Finding Dense Subgraphs

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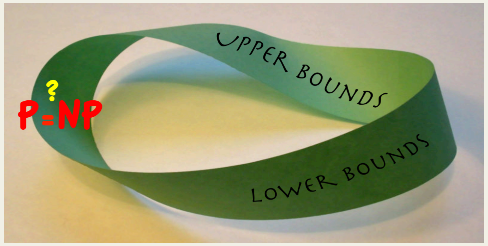

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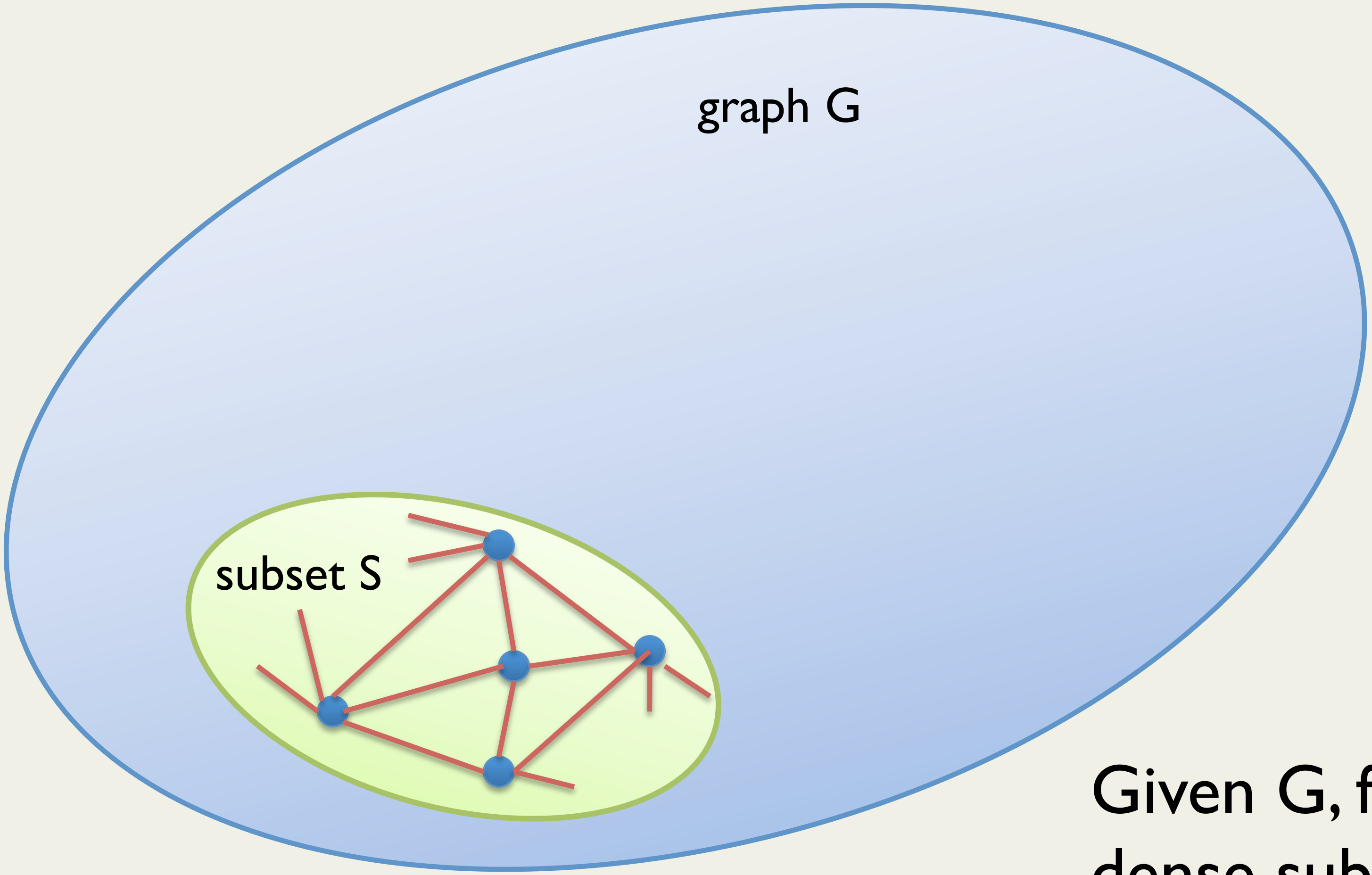
The Dense Subgraph Problem



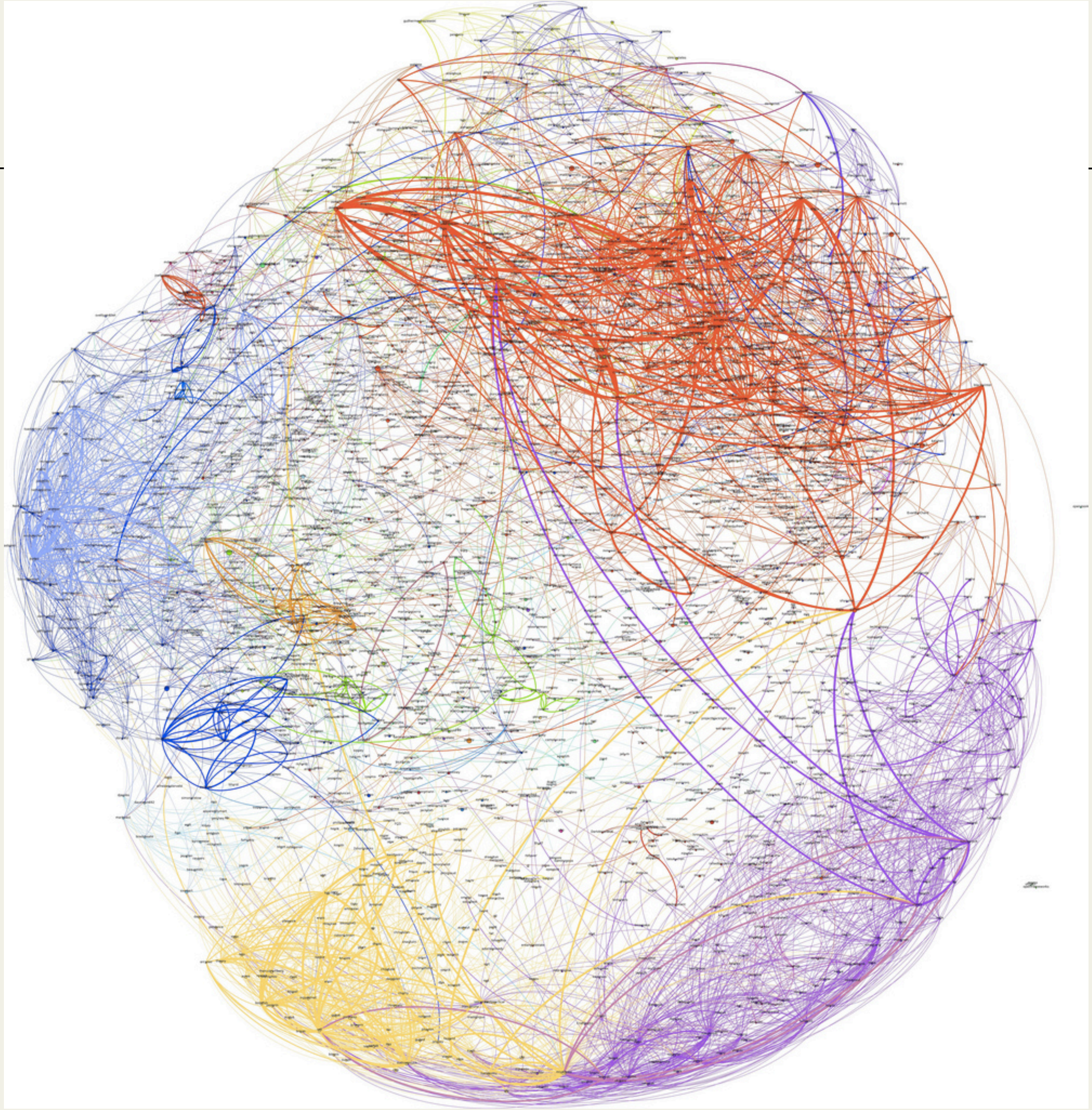
Given G, find
dense subgraph S

Dense subgraphs are everywhere !

- A useful subroutine for many applications.

Social Networks

- Trawling the Web for emerging cyber-communities [KRRT '99]
  - *Web communities are characterized by dense bipartite subgraphs*



Communities on gitweb

Computational Biology

- Mining coherent dense subgraphs across massive biological networks for functional discovery [HYHHZ '05]

  – *dense protein interaction subgraph corresponds to a protein complex* [BD'03] [SM'03]

  – *dense co-expression subgraph represent tight co-expression cluster* [SS '05]

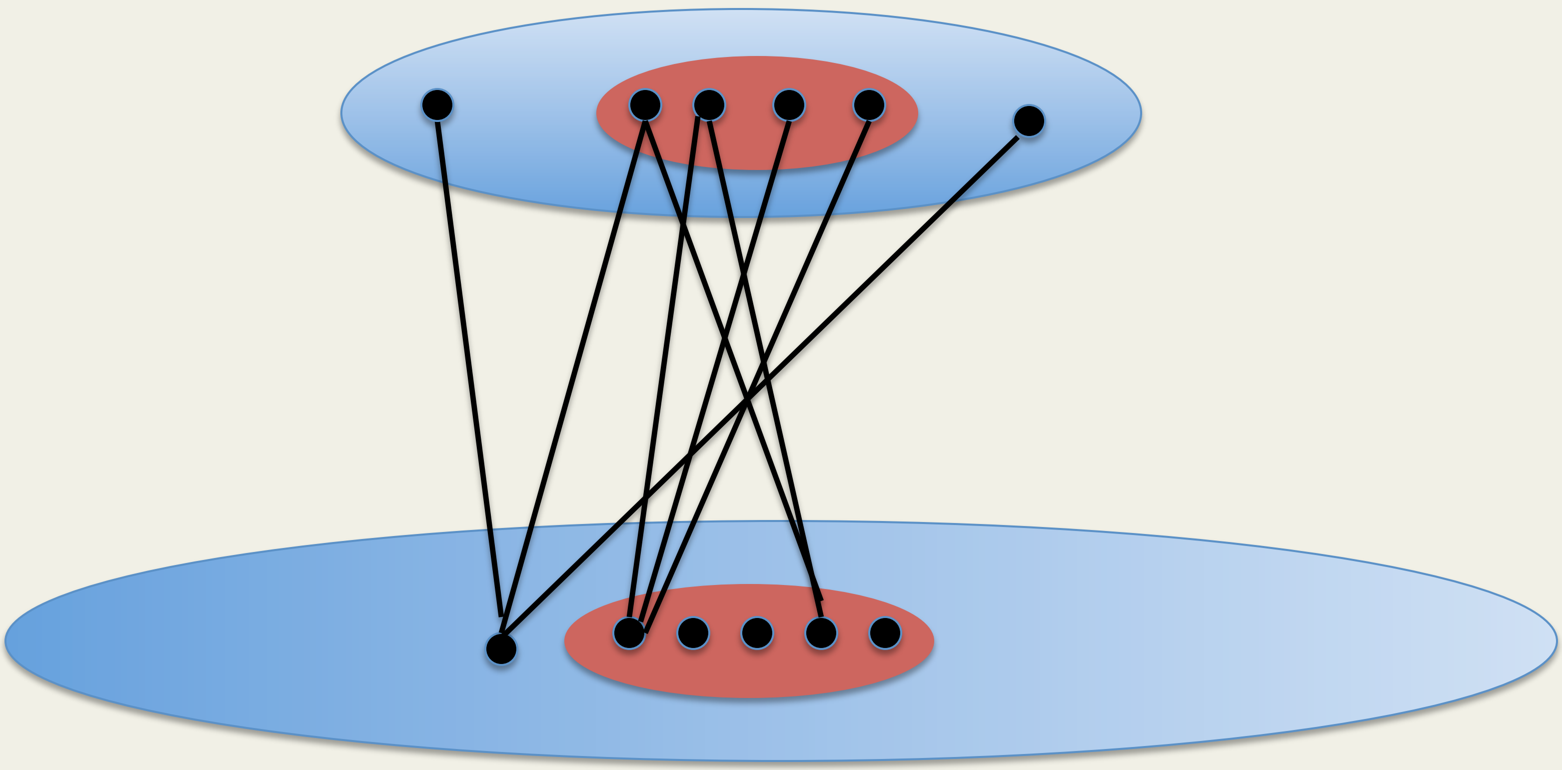
Dense subgraphs are everywhere !

- A useful subroutine for many applications.

- A useful candidate hard problem with many consequences

Public Key Cryptography [ABW '10]

- Hardness assumption

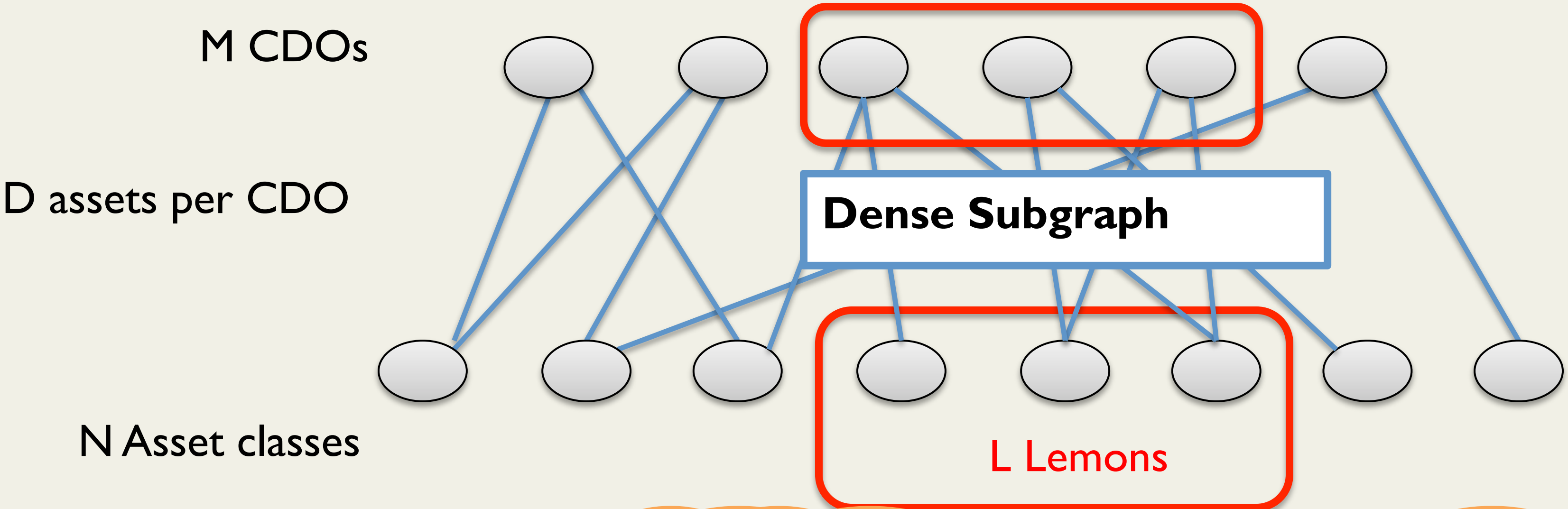


Complexity of Financial Derivatives

- Computational Complexity and Information Asymmetry in Financial Products [ABBG '10]
  - *Evaluating the fair value of a derivative is a hard problem*
  - *Tampered derivatives (CDOs) can be hard to detect.*
  - *Derivative designer can gain a lot from small asymmetry in information (lemon cost).*

Simplest Model

6σ lemons, default w.p. ½



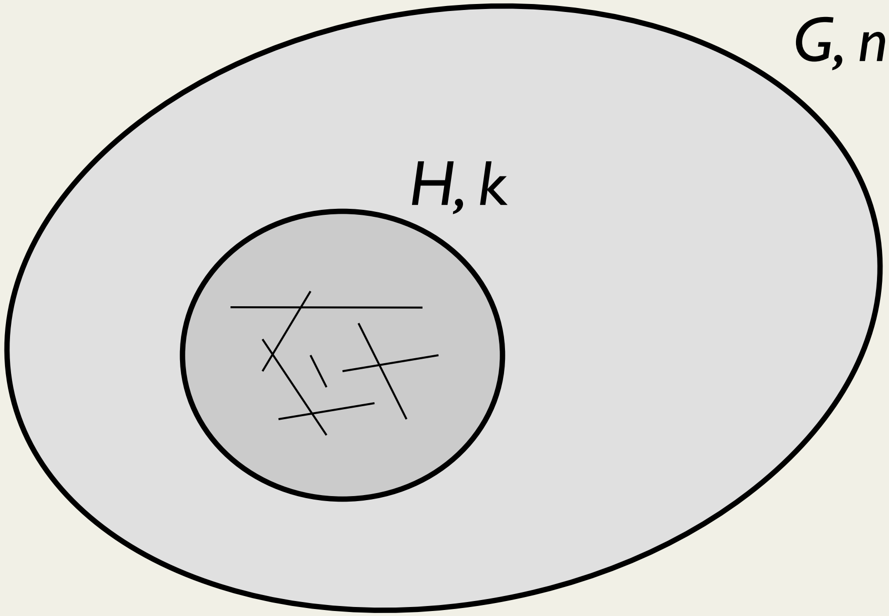
Summary so far

- Finding dense subgraphs is useful, both as a subroutine as well as a candidate hard problem

- So, what do we know about the problem ?
  – Formal definition
  – New results
  – New results on related problems

Densest *k*-subgraph

Problem. Given *G*, find a subgraph of size *k* with the maximum number of edges (think of *k* as $n^{1/2}$)



Problems of similar flavor

- Max clique
- Max density subgraph – find *H* to maximize the ratio:

$$\frac{\#\text{edges}(H)}{|H|}$$

Approximation Algorithm

- Exact problem is hard, prove that efficient heuristic finds good solution.

- Approximation ratio = $\dfrac{\text{Value of heuristic solution}}{\text{Value of optimal solution}}$

- Solution value = number of edges in subgraph

Densest *k*-subgraph

Problem. Given *G*, find a subgraph of size *k* with the maximum number of edges (think of *k* as $n^{1/2}$)

[Feige, Kortsarz, Peleg 93] $O(n^{1/3 - 1/90})$ approximation

[Feige, Schechtman 97] $\Omega(n^{1/3})$ integrality gap for natural SDP

[Feige 03] Constant hardness under the Random 3-SAT assumption

[Khot 05] There is no PTAS unless NP $\subseteq$ BPTIME(sub-exp)

Main Result

[Bhaskara, C, Chlamtac, Feige, Vijayaraghavan '10]

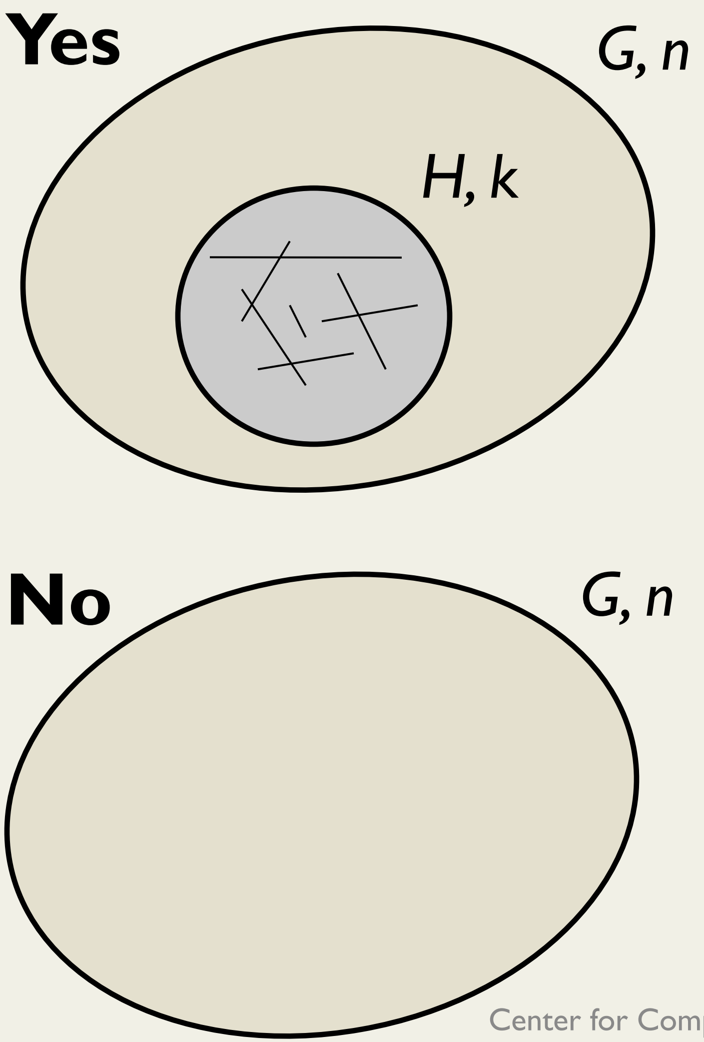
Theorem. $O(n^{1/4+\varepsilon})$ approximation for DkS in time $O(n^{1/\varepsilon})$

(Informal) Theorem. Can efficiently detect subgraphs of high log-density.

Outline

- Introduce two average case problems
- ‘Local counting’ based algorithms for these
- Notion of log-density
- Techniques lead to algorithms for the DkS problem

Planted problems related to DkS



- Assume *G* does not have dense subgraphs
- Good algorithm for DkS ⇒ we can distinguish

Two natural questions:

1. Random in Random: *G(k,q)* planted in *G(n,p)*
2. Arbitrary in Random: *Some* dense subgraph planted in *G(n,p)*

Random in Random

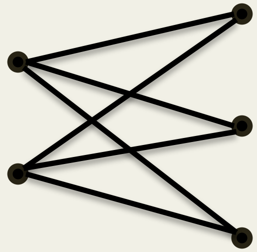
Question. How large should q be so as to distinguish between

YES: $G(n,p)$ with $G(k,q)$ planted in it

NO: $G(n,p)$

When would looking for the presence of a subgraph help distinguish?

Eg. $K_{2,3}$



Random in Random

Question. How large should q be so as to distinguish between

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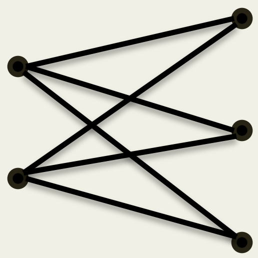


[Erdos-Renyi]:

- Appears w.h.p. in $G(n,p)$ if $n^5p^6 >> 1$, i.e., degree $>> n^{1/6}$
- Does *not* appear w.h.p. in $G(n,p)$ if $n^5p^6 << 1$, i.e., degree $<< n^{1/6}$

Valid distinguishing algorithm if: $k^5q^6 >> 1$, and $n^5p^6 << 1$

I.e., degree $<< n^{1/6}$, and planted-degree $>> k^{1/6}$

Random in Random

Question. How large should q be so as to distinguish between

YES: $G(n,p)$ with $G(k,q)$ planted in it

NO: $G(n,p)$

In general, suppose degree < n^{δ}, and planted-degree > $k^{\delta+\varepsilon}$

Find a rational number $1-r/s$ between δ and $\delta+\varepsilon$, and use a graph with r vertices and s edges to distinguish.

Log density

A graph on n vertices has **log-density** δ if the average degree is n^δ

$$\delta = \frac{\log d_{avg}}{\log |V|}$$

Question. Given G, can we detect the presence of a subgraph on k vertices, with higher log-density?

Dense vs. Random

Problem. Distinguish $G \sim G(n,p)$, log-density δ from a graph which has a *k*-subgraph of log-density δ+ε

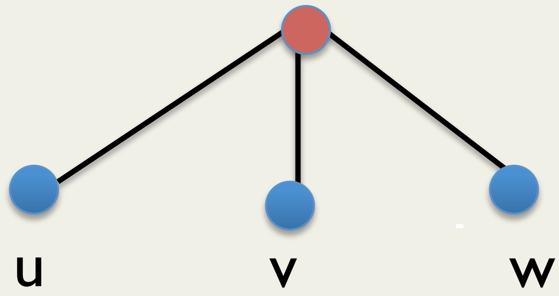
(Note. $kp = k(n^{\delta}/n) = k^{\delta}(k/n)^{1-\delta} < k^{\delta}$)

More difficult than the planted model earlier (graph inside is no longer *random*)

Eg. *k*-subgraph could have log-density=1 and not have triangles

Main idea

Example. Say $\delta = 2/3$, i.e., degree = $n^{2/3}$



random graph $G(n, n^{-1/3})$:

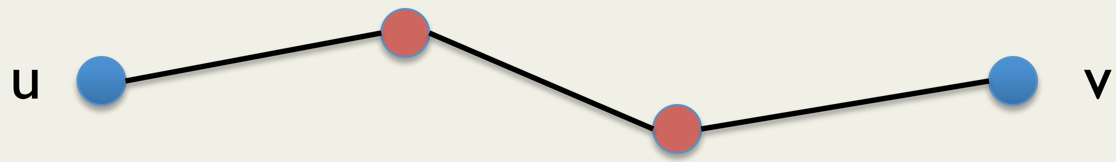
any three vertices have O(log n) common neighbors w.h.p.

planted graph: size k, log-density $2/3+\varepsilon$:

triple with $k^{3\varepsilon}$ common neighbors

Main idea (contd.)

Example 2. δ = 1/3, i.e., degree = $n^{1/3}$



random graph $G(n, n^{-1/3})$:

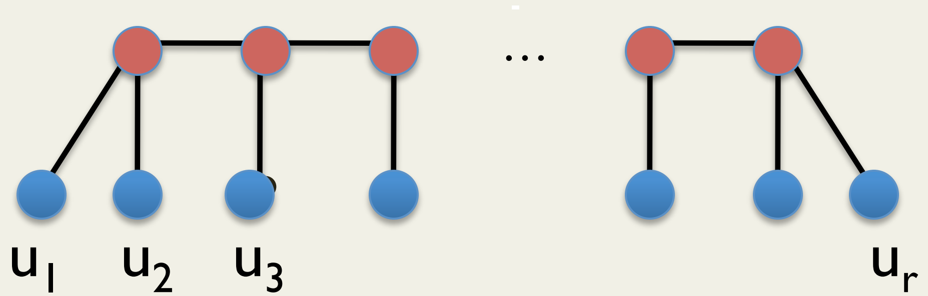
any pair of vertices have $O(\log^2 n)$ **paths of length 3**, w.h.p.

planted graph: size k, log-density 1/3+ε:

exists a pair of vertices with k^ε paths

Main idea (contd.)

General strategy: For each rational δ, consider appropriate `caterpillar' structures, count how many `supported' on fixed set of leaves



- Random graph *G(n,p)*, log-density δ:
  every leaf tuple supports polylog(*n*) caterpillars
- Planted graph, size *k*, log-density δ+ε :
  some leaf tuple supports at least k^{ε} caterpillars

Dense vs. Random – conclusion

Theorem. For every $\varepsilon>0$, and $0<\delta<1$, we can distinguish between $G(n,p)$ of log-density δ, and an arbitrary graph with a k-subgraph of log-density $\delta+\varepsilon$, in time $n^{O(1/\varepsilon)}$.

(Pick a rational number between δ and $\delta+\varepsilon$, and use the caterpillar corresponding to it)

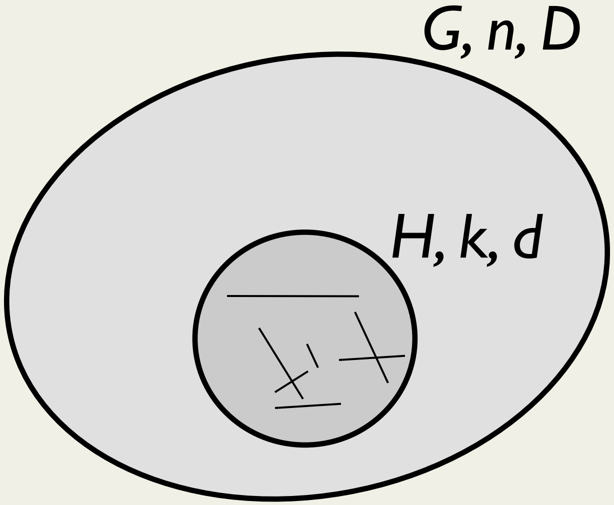
DkS in general graphs

Preliminaries

Aim. Obtain a k-subgraph of avg degree ρ

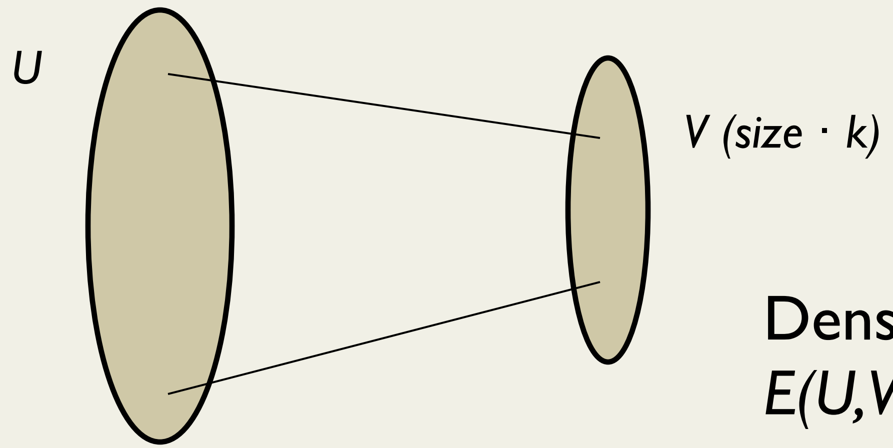
Observation 1. It suffices to return a ρ-dense subgraph with $\leq k$ vertices

(remove and repeat)



Preliminaries

Observation 2. It suffices to return a bipartite subgraph with density ρ, and ≤ *k* vertices on one side

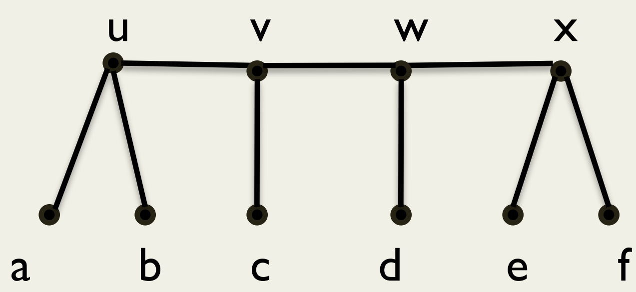


Density is ρ, so
$E(U,V) = \rho(|V|+|U|)$

- Pick the $|V|$ vertices in U of largest degree
- Density of the resulting subgraph is

$$\frac{1}{2|V|} \cdot \frac{|V|}{|U|} \cdot \rho(|V| + |U|) \geq \frac{\rho}{2}$$

Algorithm using Cat_δ



Idea. Look at the 'set of candidates' for a non-leaf after fixing a prefix of the leaves

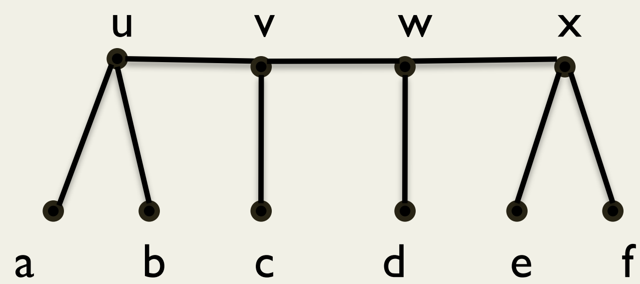
Eg., define $S_{abc}(v)$ = set of 'candidates' in G for internal vertex v after fixing a,b,c

(for instance, $S_{ab}(u)$ is the set of common nbrs of a, b)

Denote $T_{abc}(v) = S_{abc}(v) \cap H$

Given a, b, .. and the structure, we can compute the S's

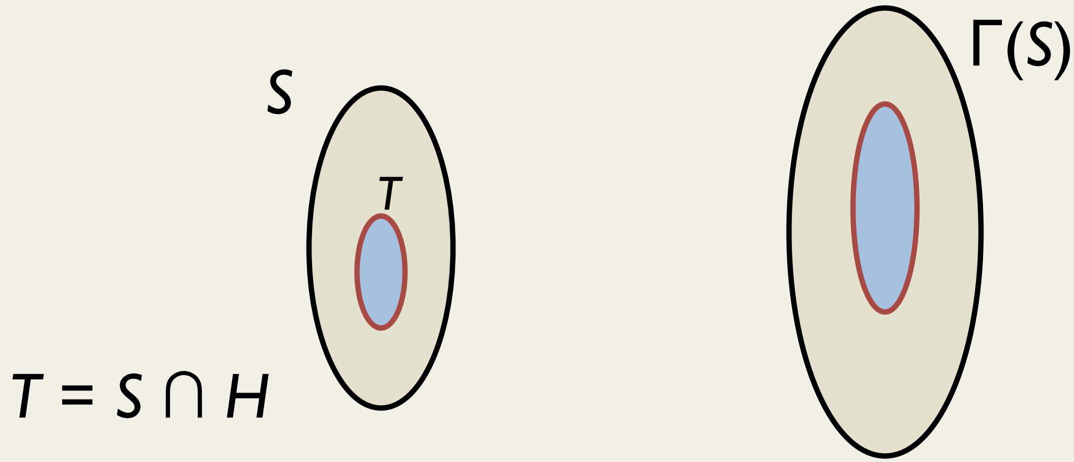
Algorithm using Cat$_\delta$ (plot outline)



Procedure LocalSearch(S)

- For every $a \in V$, perform LocalSearch($S_a(u)$)
- If it always fails, then $\exists\, a, b$, s.t. $|S_{ab}(u)| \le U_1$ and $|T_{ab}(u)| \ge L_1$
- For every a,b, perform LocalSearch($S_{ab}(u)$)
- If it fails each time, then $\exists\, a, b$, s.t. $|S_{ab}(v)| \le U_2$ and $|T_{ab}(v)| \ge L_2$
- Keep doing this … At the last step, the parameters give a contradiction!

Main Component – LocalSearch(S)



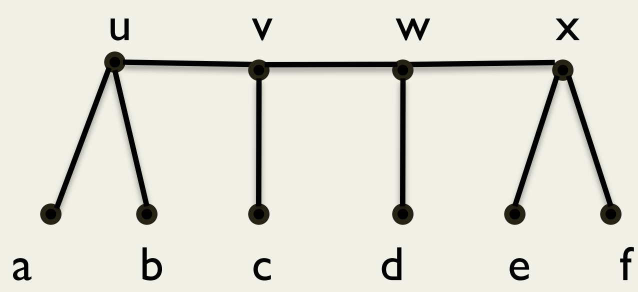
For each $i = 1 \ldots k$, do:

- Pick the i vertices on the right with the most edges to S (call this S_r). If $S \cup S_r$ has density $\geq \rho$, return it.

If no dense subgraph is found, return Fail

Linear Programming view

- Can bound the quality of the solution w.r.t value of a Lift-and-project style LP relaxation.
- Algorithm can be viewed as rounding procedure for relaxation via successive conditioning



Subexponential algorithm

- $n^{(1-\varepsilon)/4}$ approximation in time $2^{n^{6\varepsilon}}$
- Guess subsets of size n^{ε} for every leaf in caterpillar structure.

New developments

- Hardness based on non-standard assumptions
- Integrality gaps for lift-and-project relaxations

Hardness

- [AAMMW '11]
- No constant factor possible if random k-AND hard to refute.

- No constant possible if planted cliques cannot be found in polynomial time.
- Super constant hardness based on stronger assumption.

Stronger relaxations

Lasserre

Sherali-Adams

Lovasz-Schrijver

Gaps for lift-and-project

- [BCCFV '10]
  t rounds of Lovasz-Schrijver: gap $n^{\frac{1}{4}+O(1/t)}$

- [BCV '11]
  $\Omega(\frac{\log n}{\log \log n})$ rounds of Sherali-Adams:
  gap $\tilde{\Omega}(n^{\frac{1}{4}})$

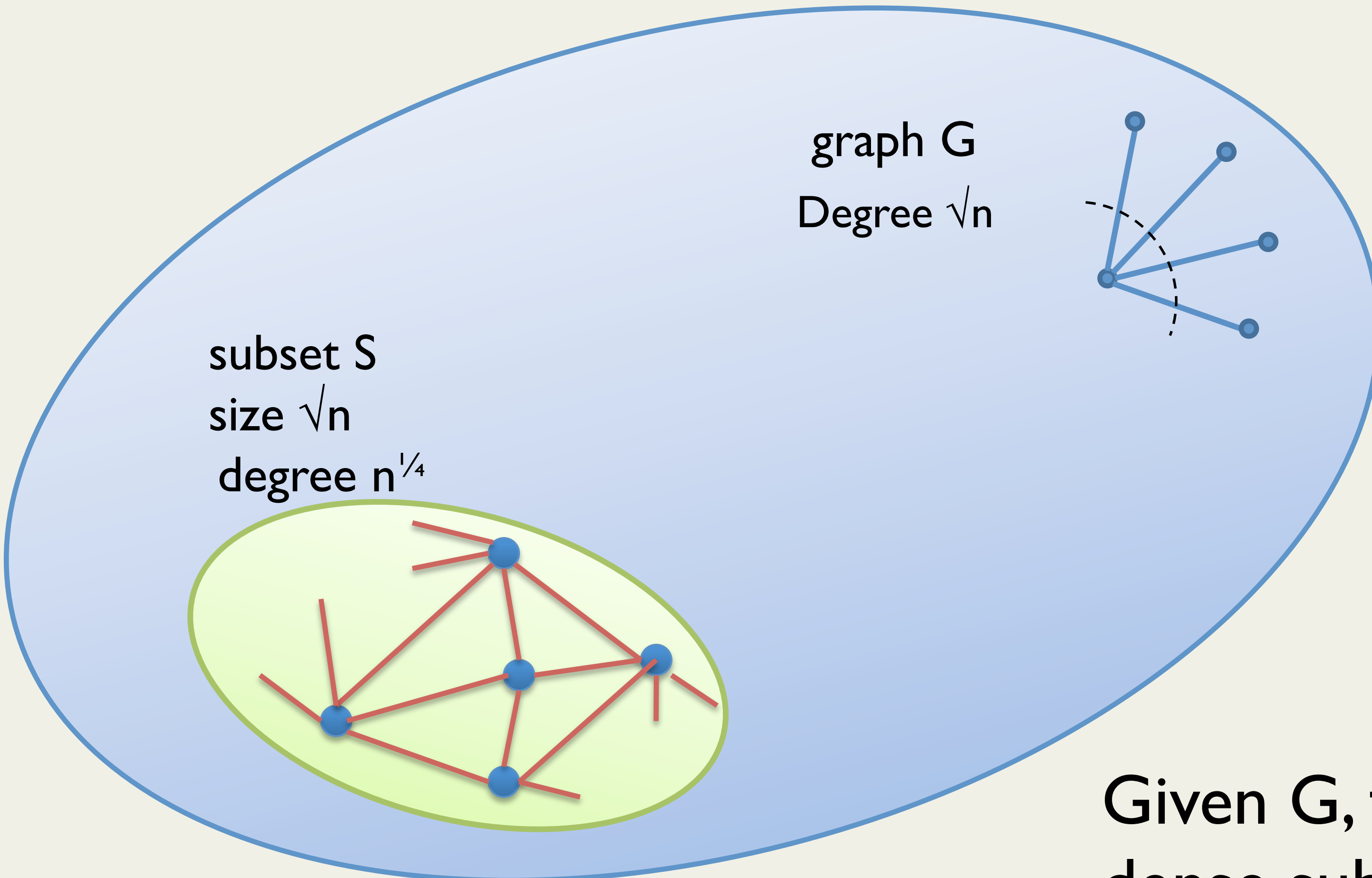
- [GZ '11]
  $n^{\Omega(1)}$ rounds of Lasserre: gap $n^{\Omega(1)}$

Open Problem

- Given random graph: n vertices, degree $n^{1/2}$
- Planted subgraph: $n^{1/2}$ vertices, degree $n^{1/4-\varepsilon}$

- Detect in polynomial time ?

Open Problem



Given G, find dense subgraph S