring: \( \frac{32}{33} \) (these are with perfect completeness)
Approximation resistance

• The tight results for CSPs showed “approx. resistance”
  – hard to beat naive random assignment.

• An exception: Max 3MAJ
  – $2/3 + \varepsilon$ hardness via reduction from 3LIN
  – Factor 2/3 algorithm [Zwick]

• In parallel with hardness revolution, sophisticated SDP rounding methods developed. Eg.
  – 7/8 algo for Max3SAT [Karloff-Zwick]
  – Factor $\frac{1}{2}$ algo for Max3CSP and factor 5/8 for satisfiable 3CSP.
Optimal PCPs: queries vs soundness

- [G.-Lewin-Sudan-Trevisan] 3-query adaptive PCP with perfect completeness & soundness $\frac{1}{2} + \epsilon$
- [Samorodnitsky-Trevisan] $1 + \epsilon$ amortized query complexity: $k$ queries, soundness $2^{-k+o(k)}$
  - $2^{-k+o(k)}$ hardness for Max $k$-CSP.
  - Later with perfect completeness [Håstad-Khot]
  - Useful starting points in some reductions, eg. Low-congestion path routing, Clique.
Low-soundness Multiprover systems

• [Arora-Sudan; Raz-Safra] $O(1)$ prover 1-round proof systems with $\exp(-(\log n)^{\Omega(1)})$ soundness
  – NP-hardness of $\Omega(\log n)$ factor for set cover
  – Similar result for 2-prover case open
    • Would have more applications, like hardness of lattice problems

• [Dinur-Fischer-Kindler-Raz-Safra] $O(1)$ prover systems with
  – $\exp(-(\log n)^{0.99})$ soundness.
  – Proof of BGLR “sliding scale” conjecture for up to $(\log n)^{0.99}$ bits read
Covering PCPs

• Notion of soundness tailored to coloring problems
  – Covering soundness = minimum number of proofs that can “cover” all constraints (for every check, at least one proof should cause acceptance)
  – [G.-Håstad-Sudan] 4-query PCP with $\omega(1)$ covering soundness
    • Super-constant hardness for coloring 2-colorable 4-uniform hypergraphs.
    • Later also for 3-uniform hypergraphs [Khot] [Dinur-Regev-Smyth]

Frontier: Rule out 5-coloring 3-colorable graphs in polytime
PCPs till ~ 2000 summary

• Label Cover hardness
  – versatile starting point for inapproximability (continues to be prominent)
• Label Cover + Long Code + Fourier analysis paradigm
• Tight hardness results for several CSPs of arity \( \geq 3 \)
• Arity 2 CSPs not well understood (results only via gadgets)
New proofs, notions:
- Dinur’s gap amplification
- Robust PCPs
- PCP of proximity (PCPP)

Short PCPs ($n^{1+o(1)}$ size):
- Best known $n (\log n)^{O(1)}$

低 soundness error 2-query PCPs

New “outer” PCPs
- multilayered, smooth, mixing, Dinur-Safra, etc.
- Conjectural forms:
  - Unique Games, 2-to-1, ...

Dictatorship tests and new “inner” PCPs
- New analytic machinery

New Proof Composition methods

PCPs in the last decade
New proofs and notions
Dinur’s proof

• **Gap amplification**: Reduce Gap-3Color\(_{1,1-\delta}\) to Gap-3Color\(_{1,1-2\delta}\) provided \(\delta < 10^{-6}\)
  
  – Apply \(O(\log n)\) times starting with \(\delta = 1/m\)
  
  – Shows that Gap-3Color\(_{1,\alpha}\) is NP-hard for some constant \(\alpha < 1\) (this implies the PCP theorem)

• **PCP via inapproximability instead of other way around**

• Requires elements of old PCP in alphabet reduction
  
  – Constant sized PCP: variant called “assignment tester” that checks if assignment \(x\) is close to satisfying circuit \(C\)
Robust PCPs

- PCP soundness: When φ \not\in SAT, for all proofs π, with probability \rho, check C_I rejects π_I (I=randomly chosen query positions)

\[ \text{Check } C_I(\pi_I) = 1 \]

- Robust PCP: stronger soundness guarantee
  - π_I far from satisfying C_I (with good prob.)
  - Formally,

\[ \mathbb{E}_I [\Delta(\pi_I, SAT(C_I))] > \eta \quad (\eta = \text{robust soundness}) \]
Robust PCPs & PCPPs

- Check if $\pi$ satisfies $C$ or is far from satisfying $C$ recursively, using another “inner” PCP

- Inner primitive: **PCP of proximity** (PCPP)
  - Input: circuit $C$
  - Proof: (purported) satisfying assignment $x$ and proof of proximity $\sigma$ that $x$ satisfies $C$
  - Verifier (Assignment Tester): read few bits in both $x$ and $\sigma$;
    - For satisfying $x$, $\exists \sigma$ with acc. prob. 1
    - If $x$ is $\delta$-far from satisfying $C$, then $\forall \sigma$, $\Omega(1)$ rej. prob.

- Useful when proximity parameter $\delta$ of inner PCPP is $< \eta$ of outer PCP
Composition streamlined

• Robust PCPs compose “syntactically” with “inner” PCPs of proximity (when PCPP proximity parameter < robustness)

• [Ben-Sasson, Goldreich, Harsha, Sudan, Vadhan] Can check that original polynomial and Hadamard based PCPs can be made robust PCPPs.
  – Simplifies details of composition
  – Used to give near-linear size PCPs

• PCPP also used in Dinur’s alphabet reduction step
  – Explicit coding used to create distance between inconsistent assignments
Talk Plan

• New proofs and notions
• Robust PCPs and PCPs for proximity
• **Short PCPs**
• Low-soundness error Label Cover
• Unique Games, Dictatorship tests, etc.
• NP-hardness via structured outer PCPs
• Some challenges
Quasi-linear PCPs

- **[BGHSV]** PCPs of length $n \cdot 2^{(\log n)^\epsilon}$ $(O(1/\epsilon)$ queries)

- **[Ben-Sasson, Sudan]** *Univariate* polynomial based PCP
  - *Proof of proximity* for Reed-Solomon codes which makes it locally testable
  - $n (\log n)^{O(1)}$ sized PCP with $(\log n)^{O(1)}$ queries
  - $O(\log \log n)$ steps of Dinur’s gap amplification gives
    $n (\log n)^{O(1)}$ sized PCP with $O(1)$ queries

- Implication for approximation: APX-hardness via reduction from 3SAT hold for $2^{\Omega(n)}$ time algos, under the ETH
• New proofs and notions
• Robust PCPs and PCPs for proximity
• Short PCPs

• **Low-soundness error Label Cover**
• Unique Games, Dictatorship tests, etc.
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Label cover with $o(1)$ soundness

- Håstad’s $7/8 + \varepsilon$ hardness for 3SAT requires soundness error of Label Cover to be $\ll \varepsilon$
- Getting this via parallel repetition makes Label Cover instance size $n^{\Omega(\log (1/\varepsilon))}$ (with large hidden constant factor)
- Can we get such soundness with LC size say $O_{\varepsilon}(n^3)$ ?

**Answer:** [Moshkovitz-Raz] YES. In fact, with *near-linear* size!
(Was very surprising to me)

For soundness $\varepsilon$, Label Cover with $n^{1+o(1)} \text{poly}(1/\varepsilon)$ vertices.
- Worse dependence on domain size: $R = \exp(\text{poly}(1/\varepsilon))$ instead of $\text{poly}(1/\varepsilon)$ in Raz.
Inapproximability consequence

• $7/8 + \varepsilon$ approx. for 3SAT requires $\exp(\Omega_\varepsilon(n^{1-o(1)}))$ time (under “Exponential Time Hypothesis”)
  – constant factor: *doubly exponentially* small in $\varepsilon$
  – Similar claims for other hardness results based on Label cover + Long code testing

• Sharp complexity *dichotomy* at the approximation threshold: polytime vs. exponential time.
  – Eg. unlikely that there’s a factor $7/8 + \varepsilon$ approx. algo for Max3SAT with runtime $\exp(n^{10\sqrt{\varepsilon}})$
[Moshkovitz-Raz] approach

• Start with Label Cover of low soundness error $\varepsilon$ but large alphabet $\Sigma$
  – $\varepsilon = (\log n)^{-\beta}$, $|\Sigma| = \text{poly}(n)$ (based on low-degree testing in list-decoding regime)
  – reduce alphabet size via composition

• New composition method for low-error regime that does not increase # provers
  – Based on “Locally Decode/Reject Codes”

Next: Few words on an alternate approach (giving polynomial instead of near-linear size)
Derandomized parallel repetition

- **$u$-parallel repetition** of Label Cover reduces soundness error from (say) 0.99 to $\varepsilon = 2^{-O(u)}$, but blows up size to $\approx n^u$
  - Can’t get $o(1)$ soundness with polytime reduction (const. $u$)

- Derandomization: Can we select a *poly-sized subset* of all possible $u$ questions?
  - Limitation [Feige-Kilian]: for $u = O(\log n)$ and poly$(n)$ size subsets, $\varepsilon \geq 1$/poly$(\log n)$

- [Dinur-Meir] Match this lower bound, combining
  - Derandomized *direct product testing* based on *subspaces* [Impagliazzo-Kabanets-Wigderson]
  - Structured “linear” PCPs
    - Identify proof coordinates (vertices) with $\mathbb{F}^m$
    - Edges corresponding to 2-query checks form a *subspace* of $\mathbb{F}^{2m}$
Composing without an extra prover

- [Dinur-Harsha] alternate composition method to reduce alphabet size keeping #provers at 2
  - Based on “decodable PCPs”
  - Exploit equivalence between robust PCPs and Label Cover

- Applying this to [Dinur-Meir] 2-query PCP, gives:
  - Label Cover of fixed polynomial (though not near-linear) size with soundness $\varepsilon$, $\forall$ constants $\varepsilon > 0$
Label Cover $\iff$ Robust PCPs
Label Cover $\Rightarrow$ Robust PCPs

Verifier
1. Selects a random “big” vertex $v \in V$
2. Reads entire neighborhood of $v$
3. Accepts iff there is a value for $v$ that would cause all edge constraints to accept.
Label Cover $\Rightarrow$ Robust PCPs

YES instances – all views are “happy”

NO instances – average view is very “unhappy”, i.e. view from a random window is at most $\delta$-close to a satisfying view.
Label Cover $\Leftrightarrow$ Robust PCPs

- This transformation is “invertible” (rotate back!)
- $|\Sigma|$ corresponds to the number of accepting configurations, which is $\leq \exp(\# \text{PCP queries})$
  - Reducing PCP queries $\Rightarrow$ reducing LC alphabet size
Low-error 2-prover systems summary

• Some very exciting recent constructions

• **Frontier**: 2-query PCP of *polynomial size and polynomial alphabet* with soundness error $1/(\log n)^{10}$
• New proofs and notions
• Robust PCPs and PCPs for proximity
• Short PCPs
• Low-soundness error Label Cover
• **Unique Games, Dictatorship tests, etc.**
• NP-hardness via structured outer PCPs
• Some challenges
Long code based “inner” PCPs

- **Task**: Checking that labels $a = A(v)$ and $b = A(w)$ satisfy some Label Cover projection constraint $\pi(a) = b$
- Check constraints on few (say $q$) bits of (purported) long codes $f, g : \{0,1\}^R \rightarrow \{0,1\}$ of labels $a$ and $b$
  - If $f$ and $g$ are long codes of “consistent” $a, b$ (i.e., $\pi(a) = b$), accept with prob. 1 (or $1 - \varepsilon$)
  - Acceptance with prob. $> s + \varepsilon$ implies one can randomly “decode” $f, g$ into labels $a', b'$ s.t. $\pi(a') = b'$ with prob. $\theta = \theta(\varepsilon)$
  - This would imply a soundness $s + 2\varepsilon q$-query PCP

Recall:
$f = \text{LONG}(a) \Rightarrow f(x) = x_a$

Let’s see Håstad’s 3-query PCP, and constructing 2-query PCPs.
Håstad’s 3-query PCP

- Pick $x, y \in \{0, 1\}^R$ u.a.r.
- Pick $\mu \in \{0, 1\}^R$ from the $\epsilon$-biased distribution.
- Define $z \in \{0, 1\}^R$ by $z_j = y_j \oplus x_{\pi(j)} \oplus \mu_j$.
- With prob. $1/2$ check $g(x) \oplus f(y) \oplus f(z) = 0$, and with prob. $1/2$ check $g(x) \oplus f(y) \oplus f(\overline{z}) = 1$ ($\overline{z} =$ coordinate-wise complement of $z$).

Completeness: If $f(y) = y_a$ and $g(x) = x_b$ and $\pi(a) = b$, then $g(x) \oplus f(y) \oplus f(z) = x_b \oplus y_a \oplus y_a \oplus x_{\pi(a)} \oplus \mu_a = \mu_a$.

Soundness: Acceptance prob. $> 1/2 + \epsilon \Rightarrow g, f$ have non-trivial agreement with “consistent” low-weight linear functions.
3 vs. 2 queries

• Get \((1-\varepsilon)\) vs. \((\frac{1}{2} + \varepsilon)\) hardness for Max3LIN \((\text{mod } 2)\)
  – Approximation resistant
• Each query point \(x, y, z\) is uniformly distributed in \(\{0, 1\}^R\)
  – \(y, z\) are correlated, but \(f\) has to give value to each separately (and each is uniform)
• What about 2-variable linear equations \(\text{mod } 2\)?
  – [Goemans-Williamson] algo finds assignment of value \(1-O(\sqrt{\varepsilon})\) in \((1-\varepsilon)\)-satisfiable instance
  – Matching hardness through a 2-query PCP?
A 2-query PCP?

Here’s a natural test, saving 1 query in Håstad’s test:

- Pick $x \in \{0, 1\}^R$ u.a.r., and $\epsilon$-biased noise vector $\mu \in \{0, 1\}^R$
- Set $y = x \circ \pi \oplus \mu$, i.e., for $j \in [R]$, set $y_j = x_{\pi(j)} \oplus \mu_j$.
- With prob. $1/2$, check $g(x) \oplus f(y) = 0$, with prob. $1/2$, check $g(x) \oplus f(\overline{y}) = 1$.

**Trouble:** query $y$ to $f$ is not uniform.
\[ y_j = y_k \text{ with prob. close to 1 when } \pi(j) = \pi(k) \]

Query $y$ reveals lots of information about projection $\pi$
Could form “cheating” f by “piecing together” many inconsistent long codes, for portions of $\{0,1\}^R$ corresponding to different projections $\pi_e$

**How to circumvent this?**
Unique Games

• Khot’s insight: if $\pi$ is a bijection, then $y = x \circ \pi$ is uniformly distributed (since $x$ is); gives no clue about $\pi$

Unique Game (UG)
* Label Cover where all projection constraints are bijections

Khot’s Unique Games Conjecture (UGC):

$\text{GapUG}_{1-\varepsilon, \varepsilon}$ is NP-hard for $R > R(\varepsilon)$

• UGC $\Rightarrow$ analysis of 2-query test reduces to $f=g$ case
  – show that if $f$ passes w.h.p, then $f$ is “like” a long code
    (modern term: dictator)
  – just codeword testing, no “consistency” checking
2-query dictator testing

The core question becomes analyzing “noise stability”

\[ \text{NS}_\varepsilon(f) = \text{Prob}_{x,\mu} \left[ f(x) = f(x \oplus \mu) \right] \] (assume \( f \) is balanced)

- If \( f = \) dictator, then \( \text{NS}_\varepsilon(f) = 1 - \varepsilon \)
- If \( \text{NS}_\varepsilon(f) \) is close to \( 1 - \varepsilon \), what can we say?

[**Bourgain**] If \( \text{NS}_\varepsilon(f) > 1 - \varepsilon^{0.51} \) then \( f \) is close to a *junta* (depends on few coordinates)

[**Mossel-O’Donnell-Oleszkiewicz**] (Majority is Stablest Thm) If \( \text{NS}_\varepsilon(f) > 1 - \Theta(\sqrt{\varepsilon}) \) then \( f \) has an *influential* coordinate.

Both of these can be used in reduction from Unique Games
UGC consequences...

- $(2-\varepsilon)$ hardness for Vertex cover [Khot-Regev]
- 0.878.. hardness for Max Cut [Khot-Kindler-Mossel-O’Donnell] (using Majority is Stablest)
- Near-tight hardness for all Boolean 2CSPs [Austrin]
- Optimal hardness for every CSP [Raghavendra] (using invariance principle of [Mossel])
- Approximation resistance of every ordering CSP [G.-Håstad-Manokaran-Raghavendra-Charikar]

Hardness matching LP integrality gaps:

- Multiway Cut, Metric Labeling [Manokaran-Naor-Raghavendra-Schwartz]
- Strict CSPs, covering problems [Kumar-Manokaran-Tulsiani-Vishnoi]
• New proofs and notions
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• **NP-hardness via structured outer PCPs**
• Some challenges

Proving UGC predictions without UGC?
Bypassing UGC?

• UGC has predicted many strong results
  – All plausible & consistent with our knowledge
  – “combinatorial” ones (like embedding lower bound) confirmed unconditionally

• Yet, no UG-completeness result so far
  – Possible that consequences of UGC are true but conjecture itself is false

• Natural question: Can we verify some of UGC’s predictions without resorting to UGC?
  – Ideally, show NP-hardness
  – Or hardness under weaker assumptions (like optimality of Goemans-Williamson, 2-to-1 conjecture)?
Smooth Label Cover

- [Khot] Label Cover where for each $v \in V$, the projections $\pi_e$ for edges $e$ incident on $v$ form a “hash family”

$$\forall a \neq a' \quad \text{Prob}_{e \ni v}[\pi_e(a) = \pi_e(a')] \rightarrow 0 \quad \text{(for large $R$)}$$

- Gap-Smooth-Label-Cover$_{1,\epsilon}$ is NP-hard
  - Reduction from Label Cover (note perfect completeness)

- Such “locally unique” projections have been useful in some NP-hardness reductions
  - 3-coloring 3-uniform hypergraphs [Khot]
  - Learning intersection of two halfspaces [Khot-Saket]
  - Agnostic learning monomials by halfspaces [Feldman-G.-Raghavendra-Wu]
A Gaussian approximation threshold

\( L_p \) Subspace approximation problem \((2 < p < \infty)\)

**Input**: A set of \( m \) points \( a_1, a_2, \ldots, a_m \in \mathbb{R}^n \).

**Goal**: Find a \((n - 1)\)-dimensional subspace (hyperplane) \( H \) minimizing

\[
\sum_{i=1}^{m} \text{dist}(H, a_i)^p
\]

where \( \text{dist}(H, a) \) is the min. Euclidean distance between \( a \) and any point in \( H \).

**[Deshpande-Tulsiani-Vishnoi]**
Factor \( \beta_p \) algorithm where \( \beta_p = \mathbb{E}_{g \sim N(0,1)} [ |g|^p ] \) is the \( p \)'th moment of the standard Gaussian.
And matching \( \beta_p - \varepsilon \) Unique-Games-hardness

**[G.-Raghavendra-Saket-Wu]** NP-hardness of factor \( \beta_p - \varepsilon \) approximation, using smooth Label Cover
Other Label Cover variants

Other structured Label Cover instances discovered in the last decade:

– Multilayered PCPs
– 2-to-1 projections (conjectural)
– Dinur-Safra
Multilayered PCP

• Multipartite label cover:
  – $L$ layers of vertices $V_1, V_2, \ldots, V_L$
  – Projection maps between every pair of layers:
    • for edge $e$ between $v_i \in V_i$ and $v_j \in V_j$ ($i < j$),
      $\pi_e : [R] \rightarrow [R]$ from label of $v_i$ to label of $v_j$
• Ensure every $\gamma$ fraction of vertices have many constraints amongst them (for $L > L(\gamma)$)
• Introduced in [Dinur-G.-Khot-Regev] to show factor $(k-1-\varepsilon)$ hardness for vertex cover on $k$-uniform hypergraphs
• Later use in hypergraph coloring, non-mixed 3SAT, etc.
2-to-1 conjecture

- Label Cover where projection maps are 2-to-1
- **Conjecture [Khot]:** Gap-2-to-1-LC\(_{1,\varepsilon}\) is NP-hard
  - Parallel repetition gives \(\text{poly}(1/\varepsilon)\)-to-1 projections

- **Consequences:**
  - \(\sqrt{2} - \varepsilon\) hardness for vertex cover [Dinur-Safra], [Khot]
  - Hardness of \(O(1)\)-coloring 4-colorable graphs [Dinur-Mossel-Regev]
  - Hardness of Gap-No-Two\(_{1,5/8+\varepsilon}\) [O’Donnell-Wu]
  - **Factor** \(1 - \frac{1}{k} + O\left(\frac{\ln k}{k^2}\right)\) hardness for Max k-coloring [G.-Sinop]
This remarkable paper showed factor 1.3606 NP-hardness for vertex cover.

Underlying this was a “2-to-1 like” Label Cover:
- label for v consistent with one or two labels for w
- Soundness: \( \forall \) labeling, every \( \varepsilon \) fraction of vertices has a 100-clique of inconsistent pairs.

Other applications?
Wrap-up

• PCPs remarkably successful in showing inapproximability (even beyond initial expectations?)
  – Breadth of problems.
    I find it amazing what all Label Cover can be reduced to!
  – Many tight results

• Some notorious problems have withstood resolution
  – Densest subgraph, minimum linear arrangement, bipartite clique, sparsest cut, graph bisection, etc.
  – Known algorithms have superconstant approx. ratio,
    but even APX-hardness not known
Cut challenges

Eg. Uniform Sparsest Cut, Minimum Bisection:
Best approximation \((\log n)^\Omega(1)\). Hardness evidence:
1. Refuting random 3SAT is hard \(\Rightarrow\) Factor 1.1 hardness [Feige]
2. Polytime \((1+\epsilon)\) approximation \(\Rightarrow\) NP has \(2^{n^{\epsilon'}}\) time algorithms [Khot, “quasi-random PCPs”]
3. Superconstant hardness under “SSE hypothesis” (stronger than Unique Games conjecture) [Raghavendra-Steurer-Tulsiani]

“Easiness” evidence [G.-Sinop]
• Factor \((1+\epsilon)/\lambda_r\) approximation in \(2^{O(\epsilon r)} n^{O(1)}\) time where \(\lambda_r\) is the \(r^{th}\) smallest eigenvalue of normalized Laplacian.
  – Factor \(3/\lambda_r\) for minimum uncut
Challenges

• Can PCP machinery (even assuming UGC) give strong hardness results for Steiner Tree, TSP, Asymmetric TSP?
• Lasserre integrality gaps beyond known hardness bound for Vertex Cover, Max Cut (or Unique Games)?
  – Just 4 rounds could improve [GW] and refute UGC!
• Unique-Games-completeness?
• Bypass UGC for some other consequences?
• Other hardness assumptions: eg. Densest subgraph?
  – Or finding indep. Sets of size $\varepsilon n$ when one of size $n/100$ exists
• Unchartered terrain for inapproximability:
  – eg., nearest codeword in algebraic codes, bin packing, …