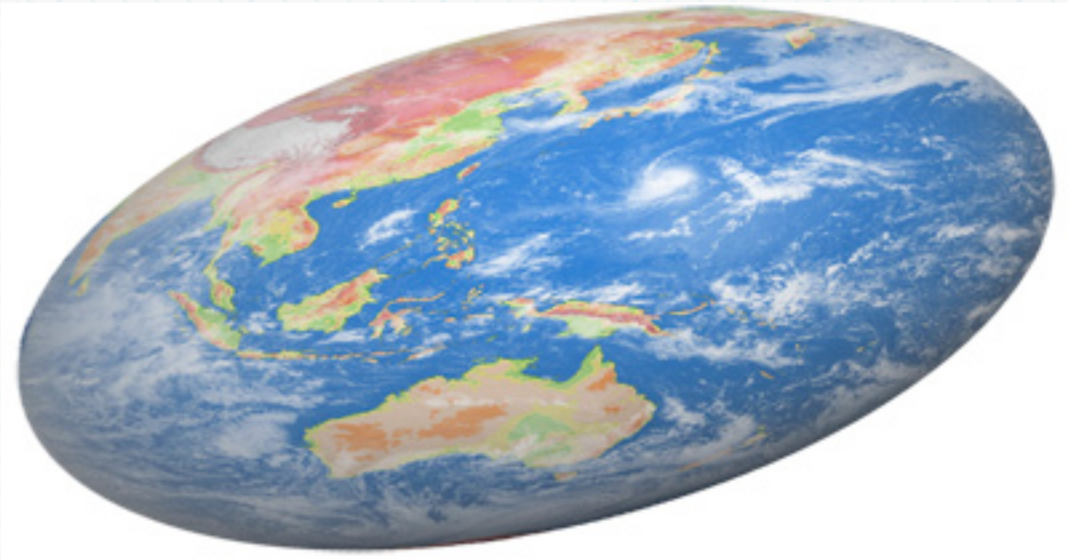
Approximation Schemes for Optimization Problems in Planar Graphs

Philip Klein

BROWN

The world is flat....



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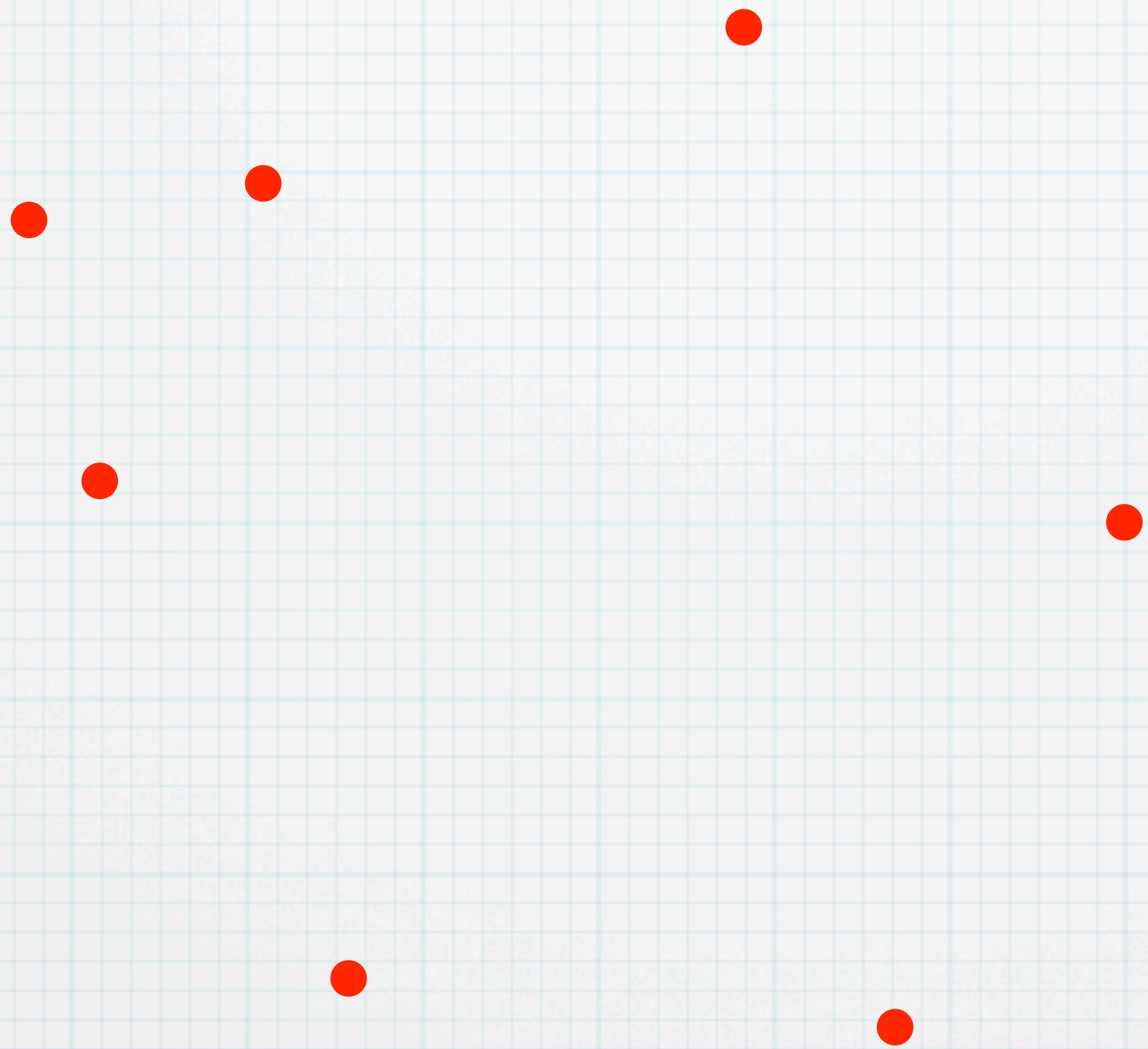

Traveling-salesman tour in the plane

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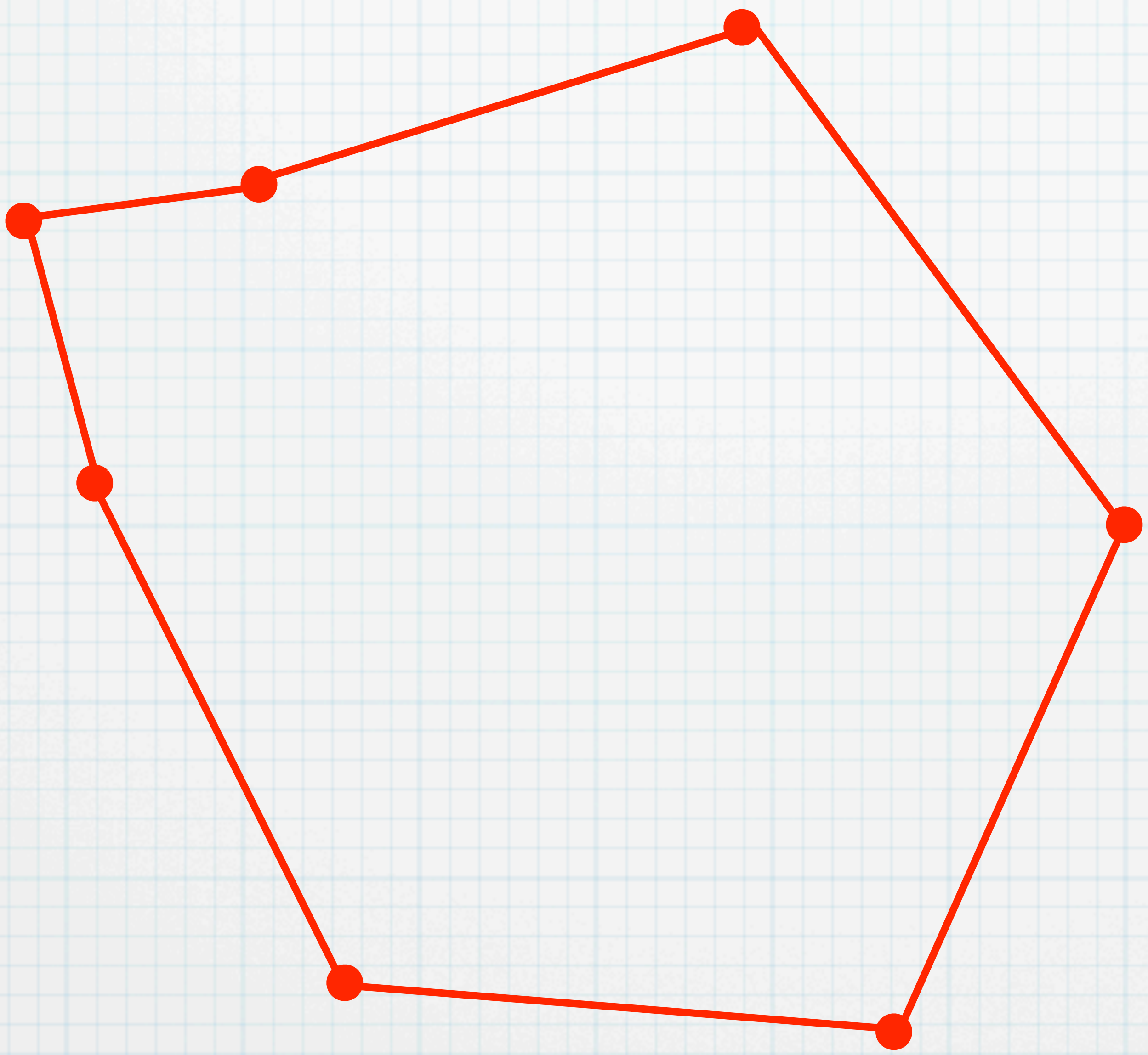

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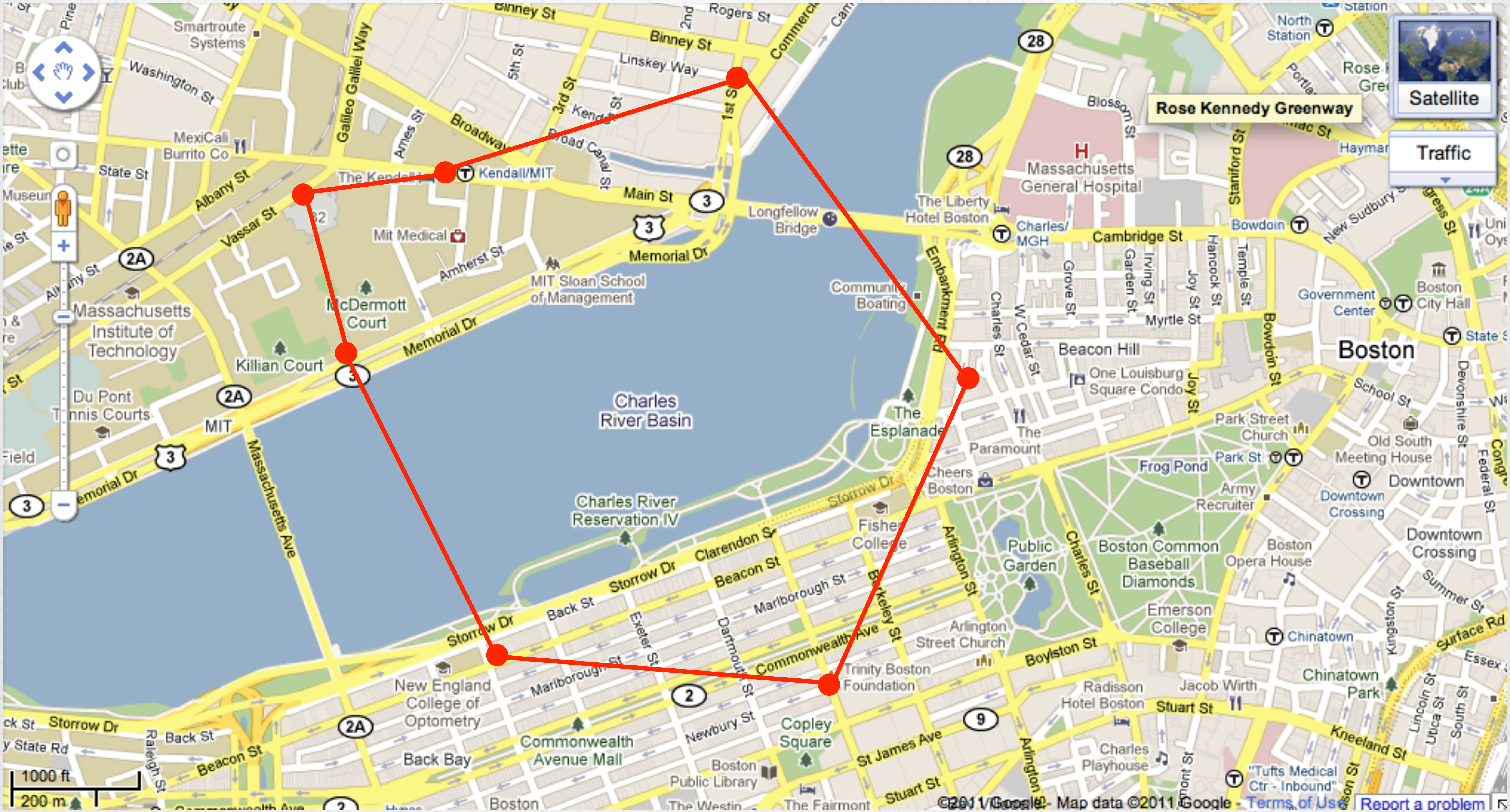

Traveling-salesman tour in ~~the plane~~ a planar embedded graph

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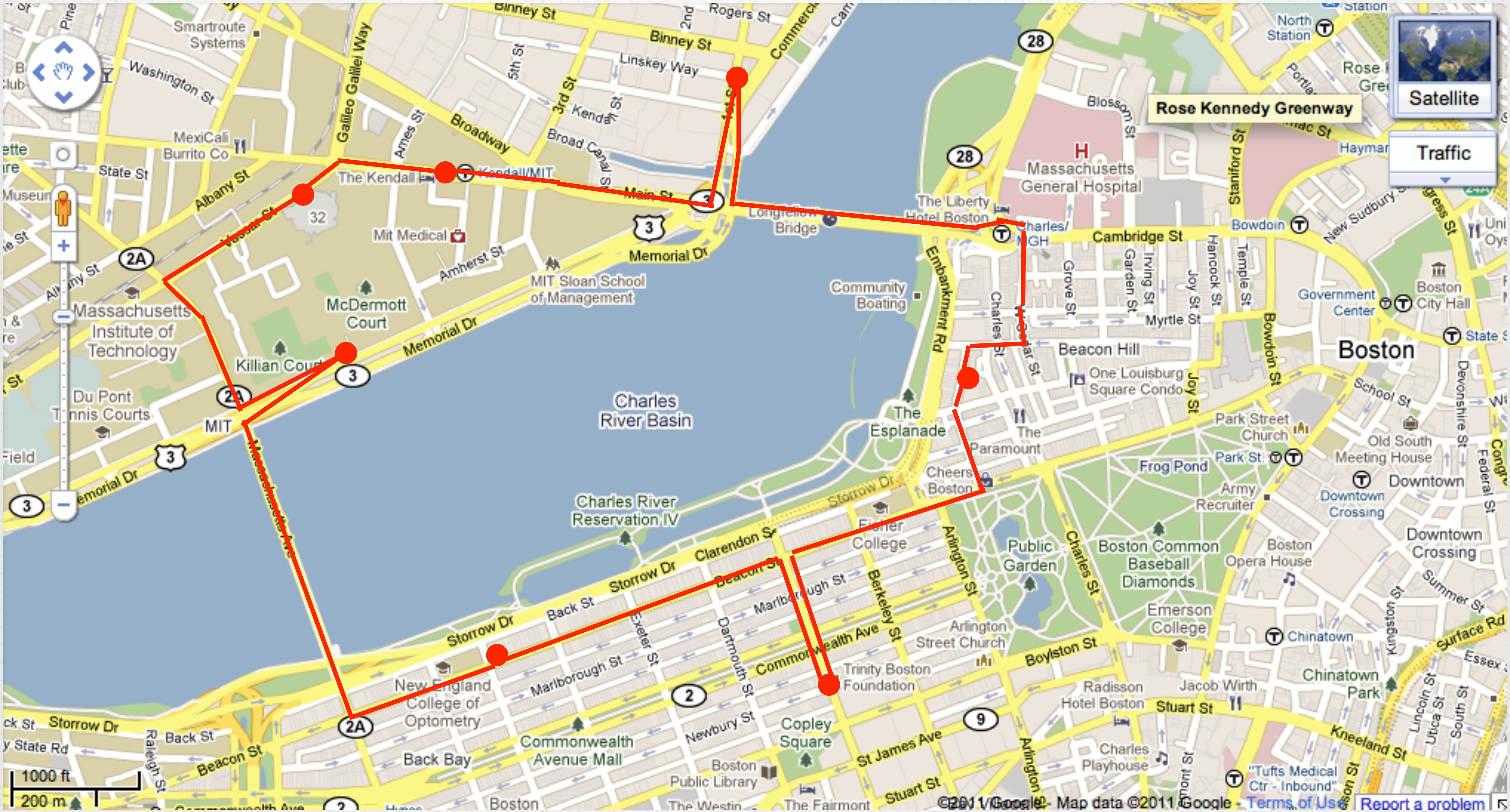
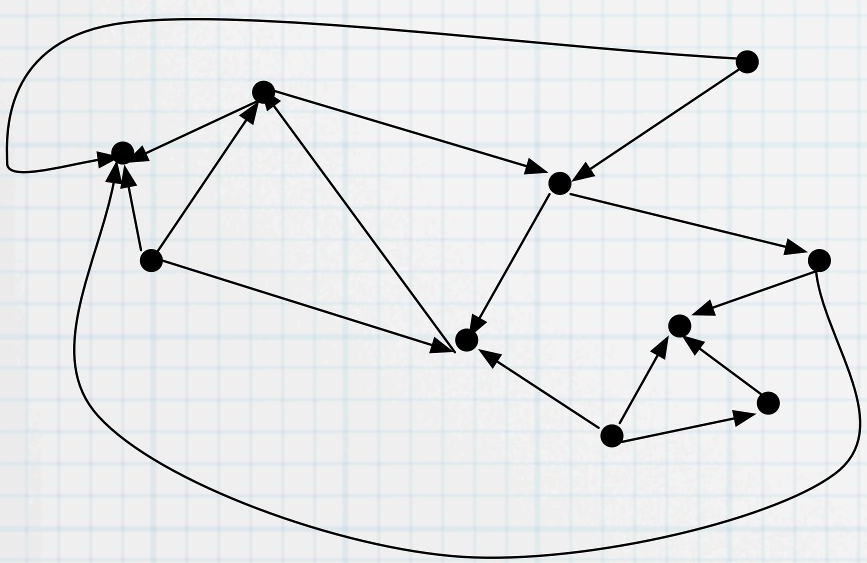
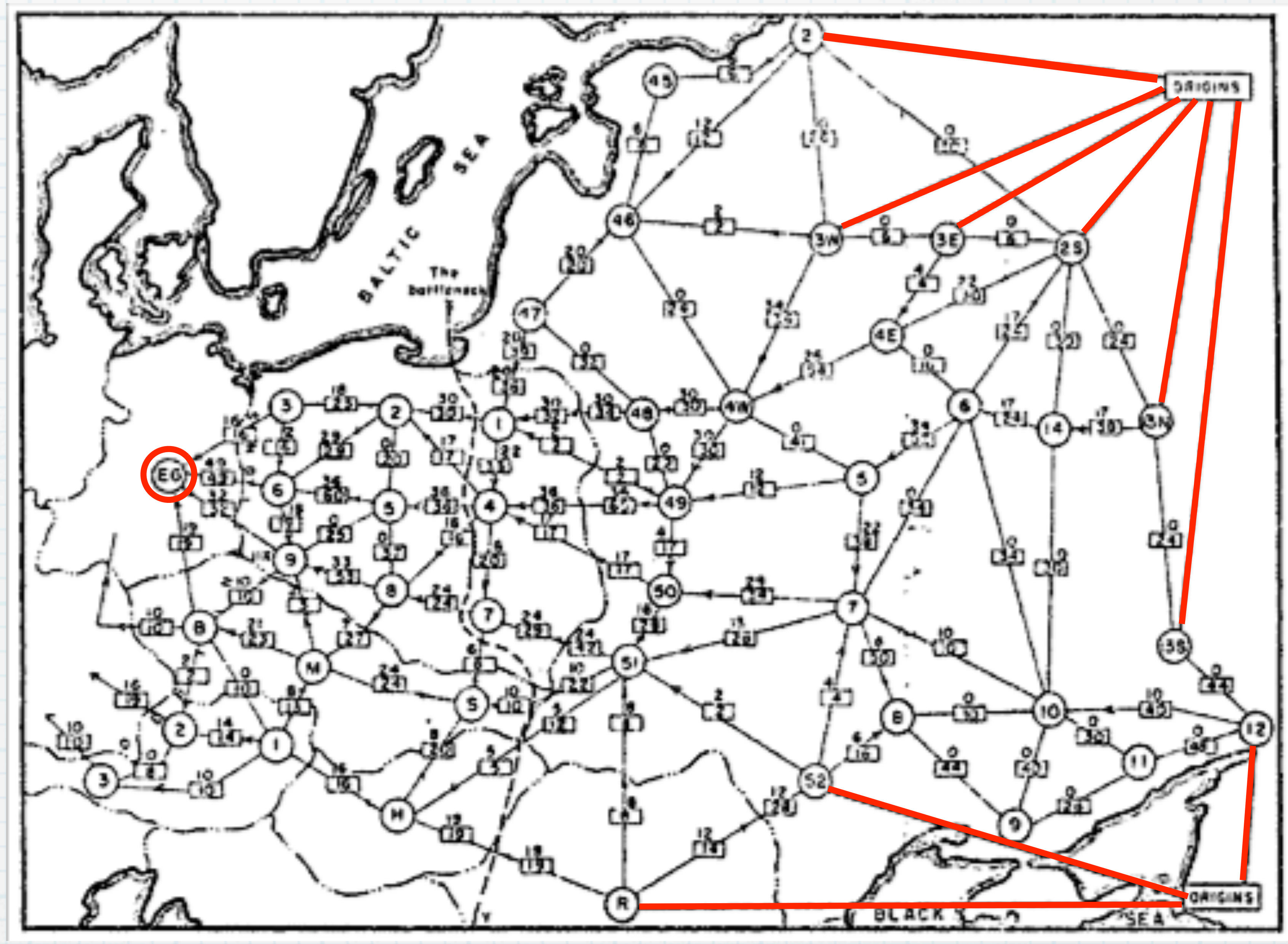


Traveling-salesman tour in ~~the plane~~ a planar embedded graph

Planar graphs

Can be drawn in the plane with no crossings





[Harris and Ross, The RAND Corporation, 1955, declassified 1999]

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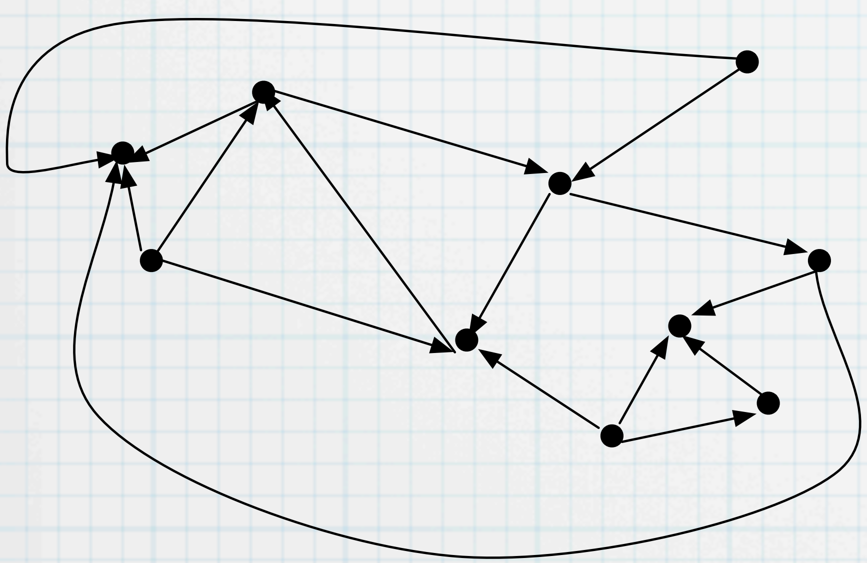

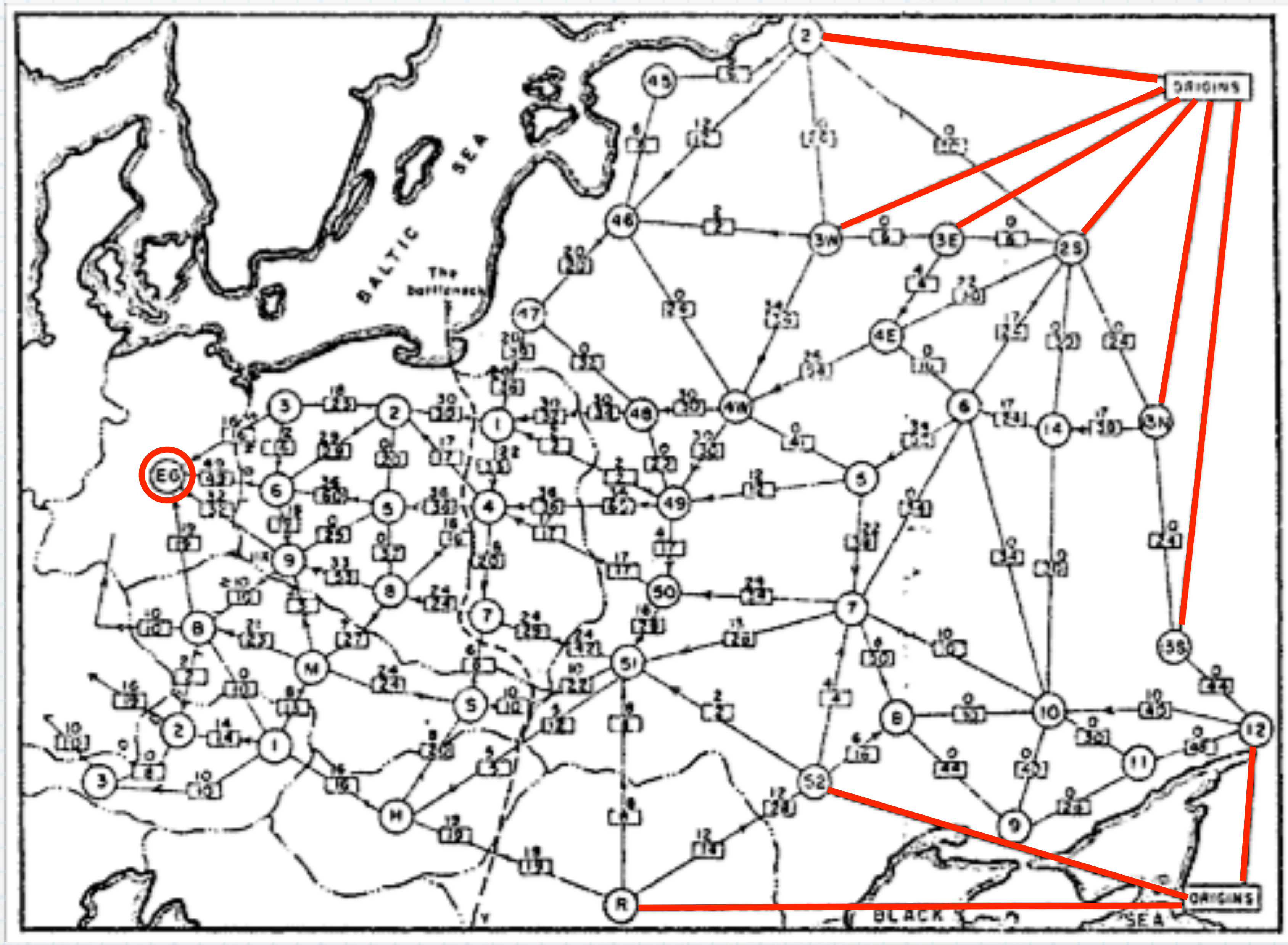

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Research Goal:

Exploiting planarity to achieve

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Faster algorithms
- Shortest paths
- Maximum flow

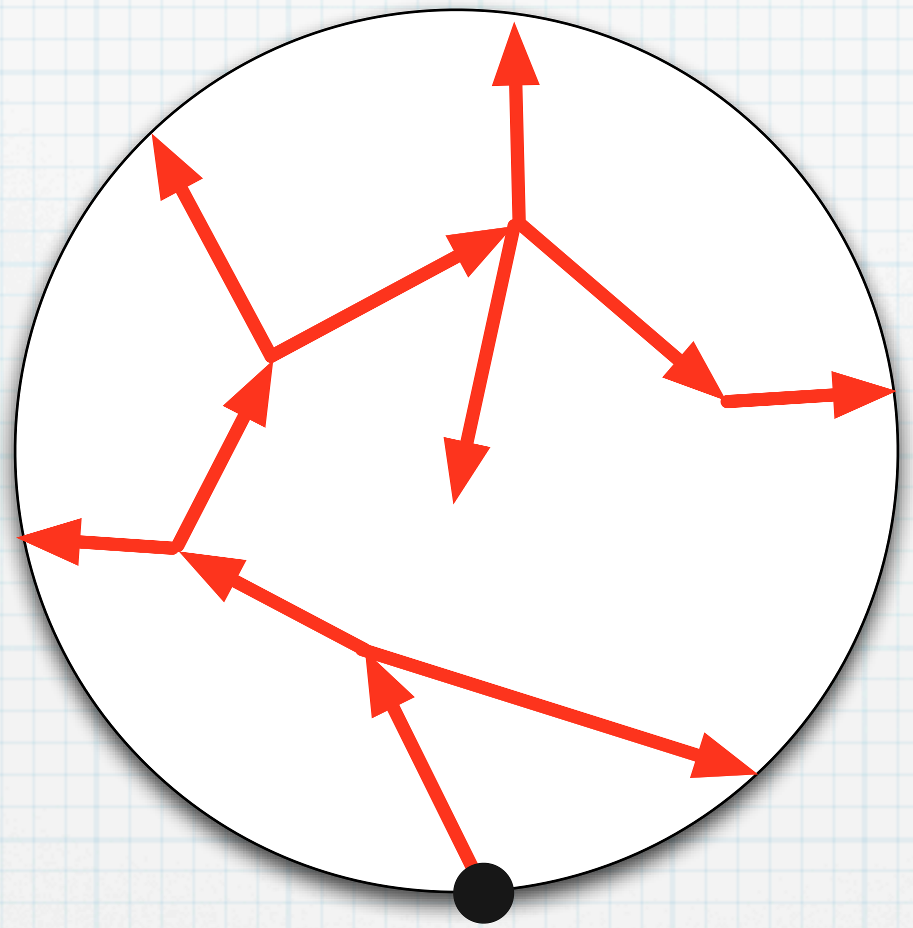
⋮

More accurate approximations
- Traveling salesman
- Steiner tree
- Multiterminal cut

⋮

Combining the two thrusts, get fast and accurate approximation algorithms.

Example of faster algorithm:
Multiple-source shortest paths (MSSP)



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Total time required: $O(n \log n)$

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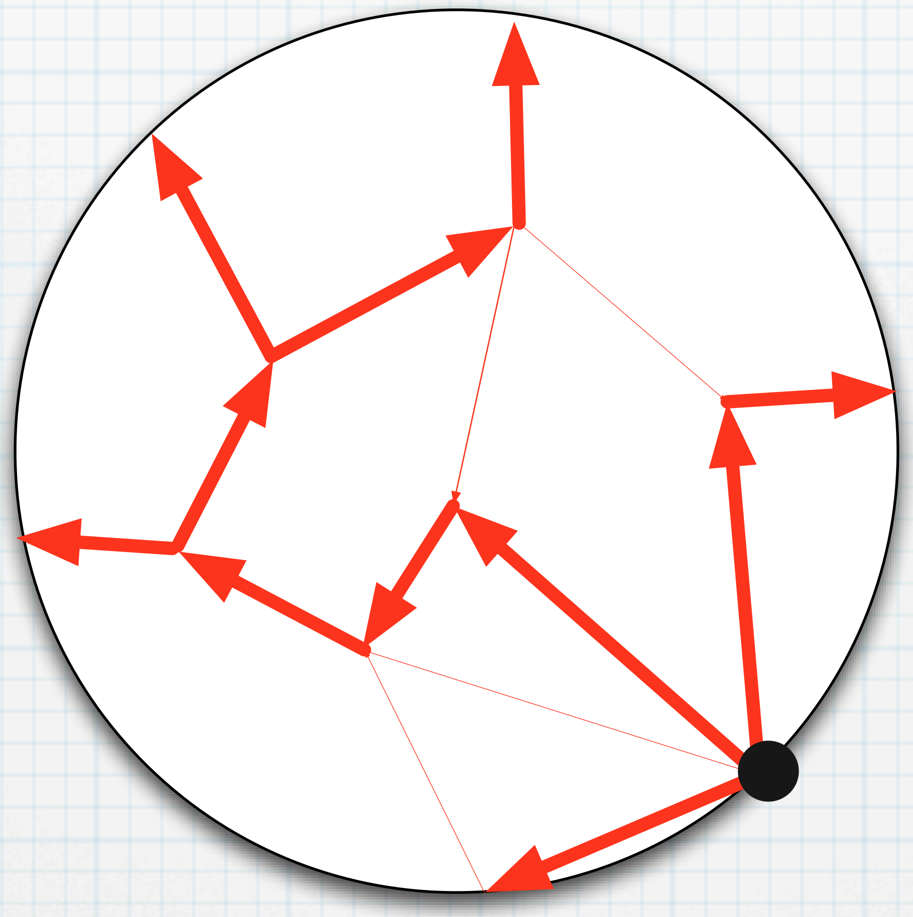

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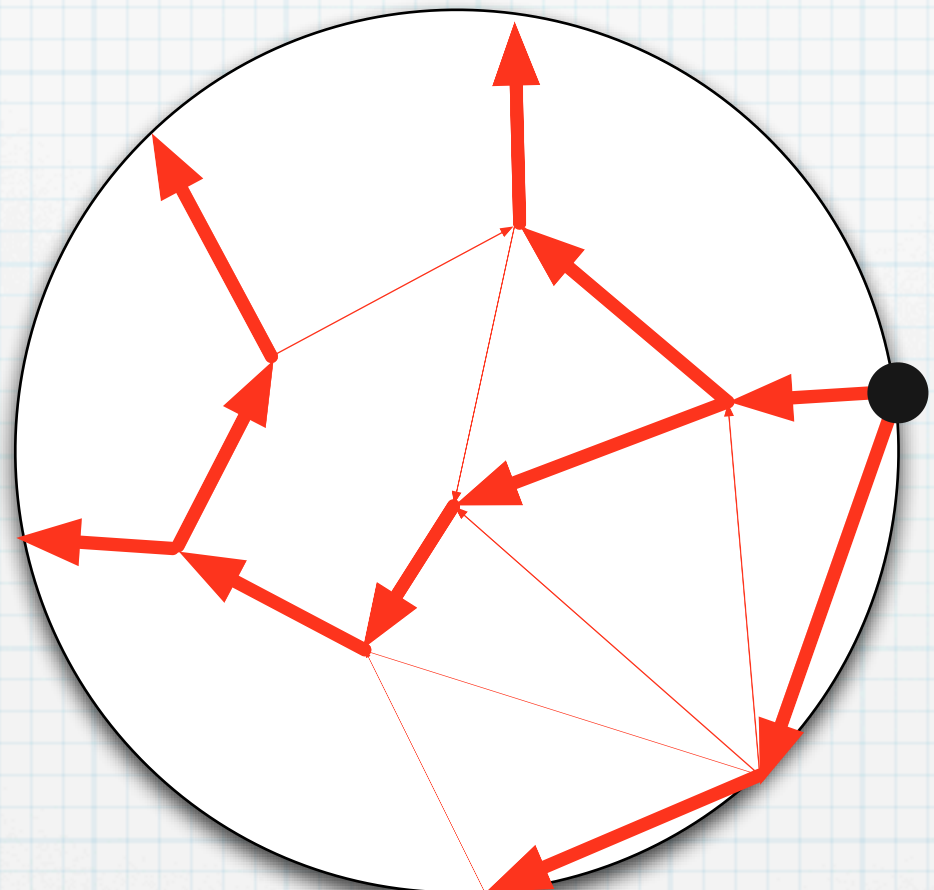

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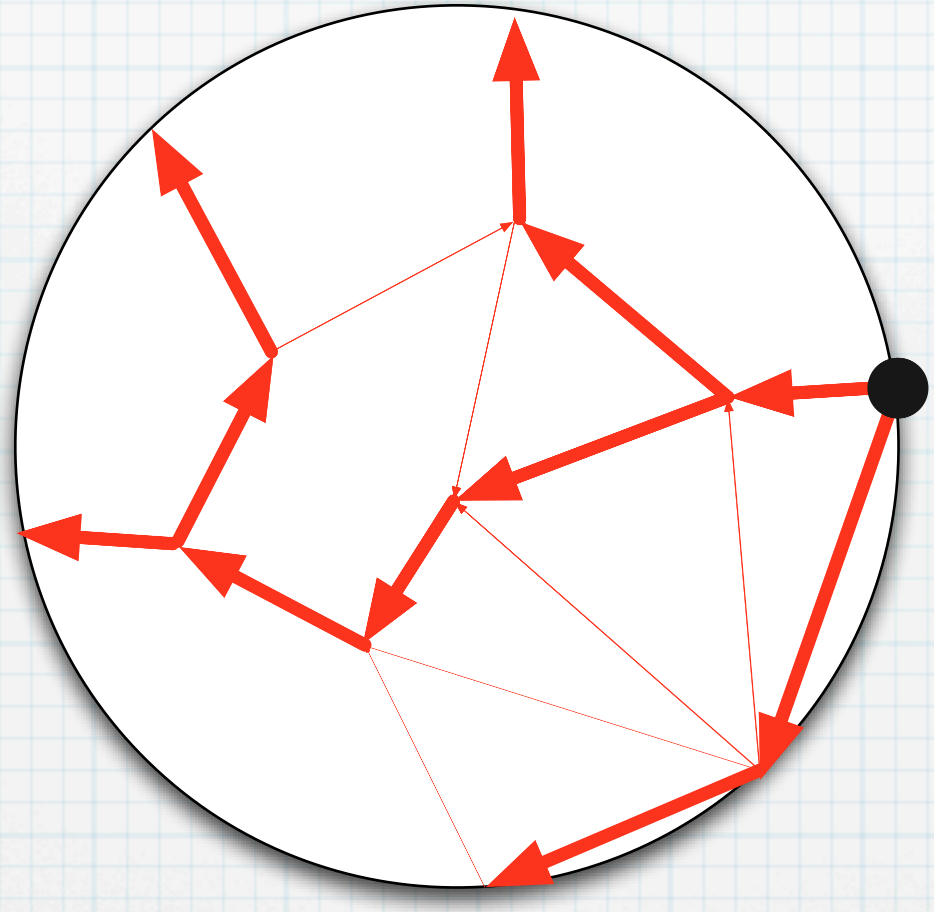

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This algorithm has turned out to have many uses----including the approximation algorithms we will discuss.

Approximation schemes for NP-hard optimization problems in planar graphs:
Greatest hits of the 70's, 80's, and 90's

1977	Lipton, Tarjan	maximum independent set	$O(n \log n)$
1983	Baker	max independent set, partition into triangles, min vertex-cover, min dominating set....	$O(n)$
1995	Grigni, Koutsoupias, Papadimitriou	Traveling salesman in unweighted graphs	$n^{O(1/\varepsilon)}$
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For the 00's: give *efficient* approximation scheme for TSP, address greater variety of traditional optimization problems.

Question: Is there an *efficient* approximation scheme for traveling salesman?

Theorem [Klein, 2005]: There is a linear-time approximation scheme for the traveling-salesman problem in planar graphs with weights

The framework introduced by this paper has since been used to address many other problems....

Use of new framework for approximation schemes for planar graphs

- Traveling salesman [Klein, 2005]
- Traveling salesman on subset of vertices [Klein, 2006]
- 2-edge-connected spanning subgraph
  [Berger, Grigni, 2007]
- Steiner tree [Borradaile, Klein, Mathieu, 2008]
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Framework generalized to broader graph classes

- Steiner tree in bounded-genus graphs [Borradaile, Demaine, Tazari, 2009]
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• Prize-collecting Steiner tree, TSP, stroll [Bateni, Chekuri, Ene, Hajiaghayi, Korula, Marx, 2011]	$O(n^c)$
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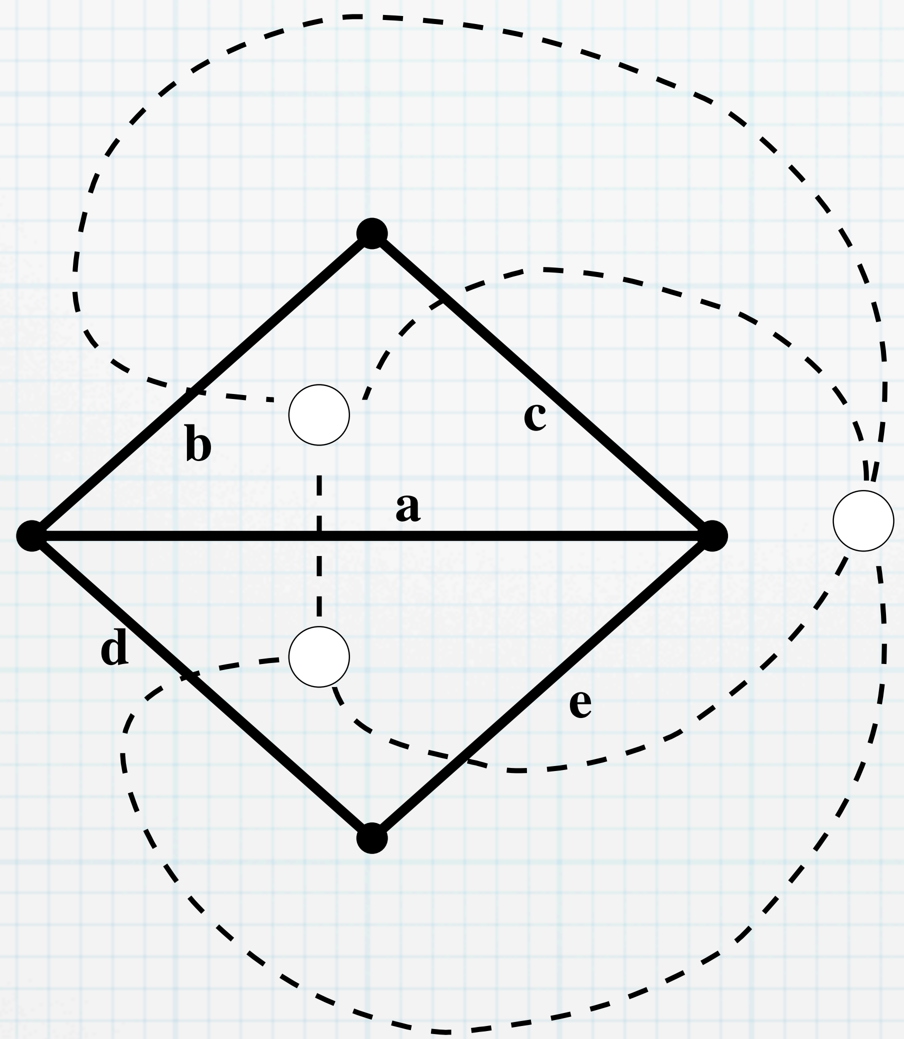
Planar duality

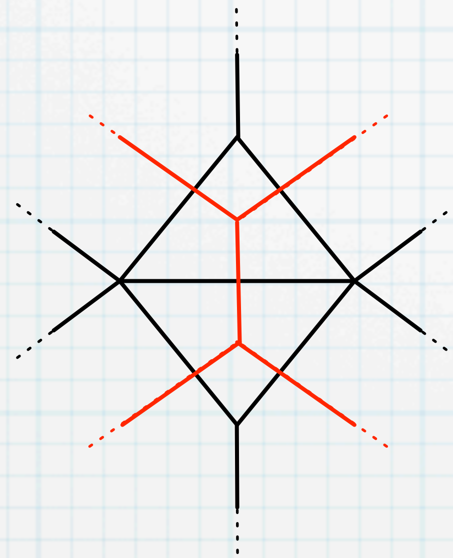


For each connected planar embedded graph, the *dual* is another connected planar embedded graph:

- Dual has a vertex for each face of the *primal* (the original graph)
- Dual has an edge for each edge of the primal.

One key idea for framework

Deletion and contraction* are dual to each other



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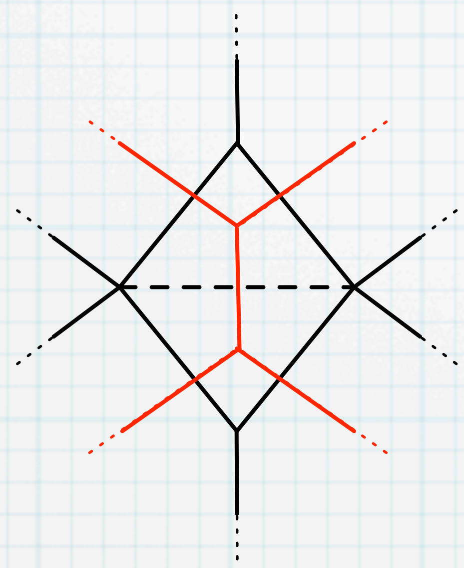

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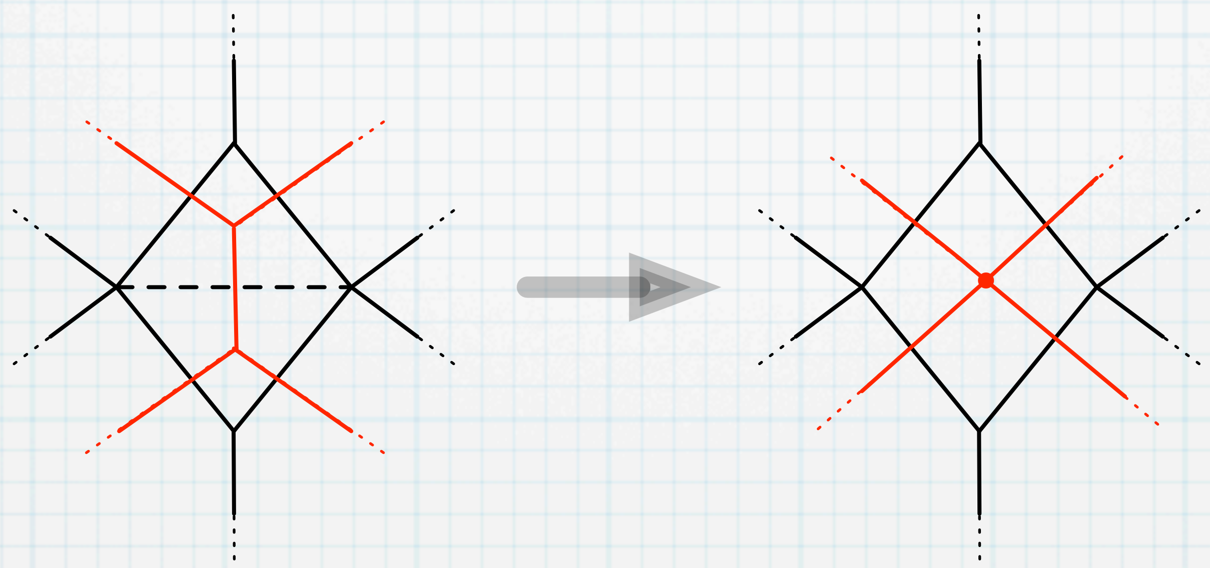

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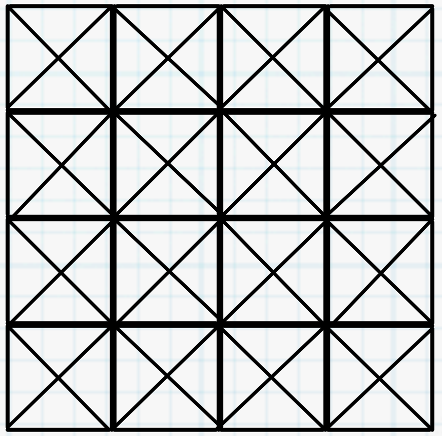

Ensure total cost of resulting graph is *O(OPT)*

2. *Contract* edges of total cost at most 1/*p* times total

Ensure resulting graph has branchwidth *O(p)*

3. Find (near-)optimal solution in low-branchwidth graph

4. Lift solution to original graph, increasing cost by $1/p \times O(OPT)$

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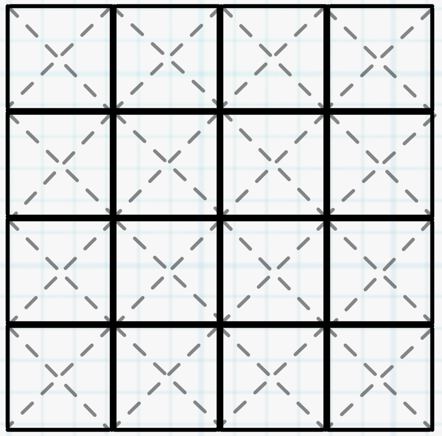

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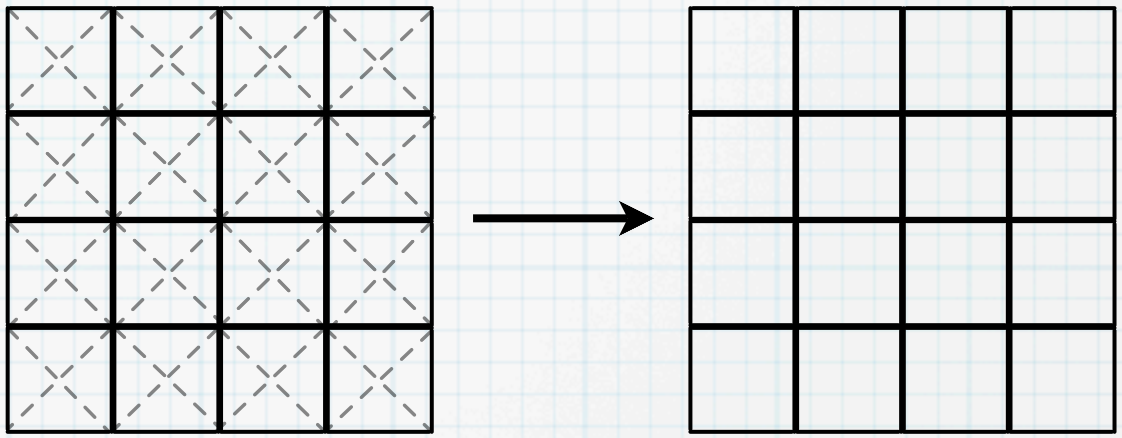

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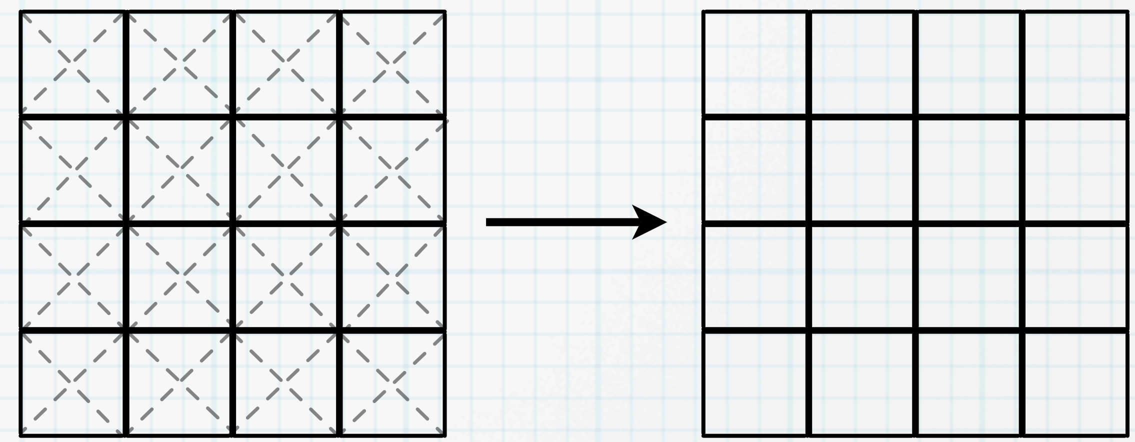

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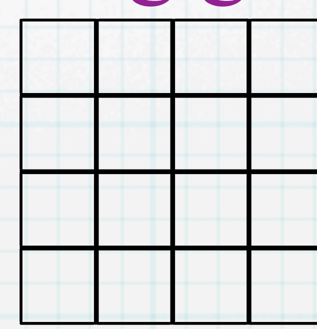

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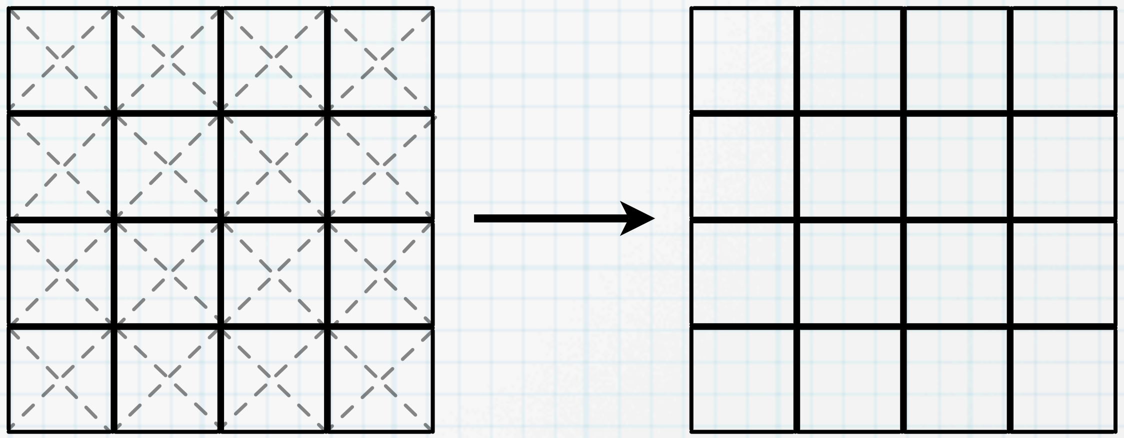

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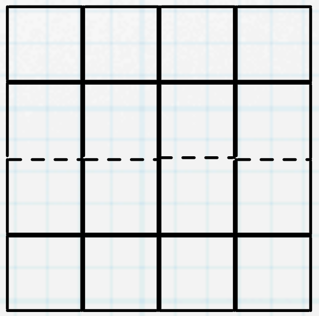

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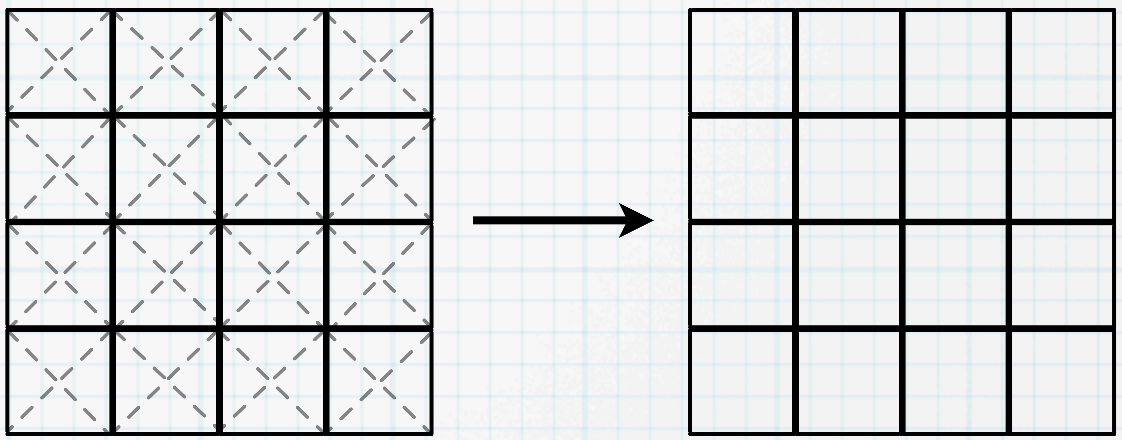

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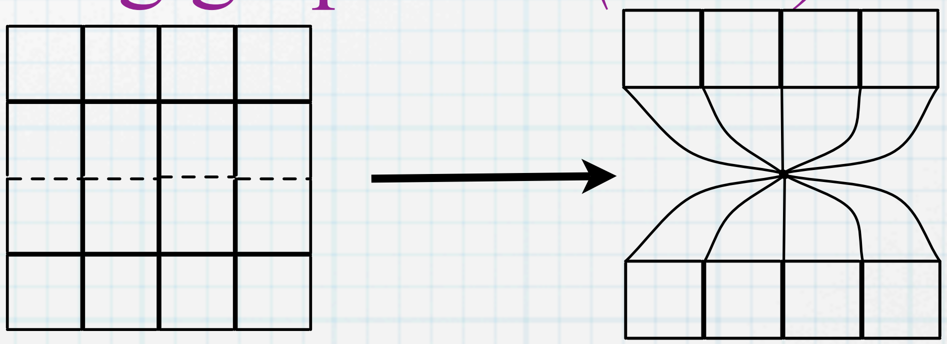

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Choose p big enough so increase is $\leq \varepsilon$ *OPT*

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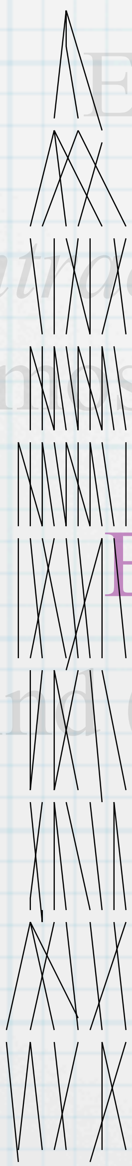

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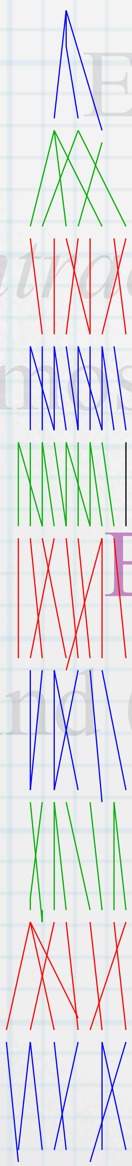

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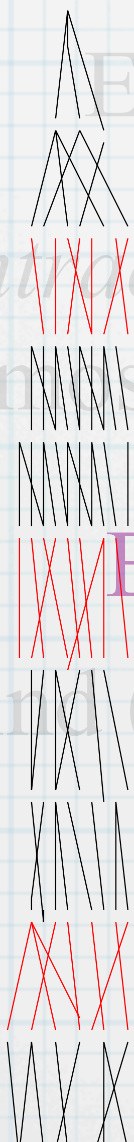

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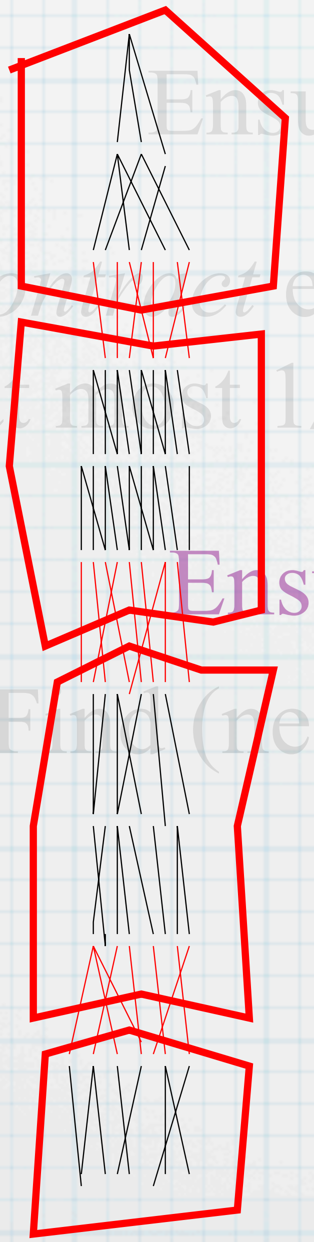

2. *Contract* edges of total cost at most 1/*p* times total

2. *Delete* edges of total cost at most 1/*p* times total

Ensure resulting graph has branchwidth *O(p)*

3. Find (near-)optimal solution in low-branchwidth graph

This was known before, implicit in Baker's work. For planar graphs, do breadth-first search, p-color the levels, and delete the cheapest level.

Framework for approximation schemes for planar graphs [Klein, 2005]

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This is just deleting in the planar dual. (In next talk, same idea in larger graph classes.)

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Key step for most problems: "spanner" construction

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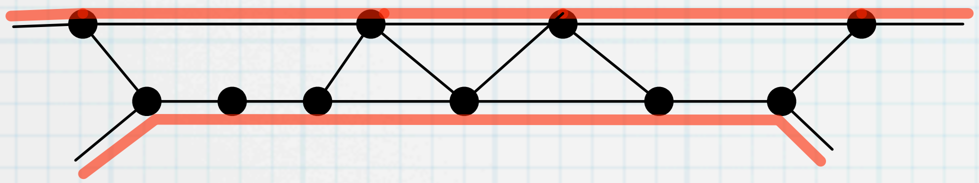
Traveling salesman problem:

How to ensure that the resulting graph approximately preserves OPT?



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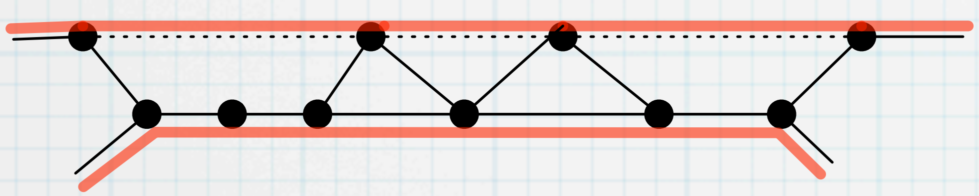

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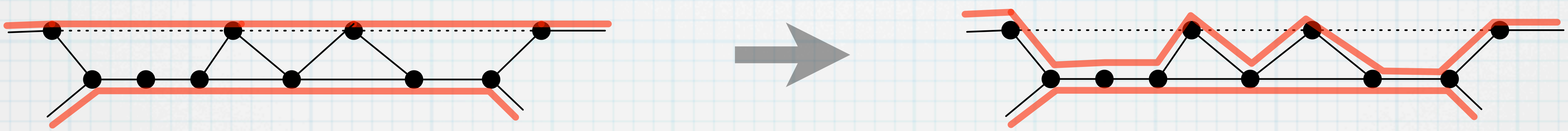

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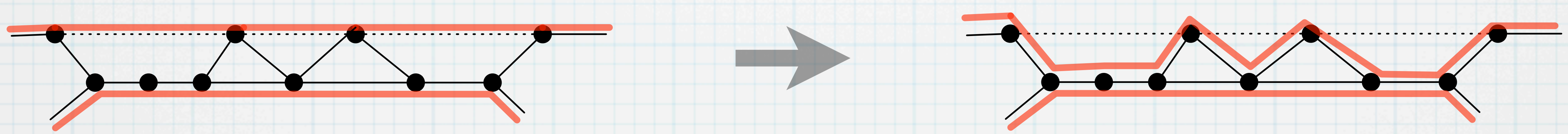

Consider optimal tour. Replace each edge by a 1+ε-approximate shortest path. Resulting tour is 1+ε-approximate.

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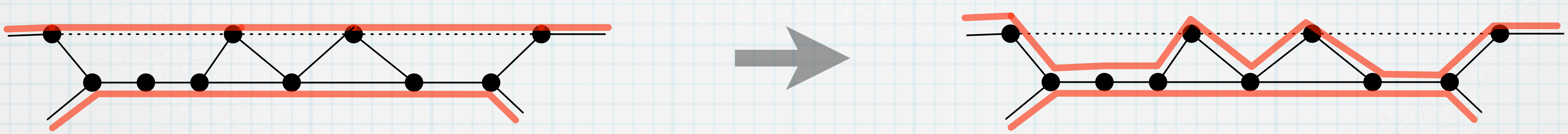


Consider optimal tour. Replace each edge by a 1+ε-approximate shortest path. Resulting tour is 1+ε-approximate.

Therefore: it suffices to select a subset of edges that approximately preserves vertex-to-vertex distances.

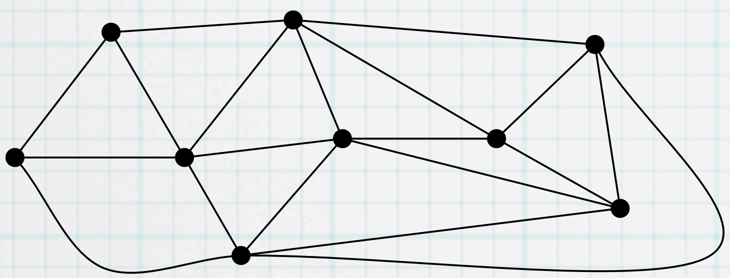
Selecting a low-weight subset of edges that approximately preserves vertex-to-vertex distances

O(n^2) time [Althoffer, Das, Dobkin, Joseph, Soares, 1993], linear time [Klein, 2005]

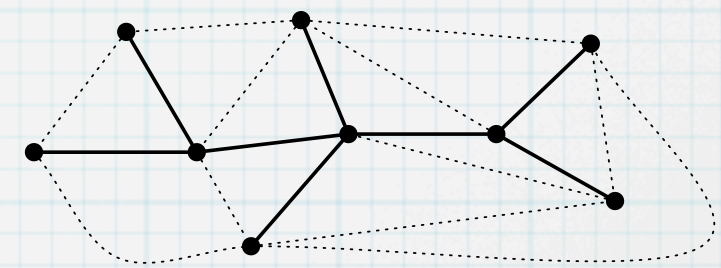
Just achieving finite distances requires a spanning tree.

To keep weight low, start with *minimum-weight* spanning tree (MST).

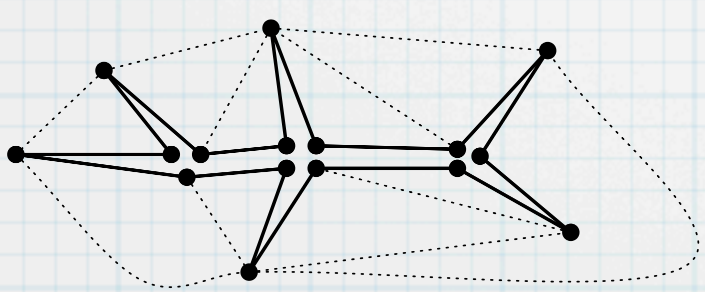
Will choose additional edges of total weight ≤ (2/ε) *weight(MST)*.

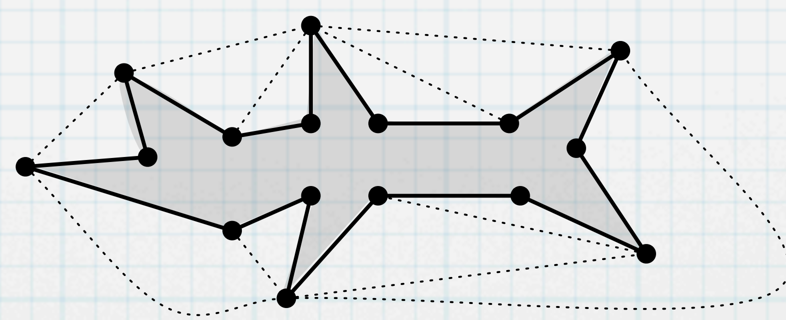


Step 1: Let *T* be the minimum-weight spanning tree. Include it in the spanner.

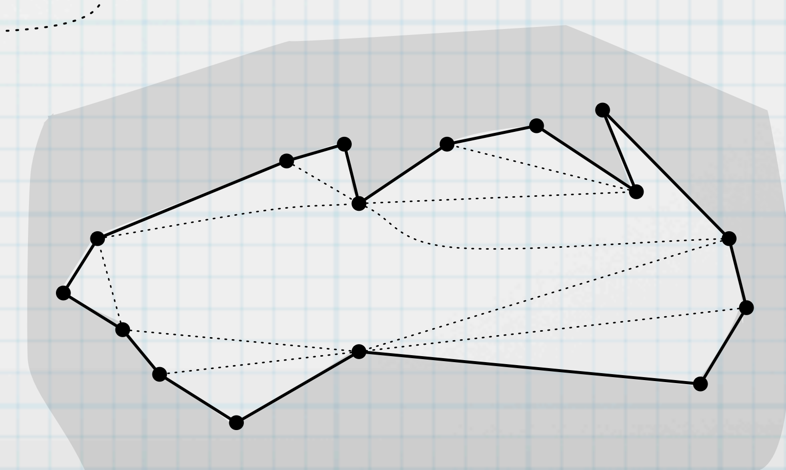


Step 2: Cut along T, duplicating edges and vertices.





Step 3: Consider resulting face as infinite face.



Selecting a low-weight subset of edges that approximately preserves vertex-to-vertex distances

$O(n^2)$ time [Althoffer, Das, Dobkin, Joseph, Soares, 1993], linear time [Klein, 2005]

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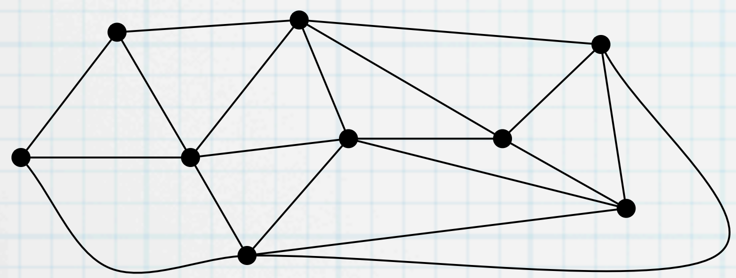

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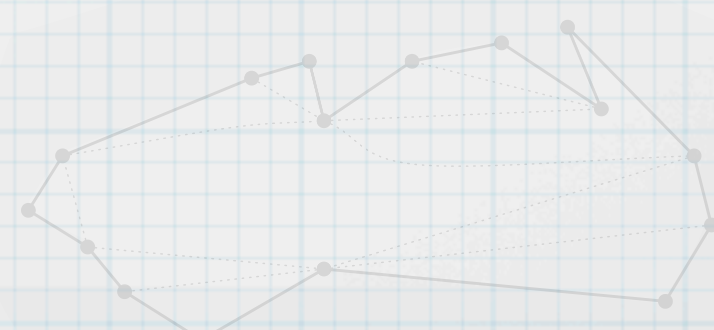

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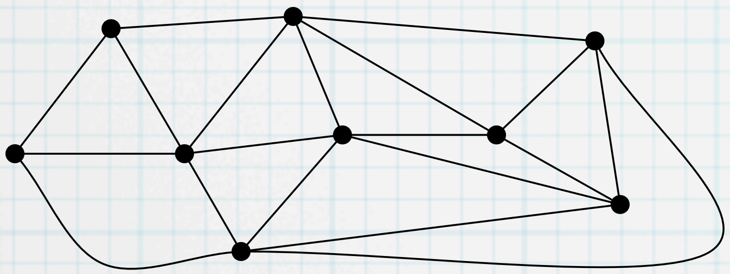

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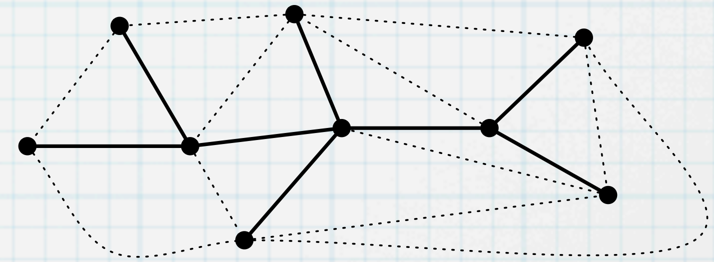

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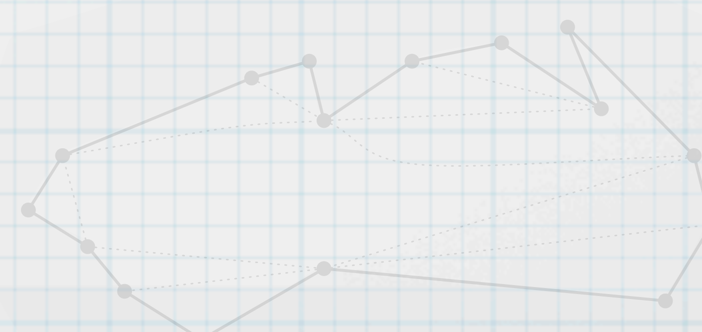

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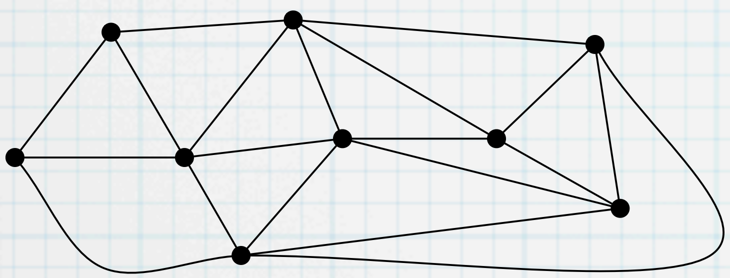

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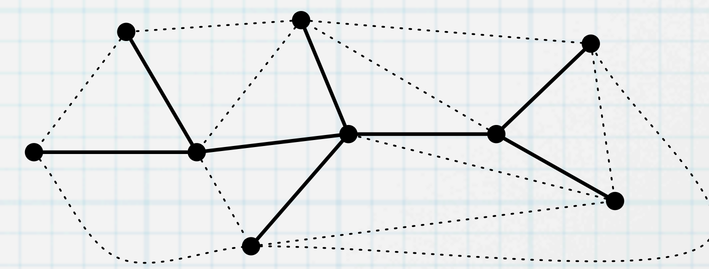

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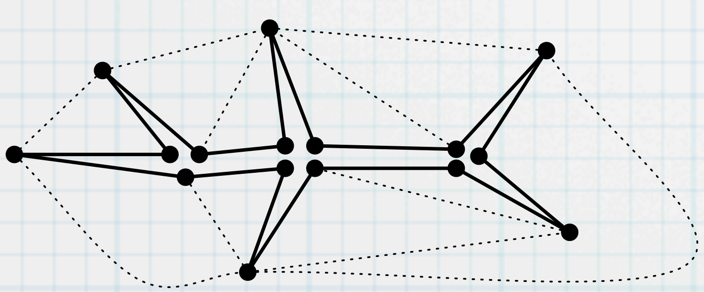

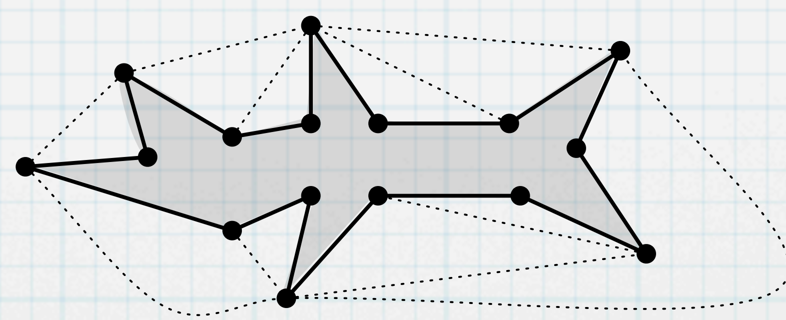

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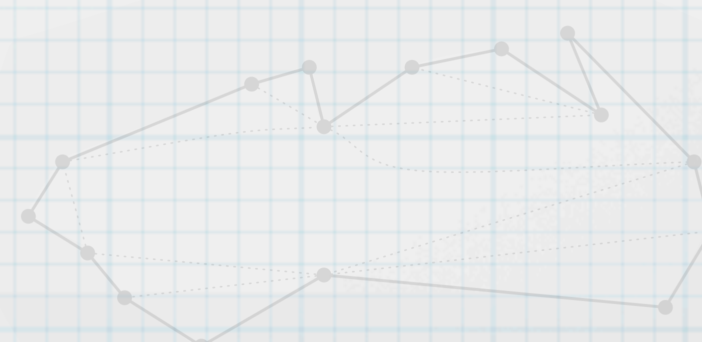

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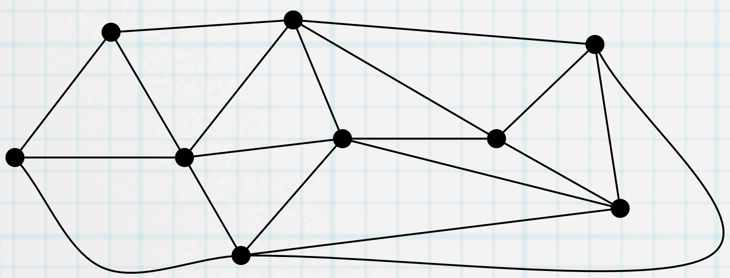

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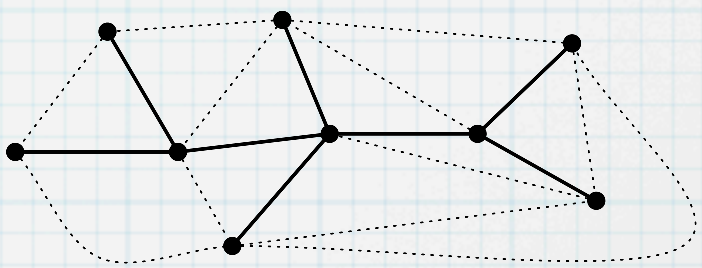

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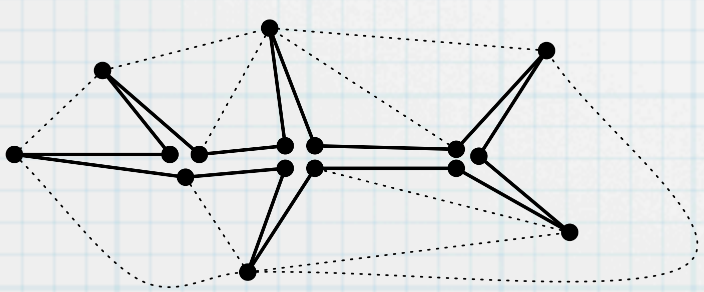

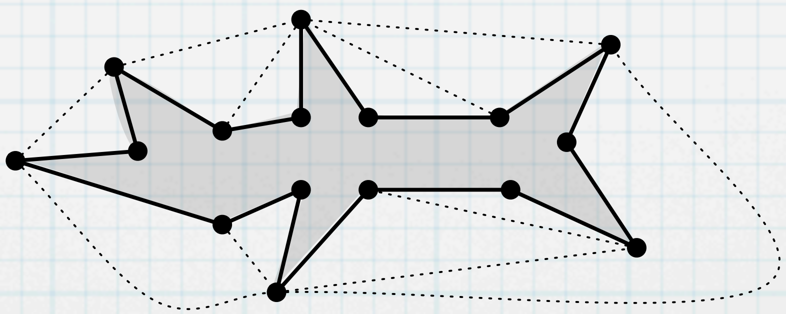

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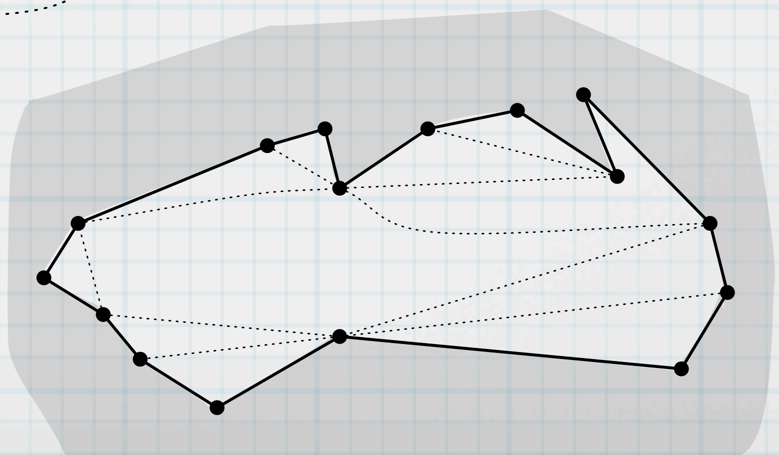
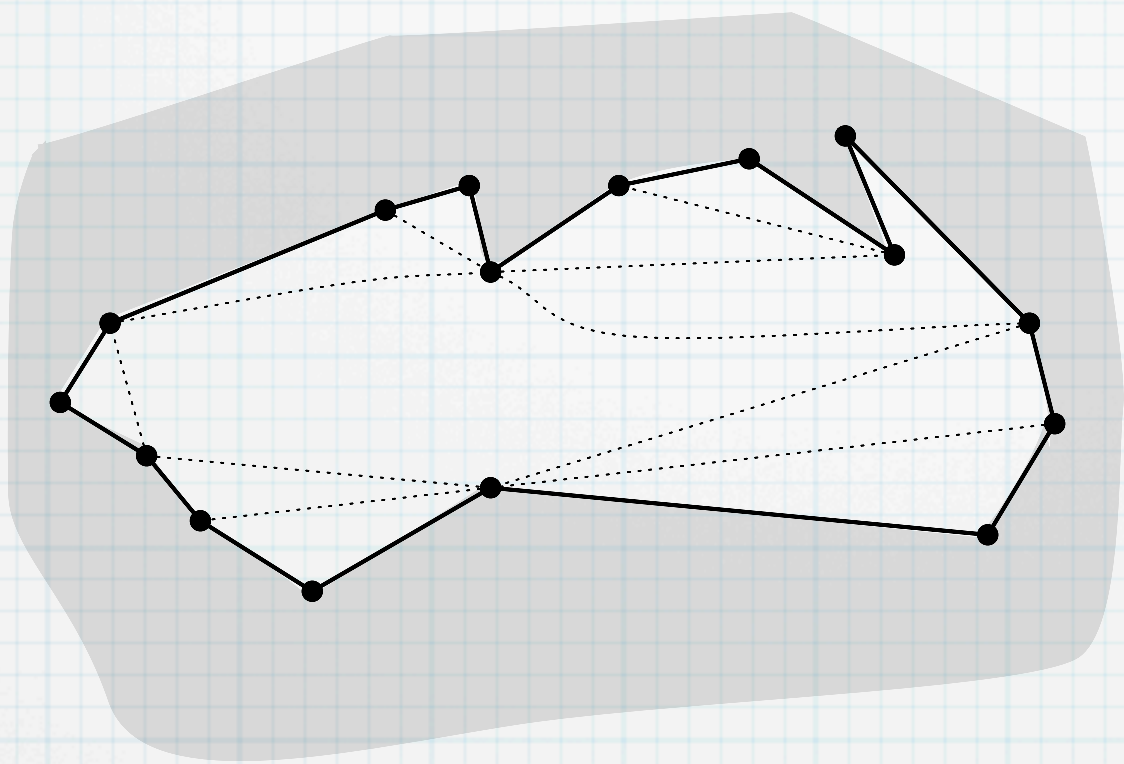


Step 4: Consider nontree edges in order.
For each such edge uv, if
$$(1+\varepsilon)\ \mathrm{weight}(uv) \le \text{weight of corresponding boundary subpath}$$
then add uv to spanner and chop along uv



For each edge uv added to spanner, boundary weight goes down by at least
$$\varepsilon\ \mathit{weight}(uv)$$
Therefore, total weight added to spanner is at most
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Initial boundary weight is 2 *weight(MST)*, so total weight added to spanner is
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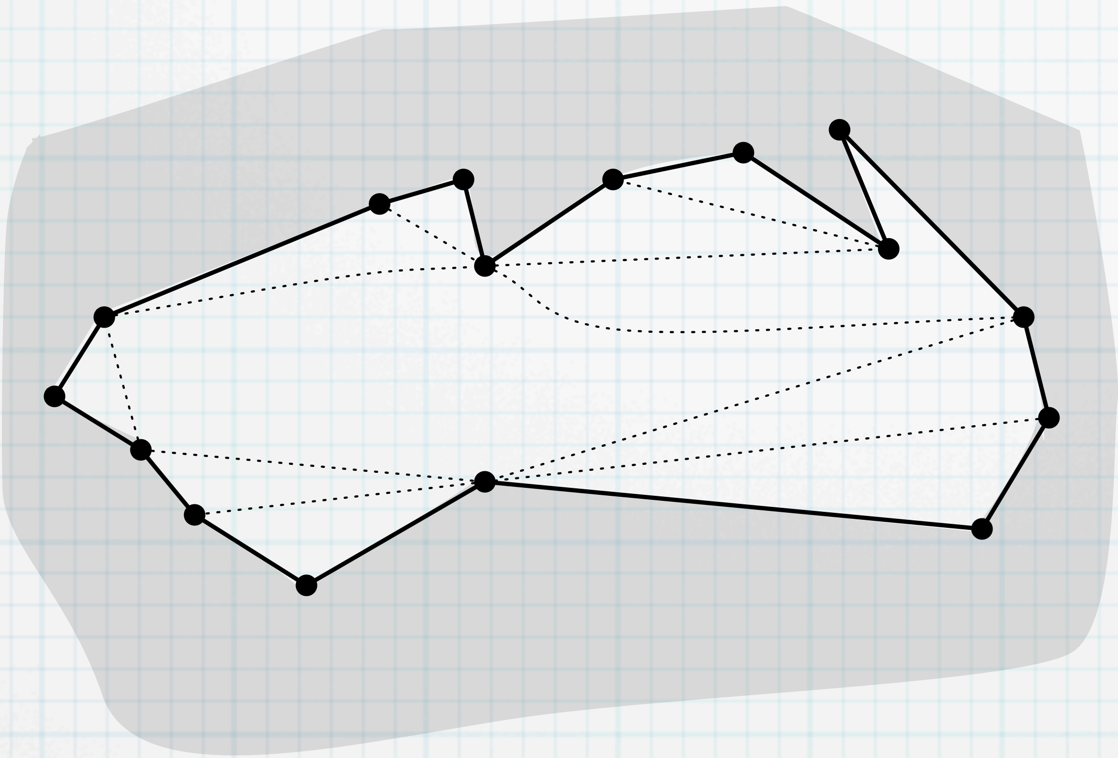

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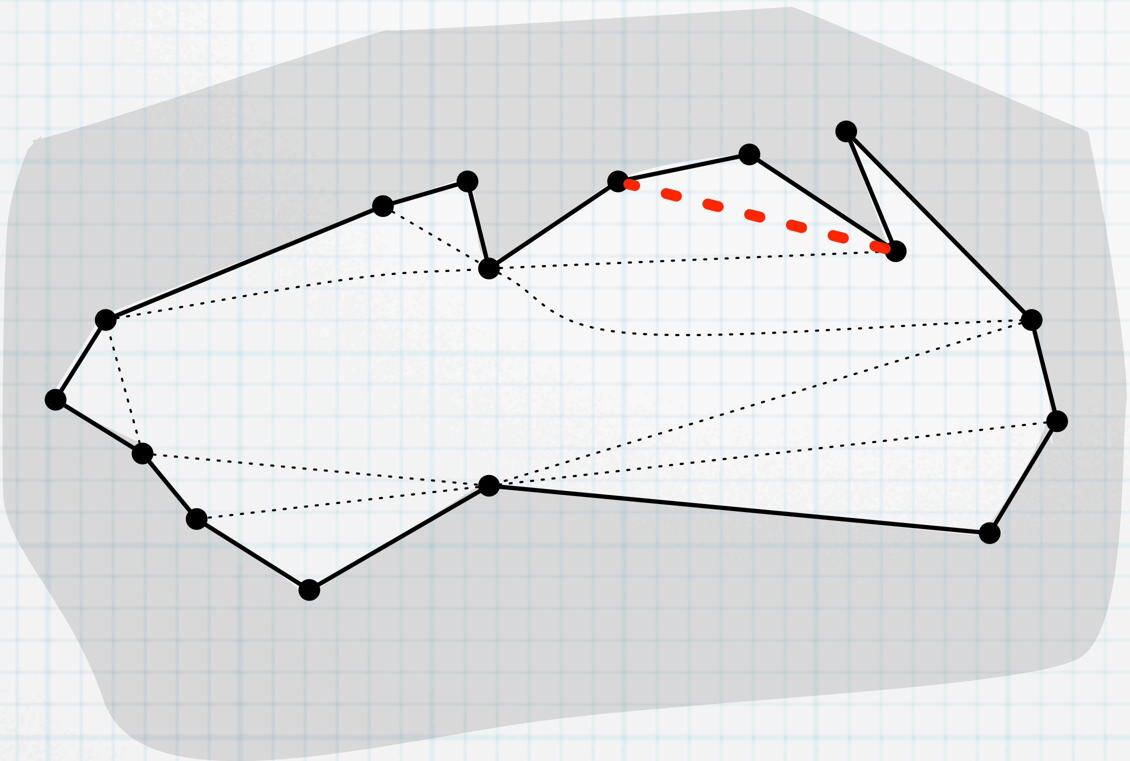

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Therefore, total weight added to spanner is at most *ε^{-1} · decrease in boundary weight*

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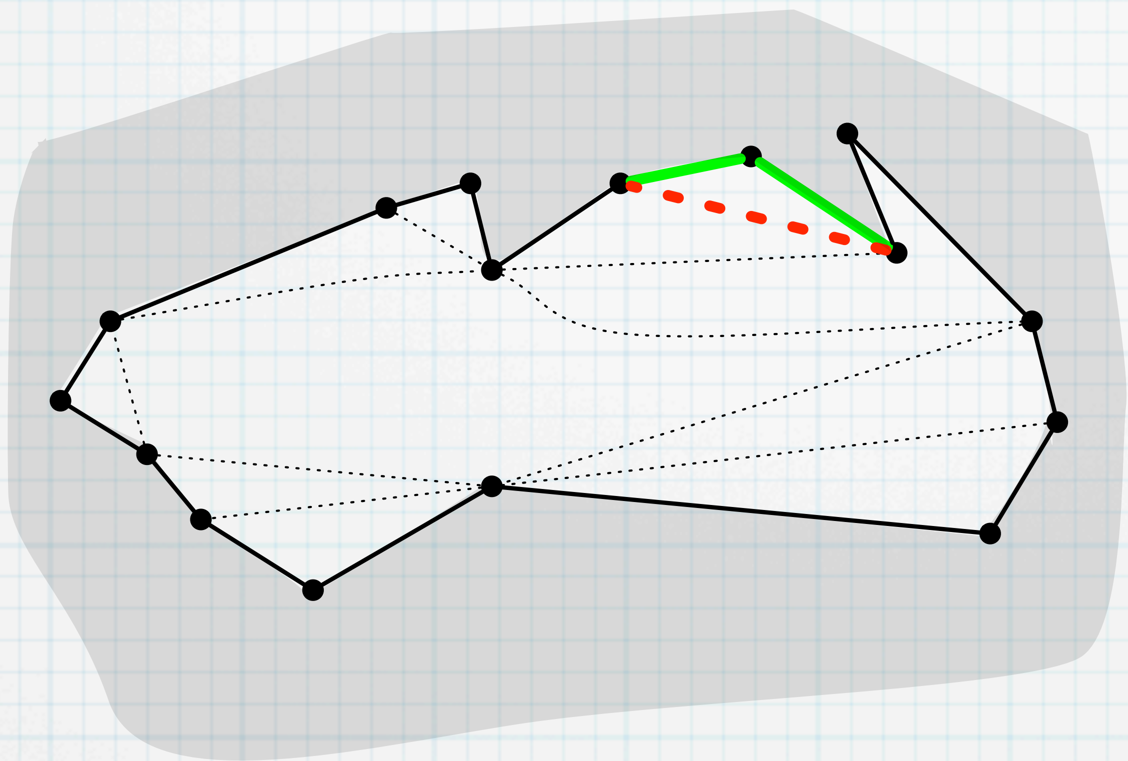

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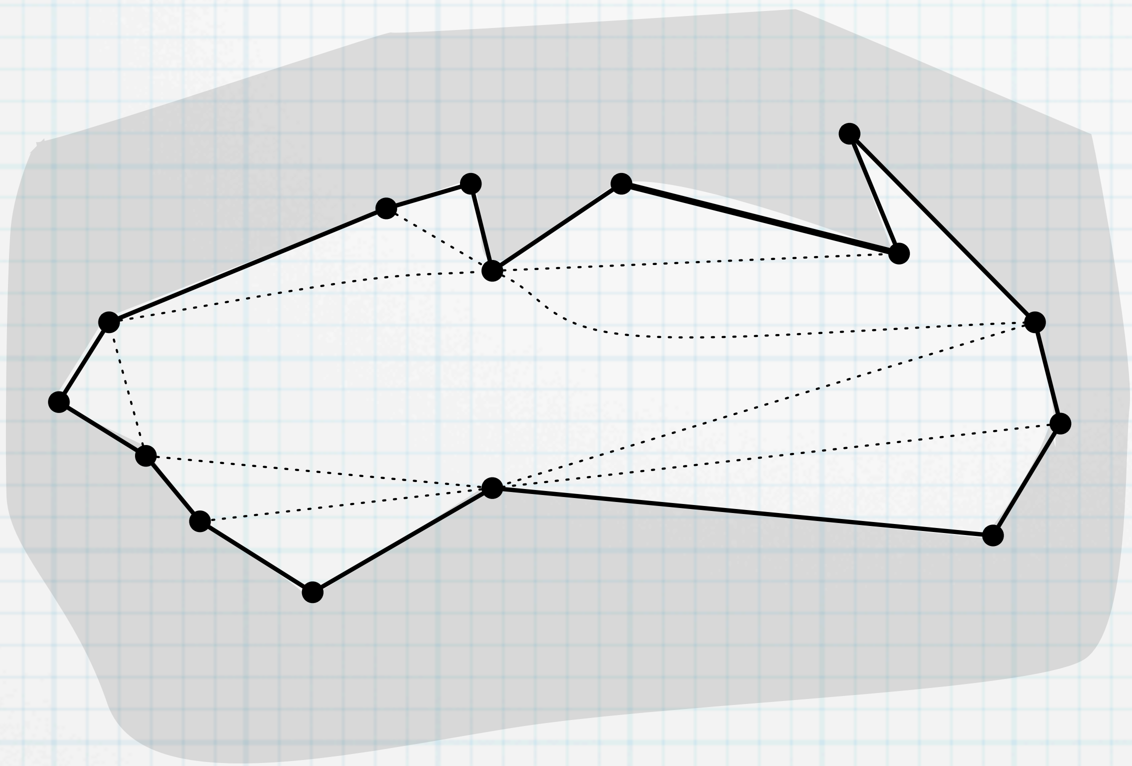

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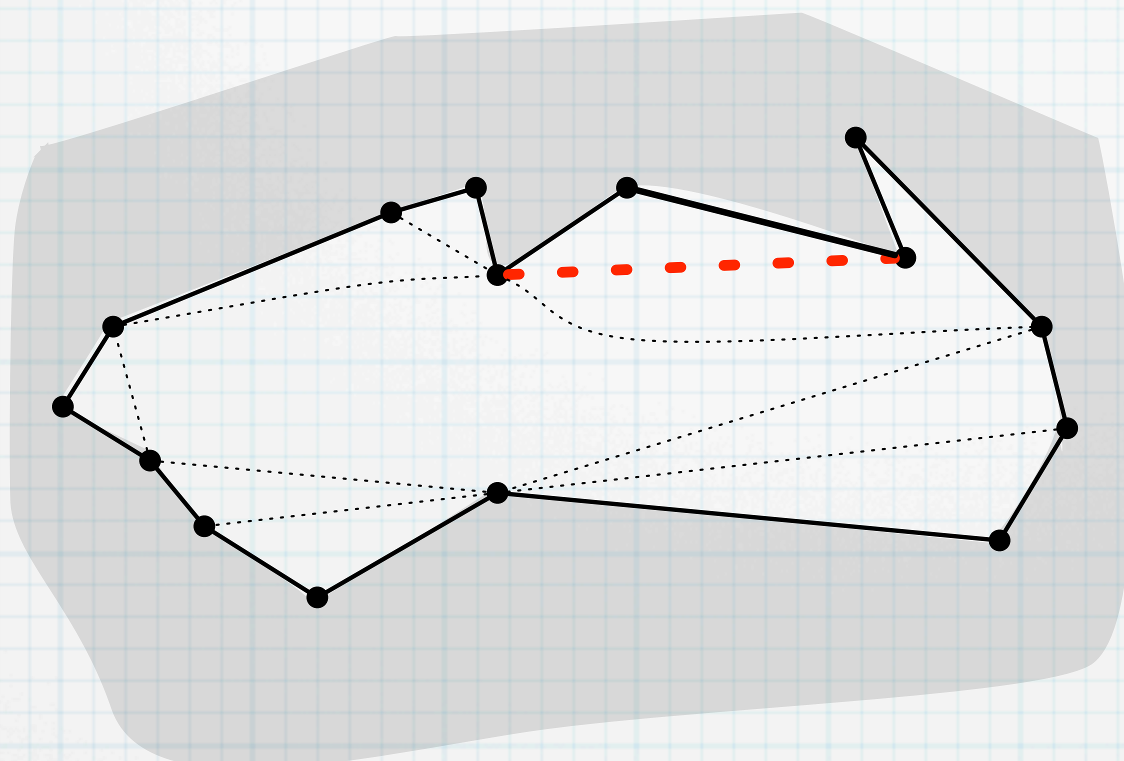

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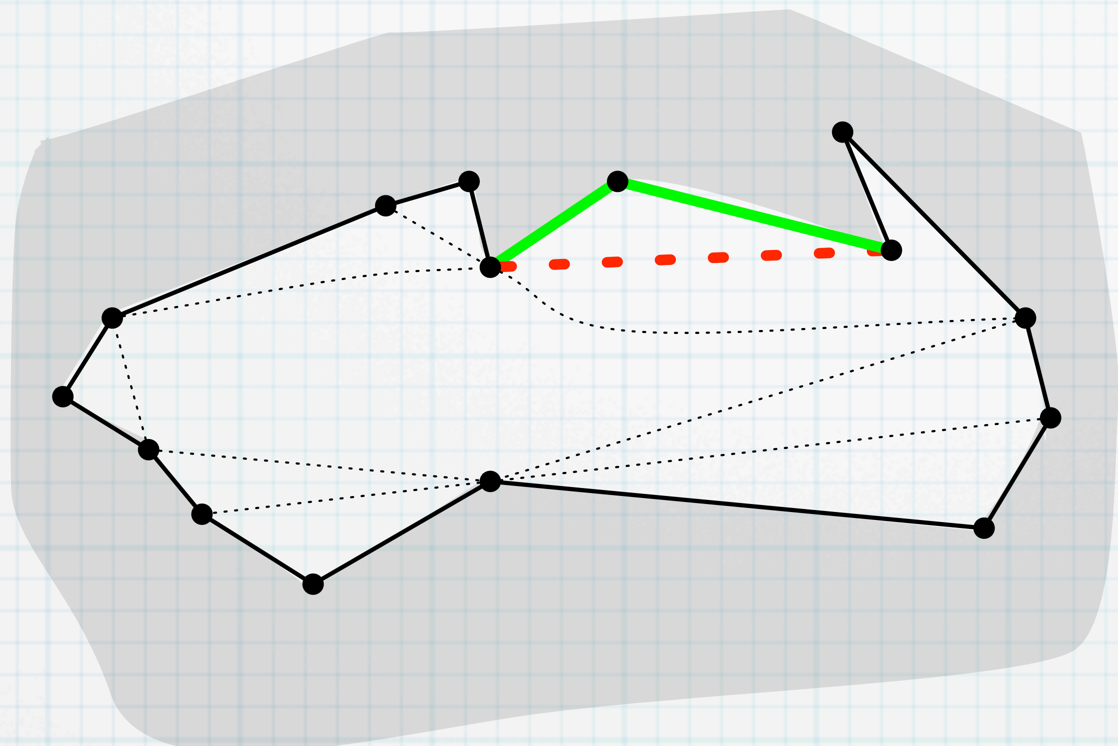

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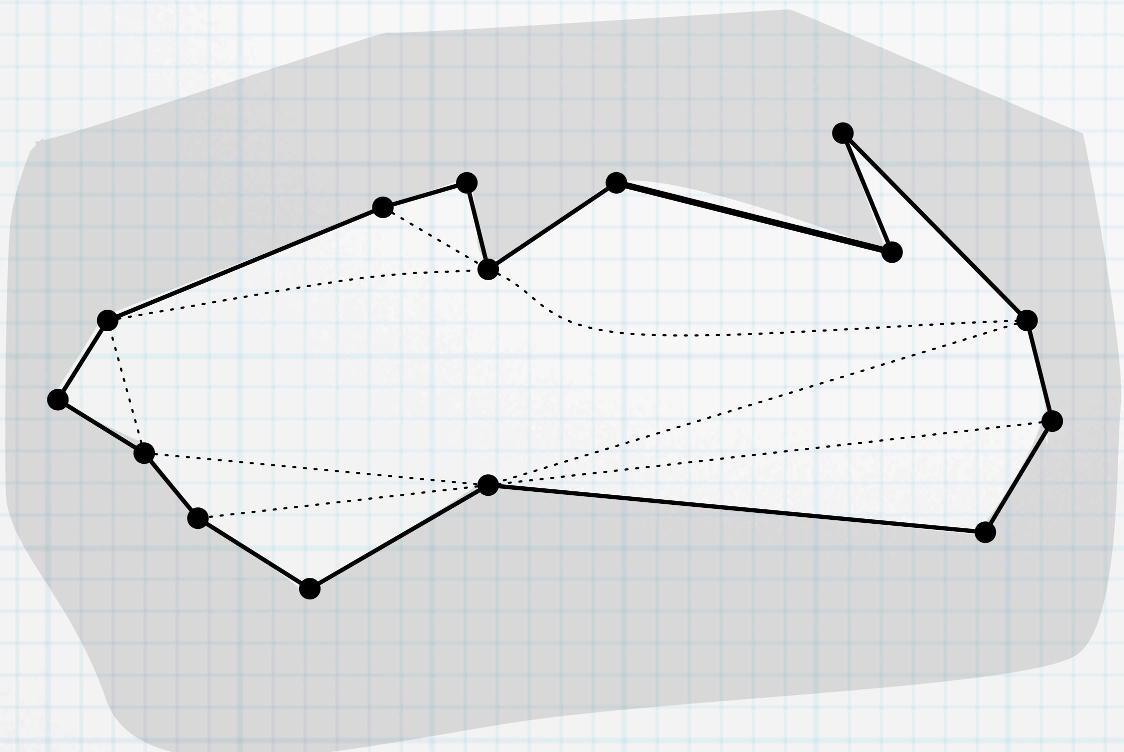

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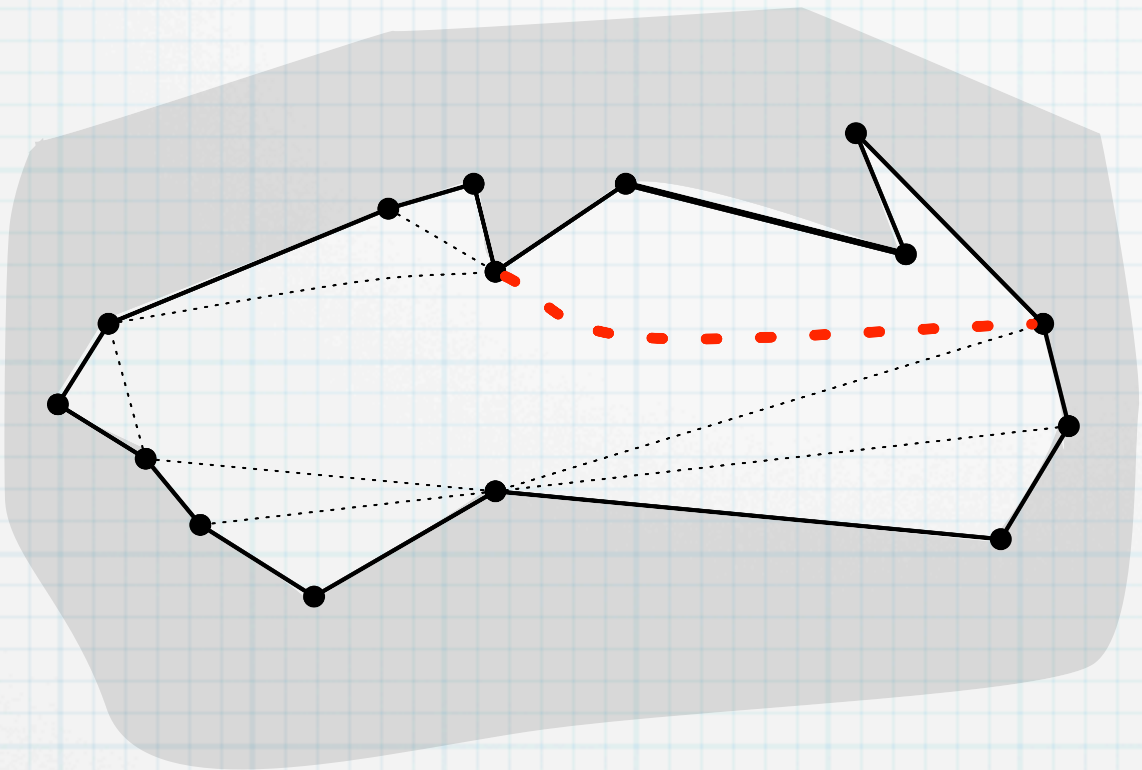

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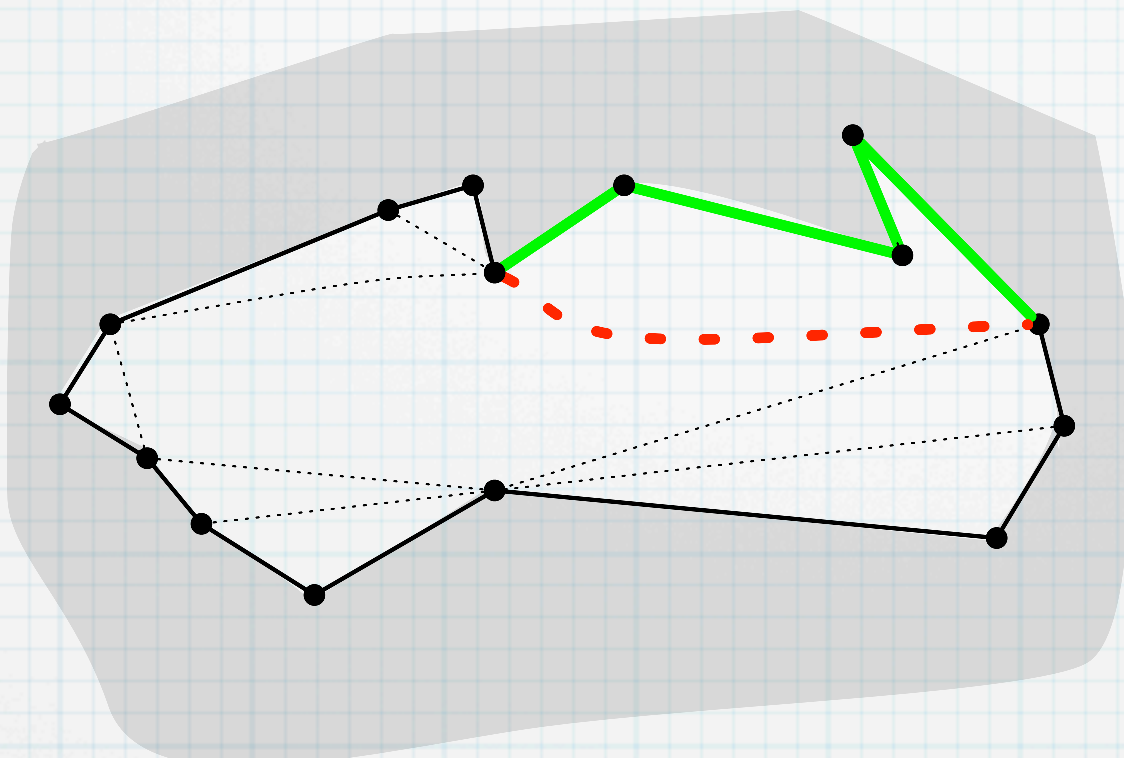

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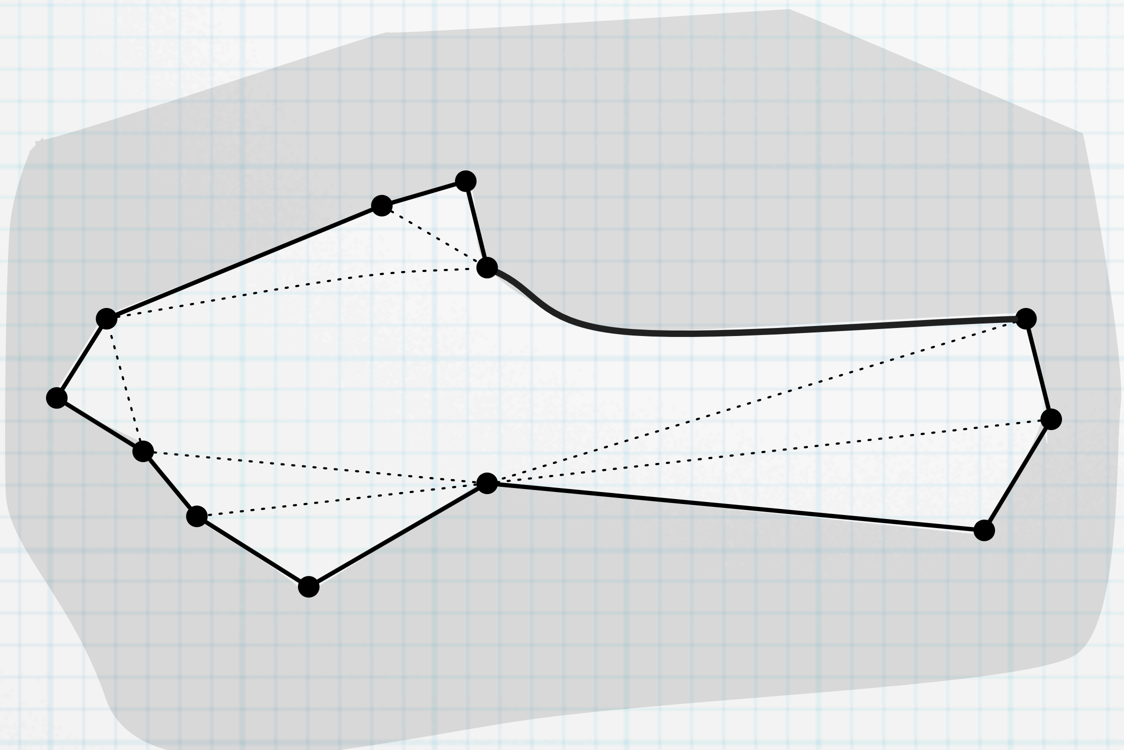

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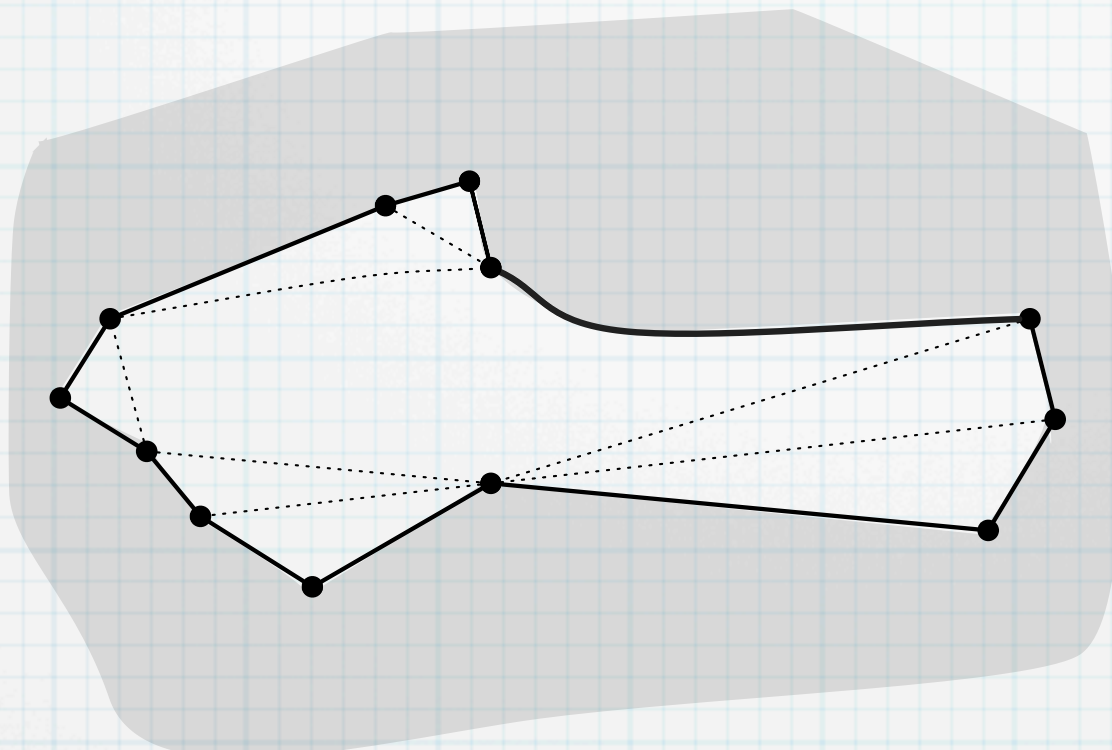

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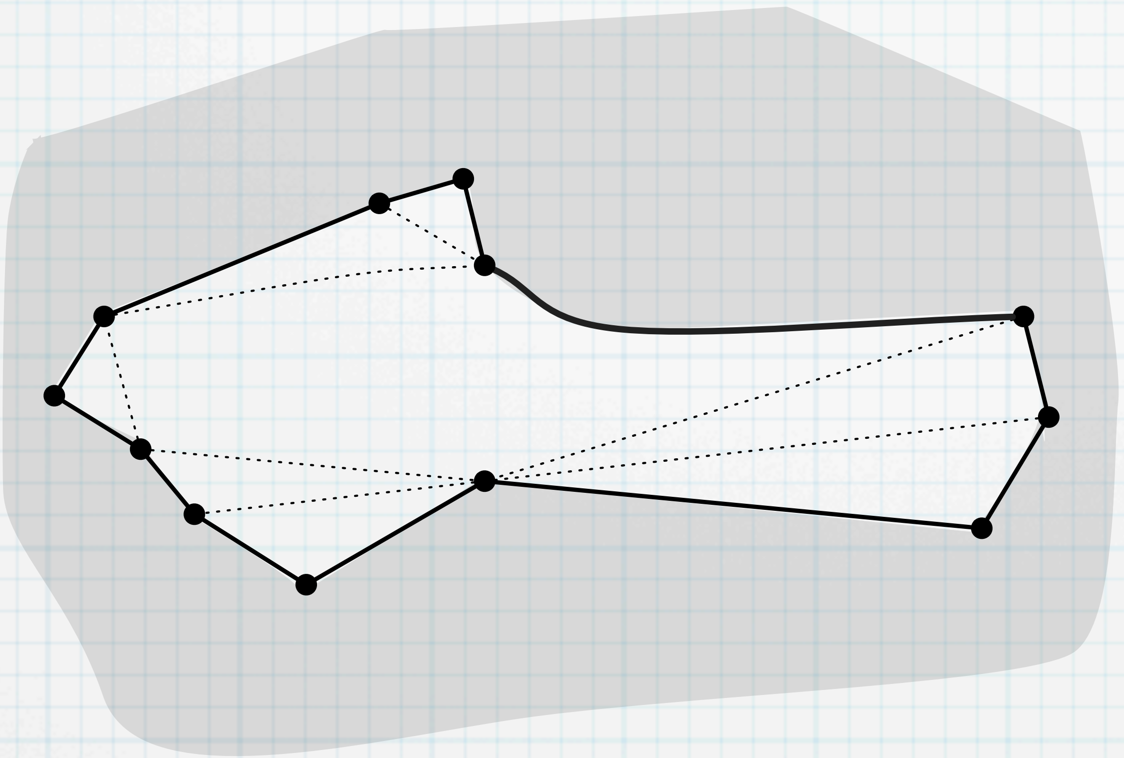

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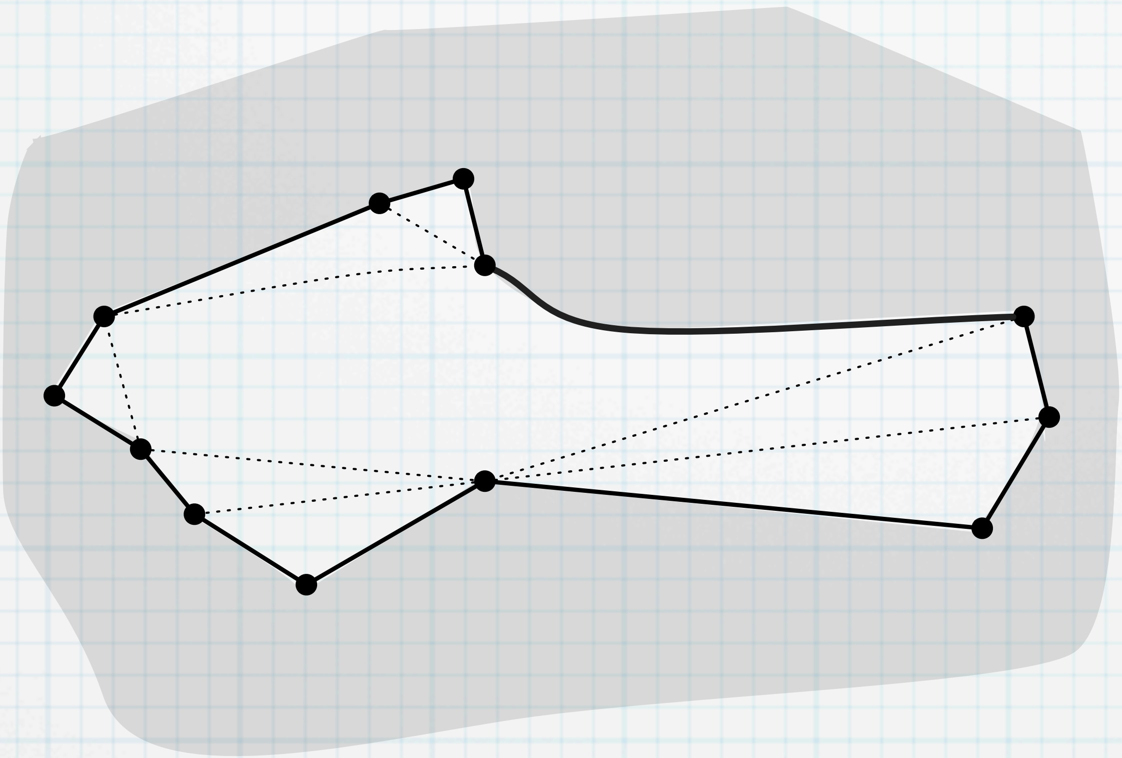

For each edge *uv* added to spanner, boundary weight goes down by at least

ε weight(uv)

Therefore, total weight added to spanner is at most

$\varepsilon^{-1} \cdot$ *decrease in boundary weight*

Initial boundary weight is 2 *weight(MST)*, so total weight added to spanner is

$2\ \varepsilon^{-1}$ *weight(MST)*

Theorem: for any undirected planar graph G with edge-weights, $\exists$ subgraph of cost $\leq 2(\varepsilon^{-1}+1) \times$ min spanning tree cost such that, $\forall u, v \in V$,
u-to-v distance in subgraph $\leq (1+\varepsilon)$ u-to-v distance in G

In framework for approximation scheme, choose $p = \varepsilon / 2(\varepsilon^{-1}+1)$ so increase in cost is at most $\varepsilon\, OPT$

Corollary: There is a linear-time approximation scheme for traveling salesman in planar graphs.

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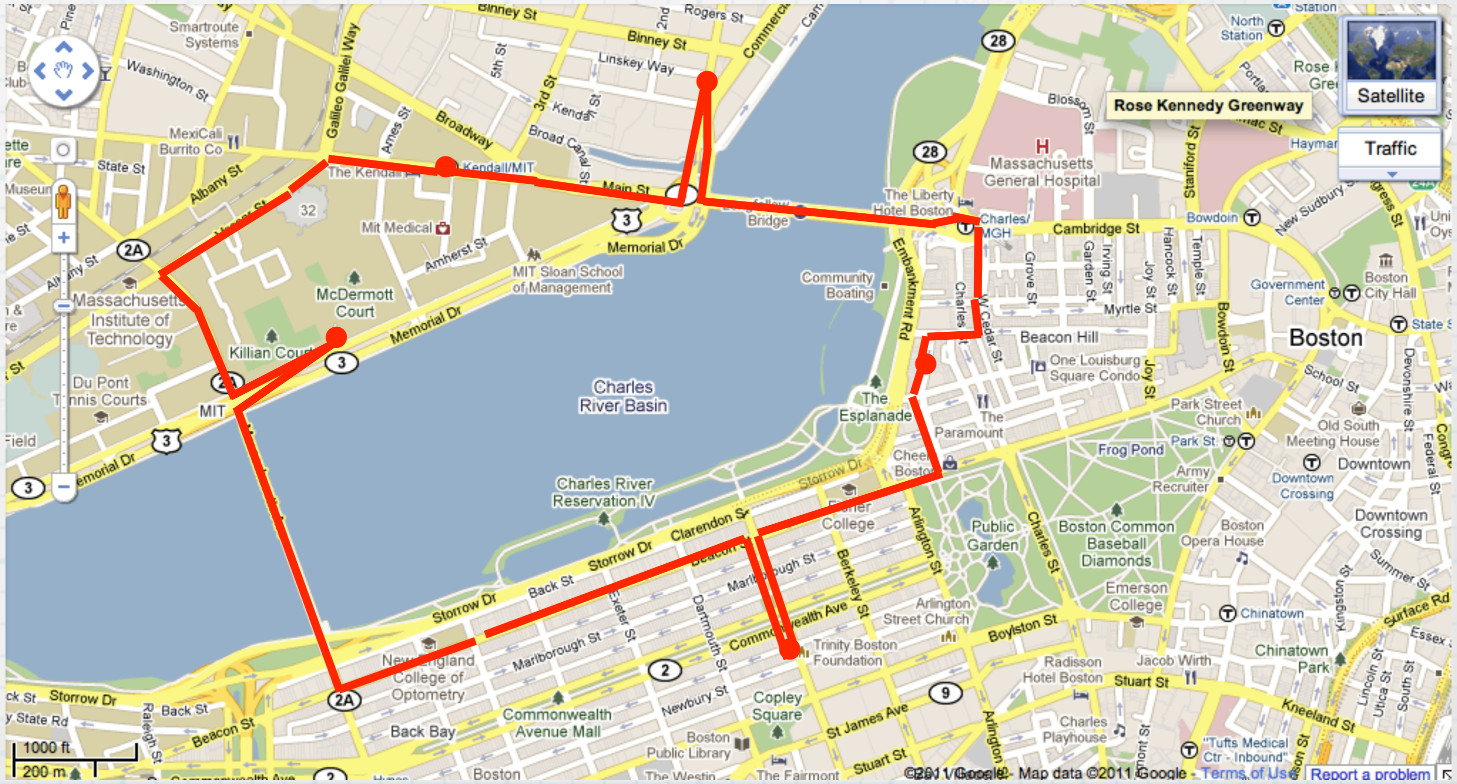
But we want to address...

Traveling salesman on a *subset* of vertices



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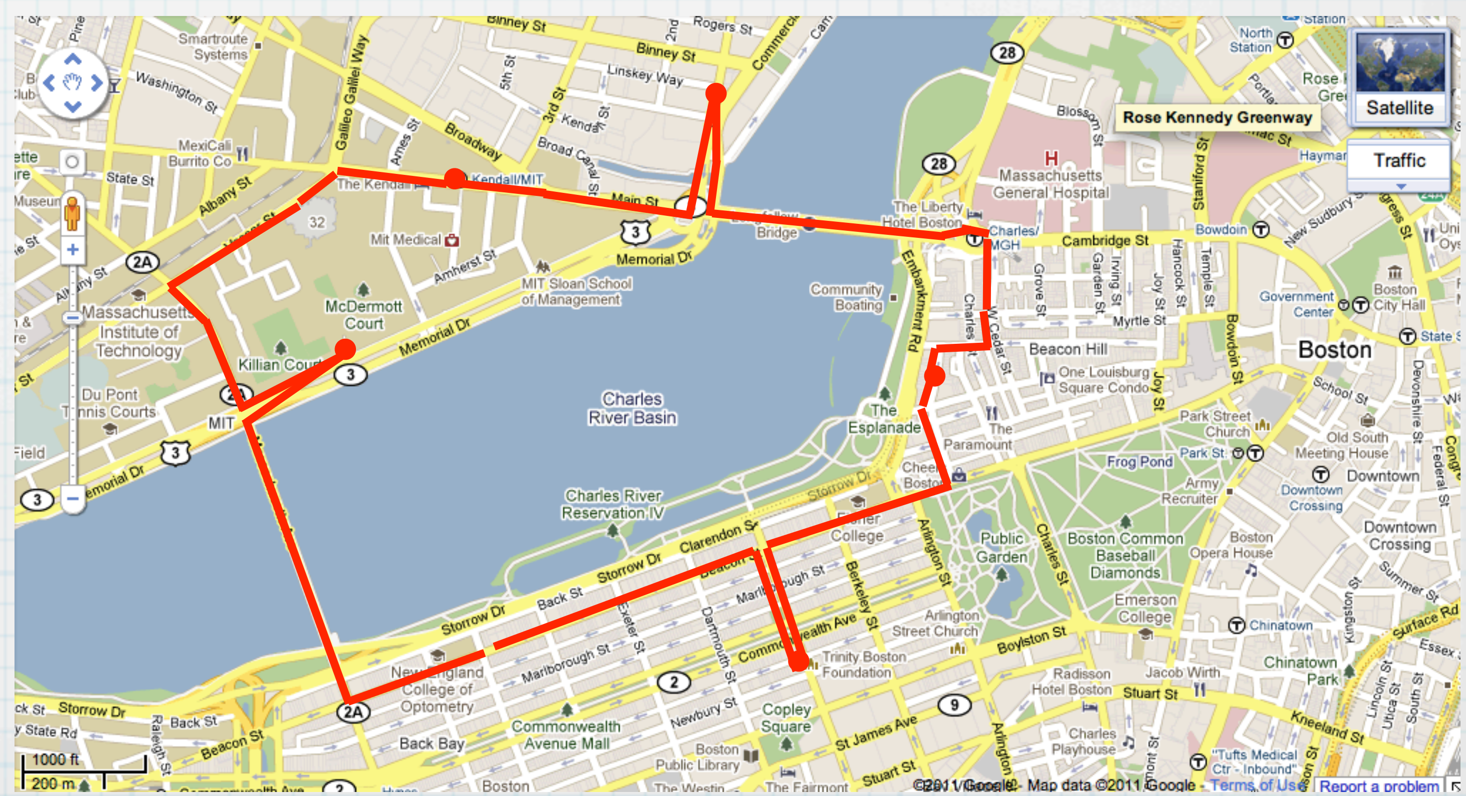
Need a more general spanner result



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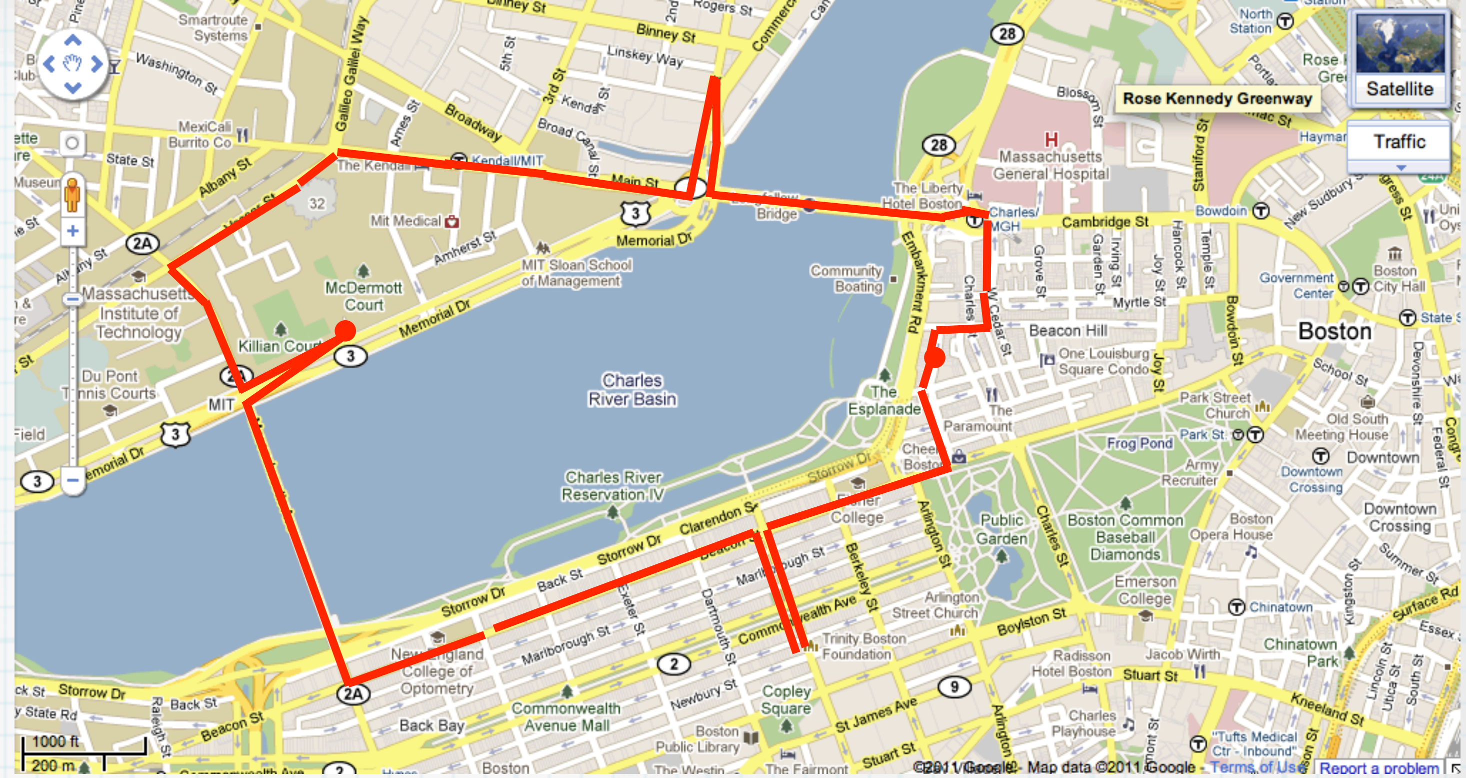

We need a subgraph that approximately preserves distances between vertices of the subset.

Minimum weight to just preserve connectivity?

weight of minimum *Steiner tree* spanning the subset.

Theorem: for any undirected planar graph G with edge-weights, and any given subset S of vertices,

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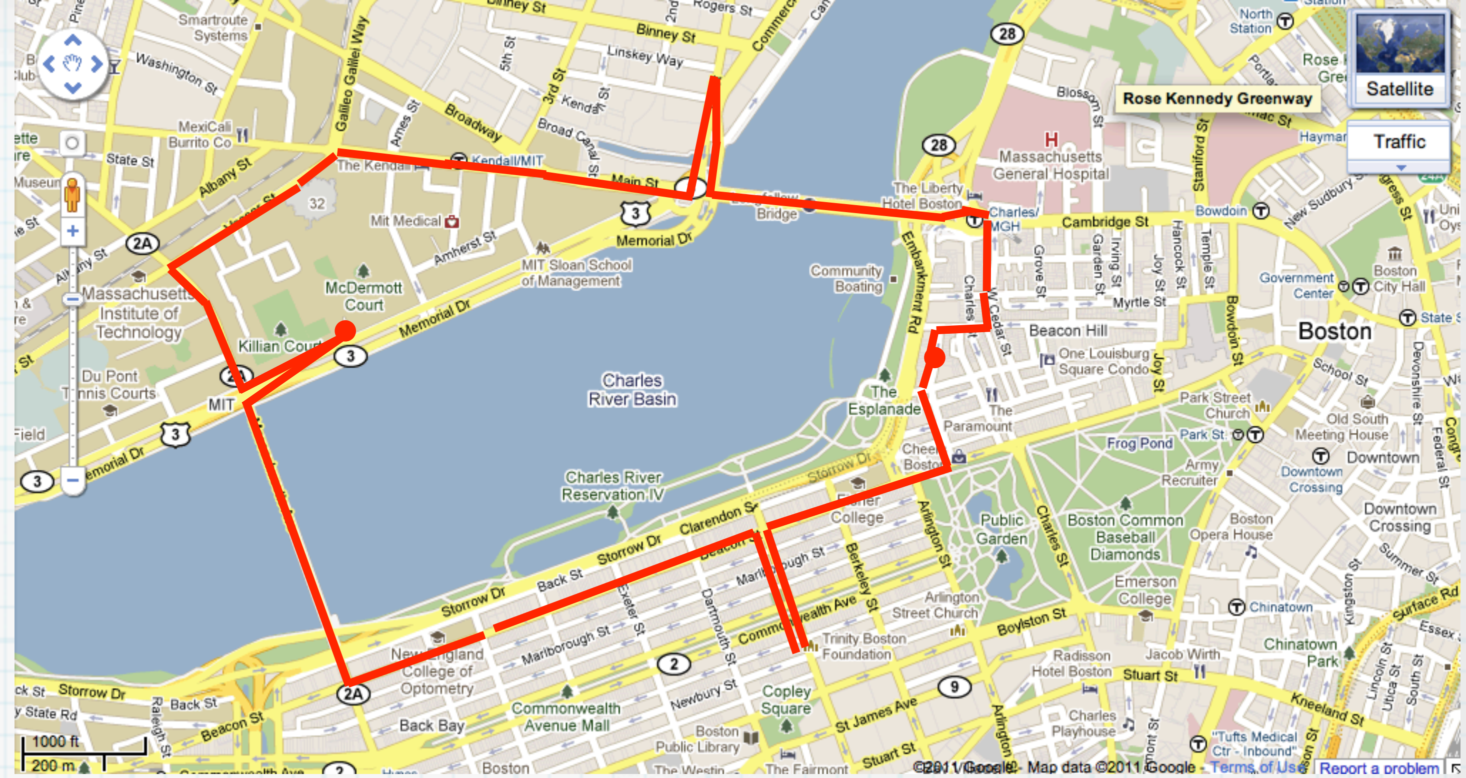

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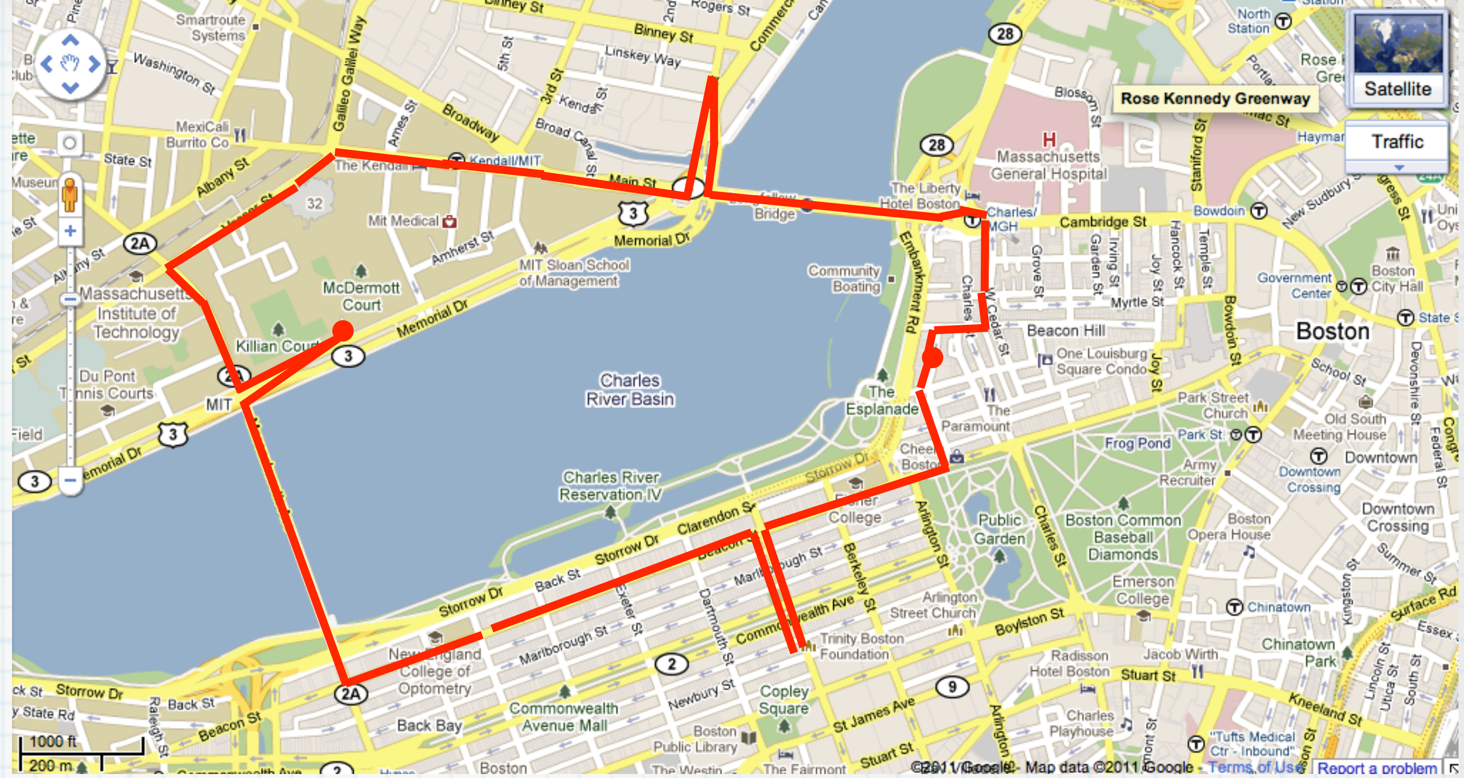

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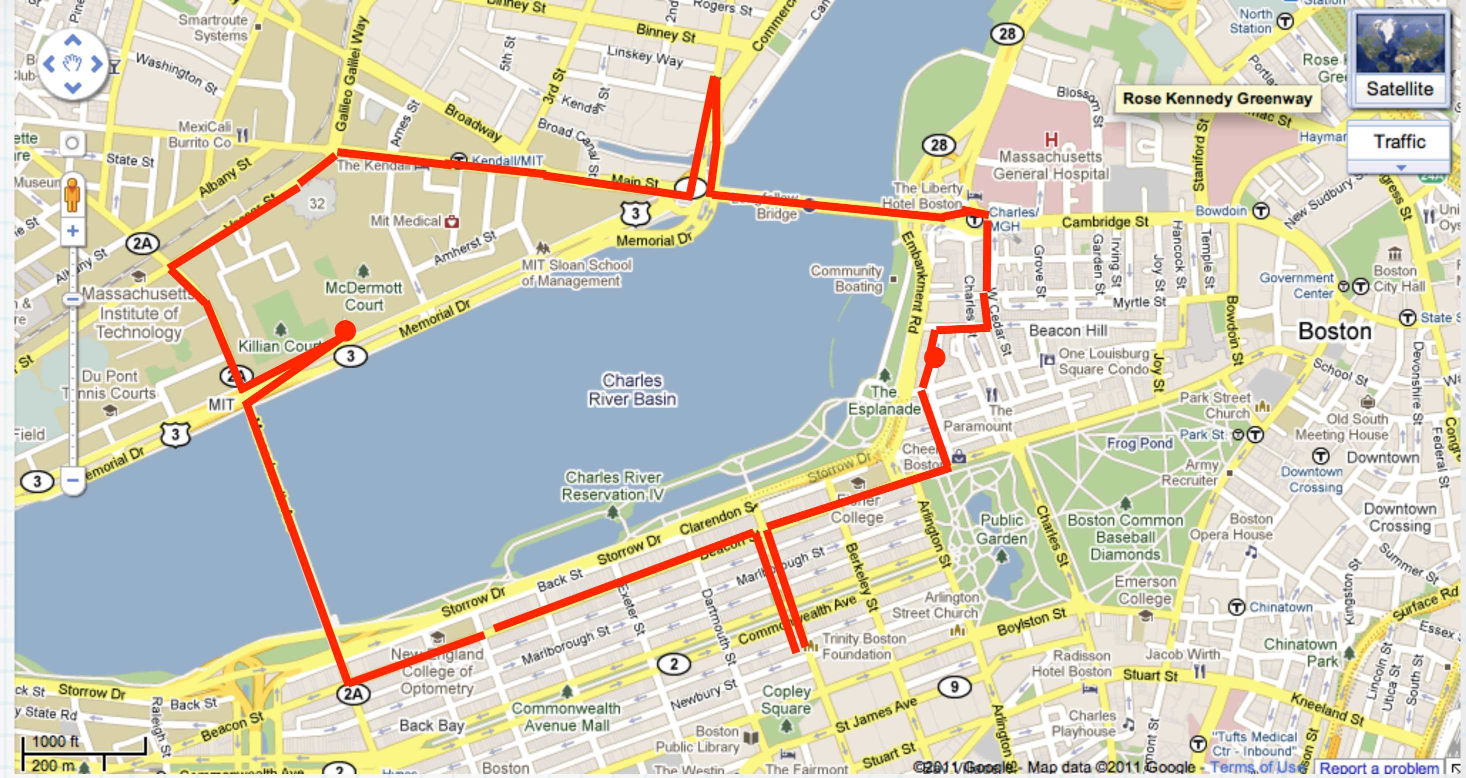

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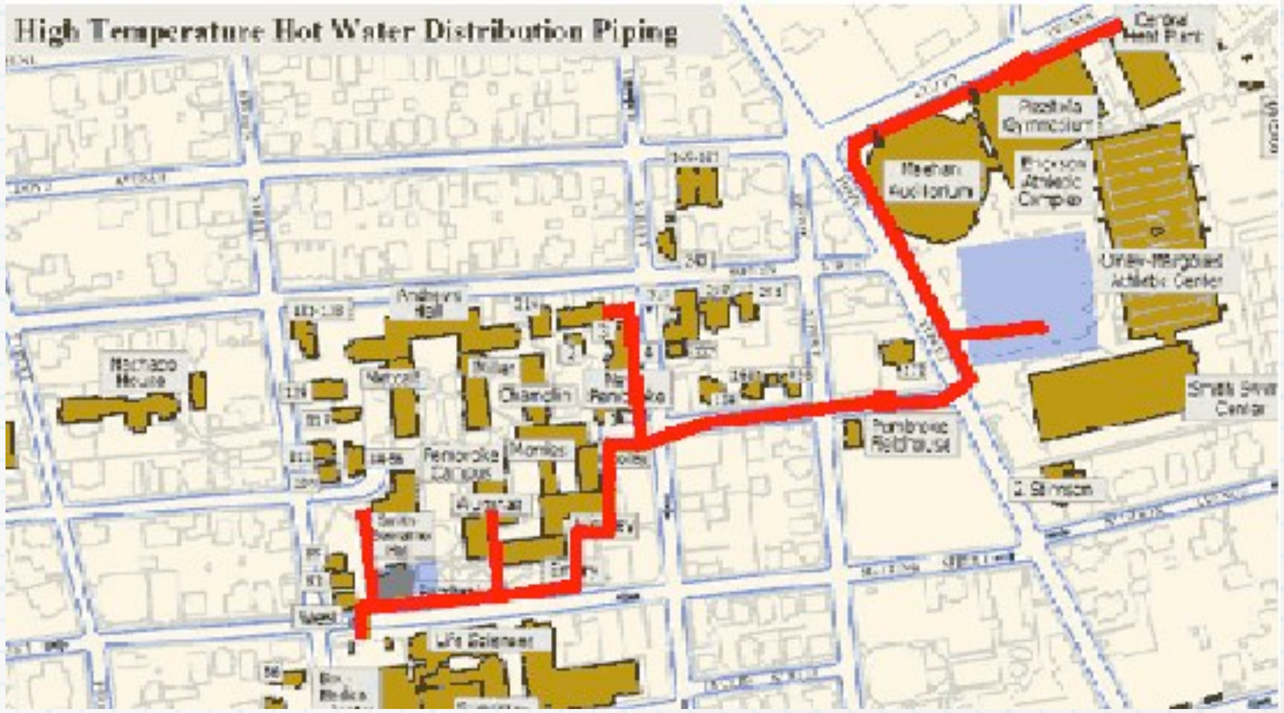
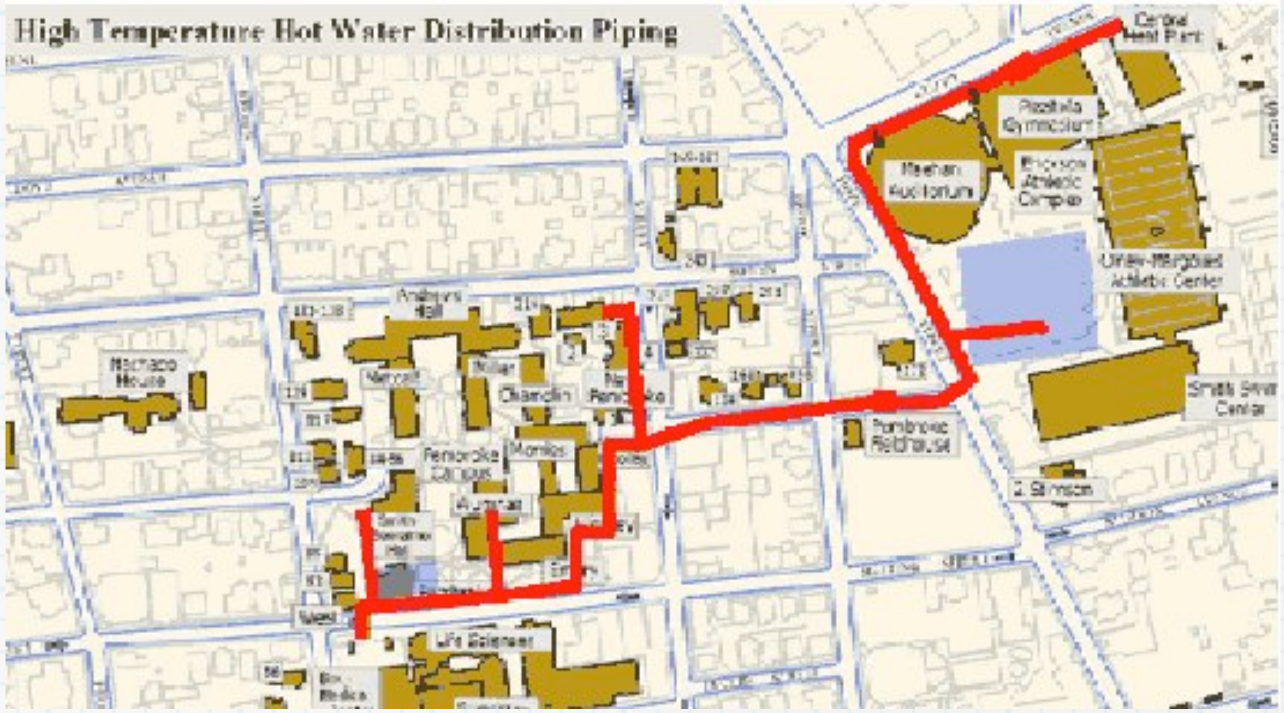


Steiner tree



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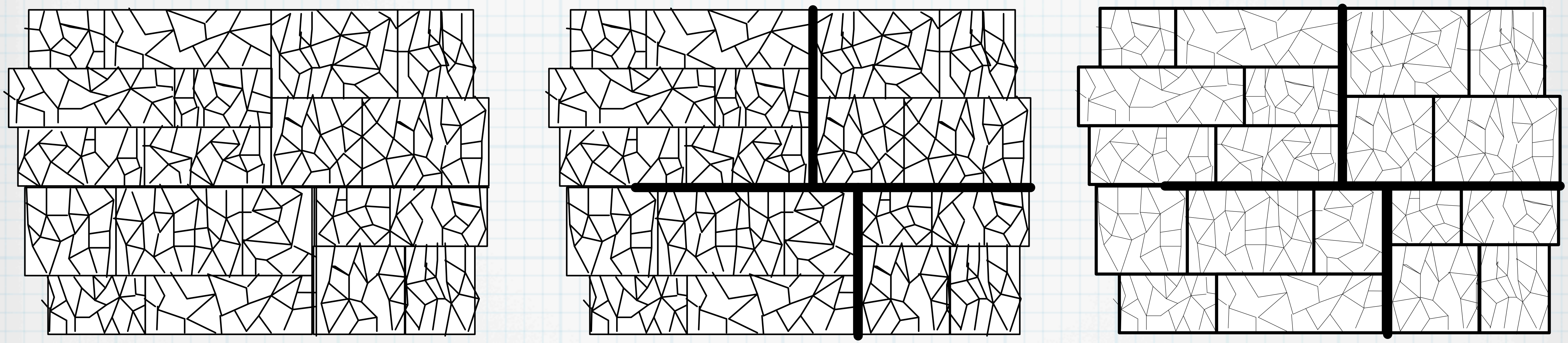
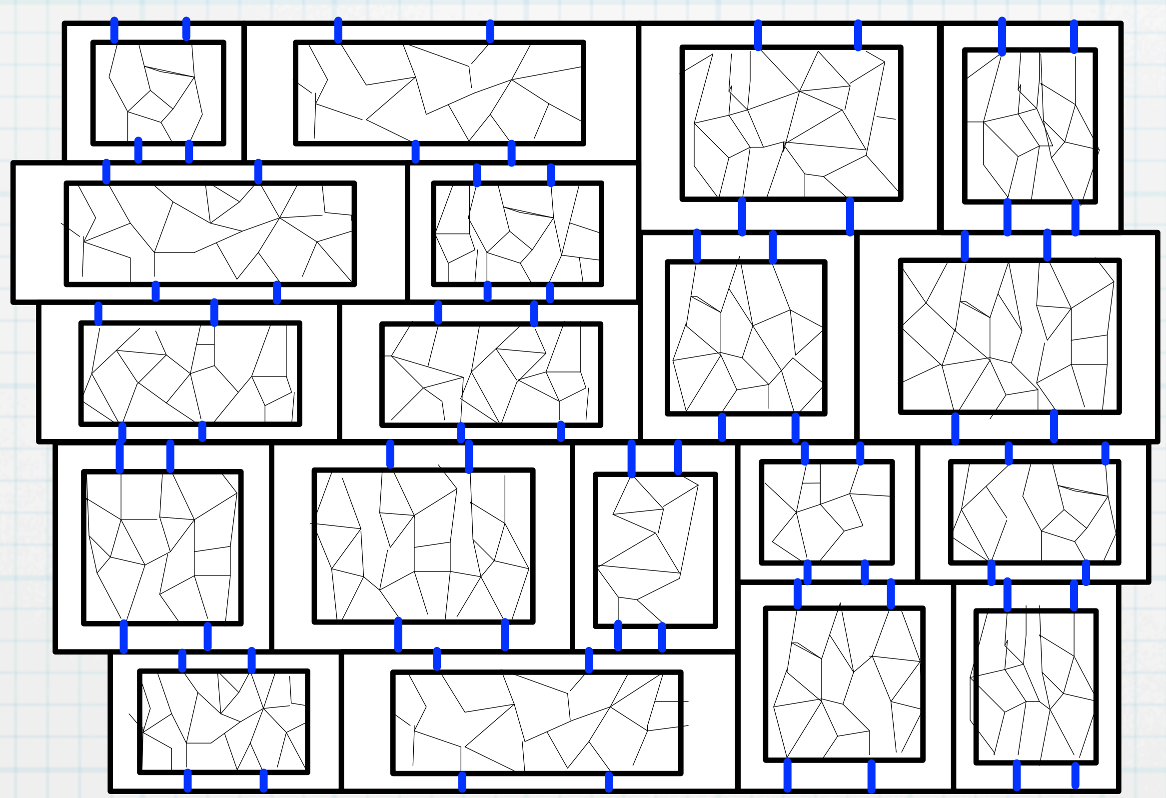
Tool for Step 1: *Brick Decomposition*

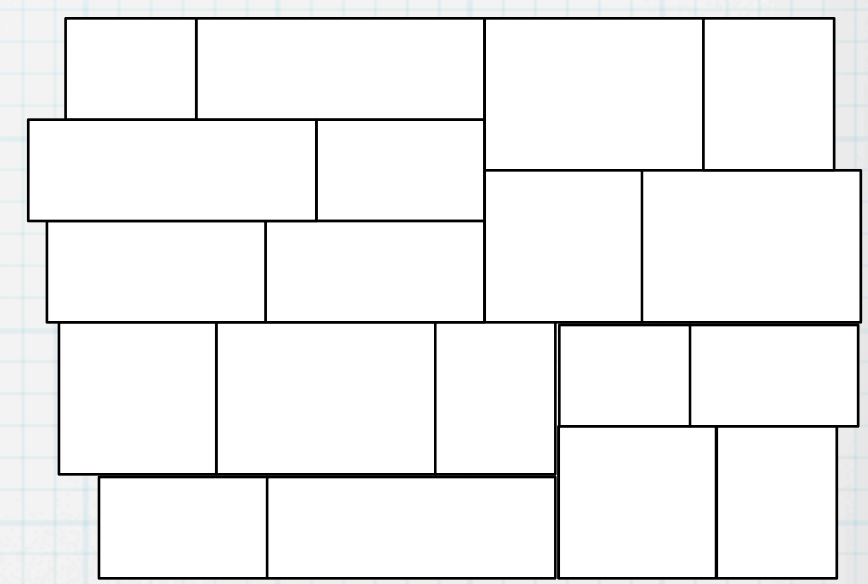


T:= 2-approx. Steiner tree

M := subgraph containing T

Bricks := faces of M





Connect each brick to copy of M using $c(\varepsilon)$ portal edges

TSP Structure Theorem: There is a 1+ ε-approx. tour that uses portal edges to go between bricks.

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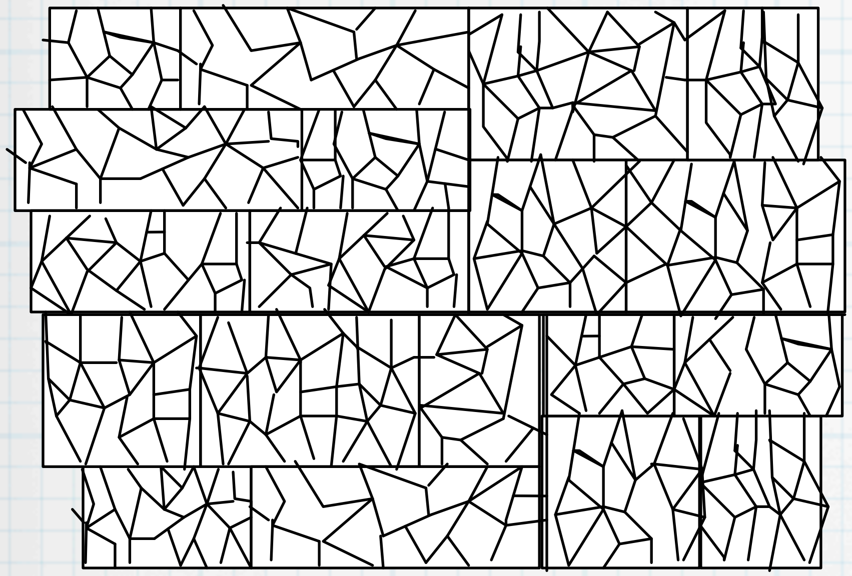

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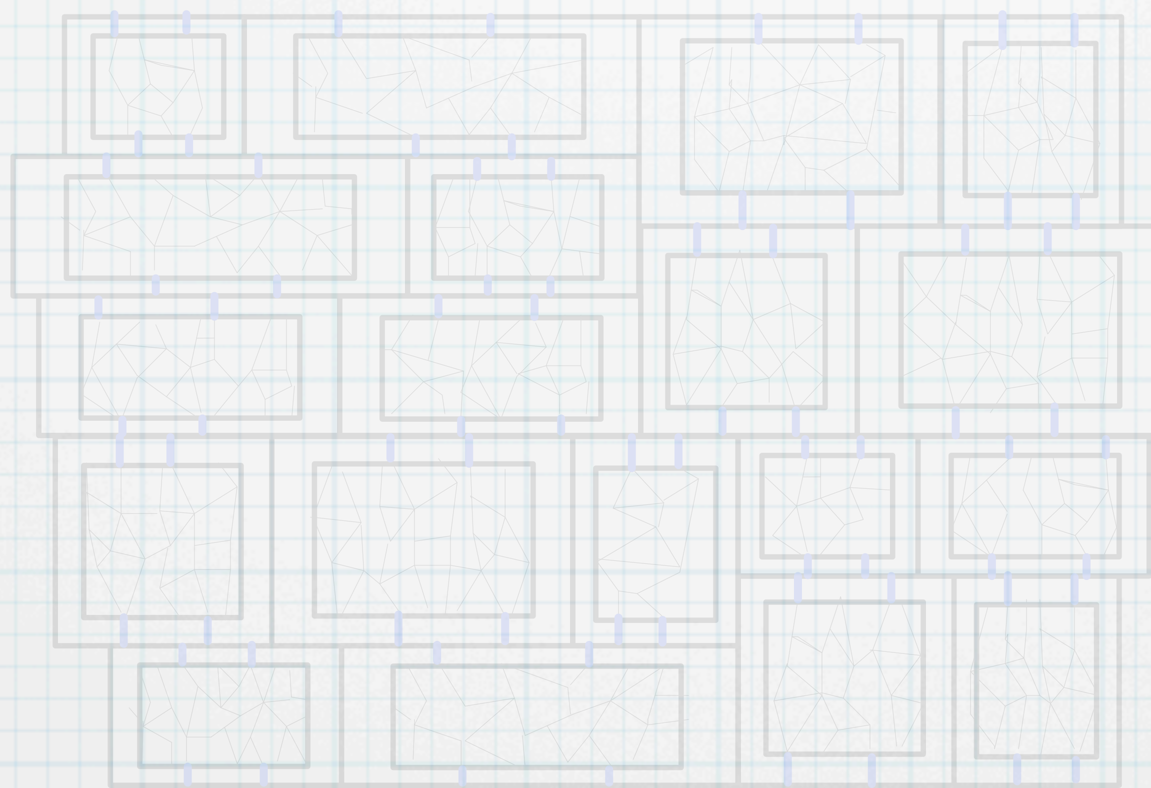

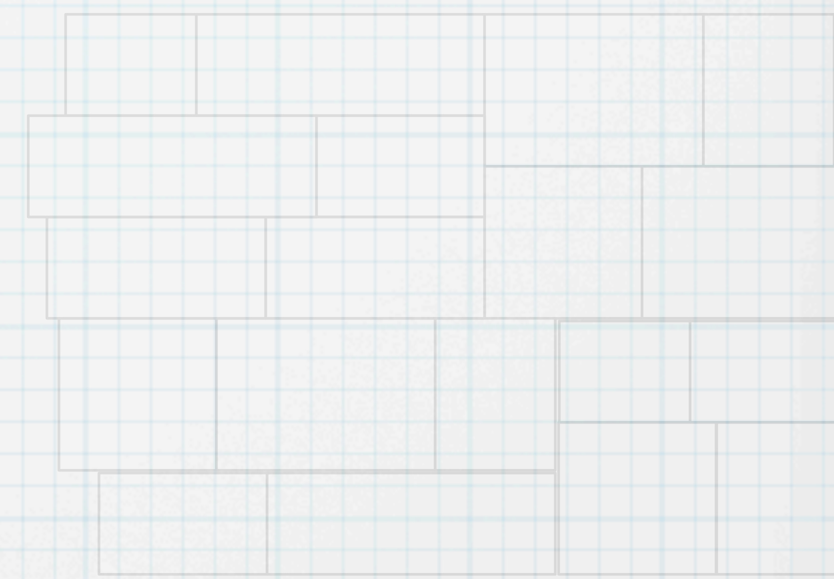

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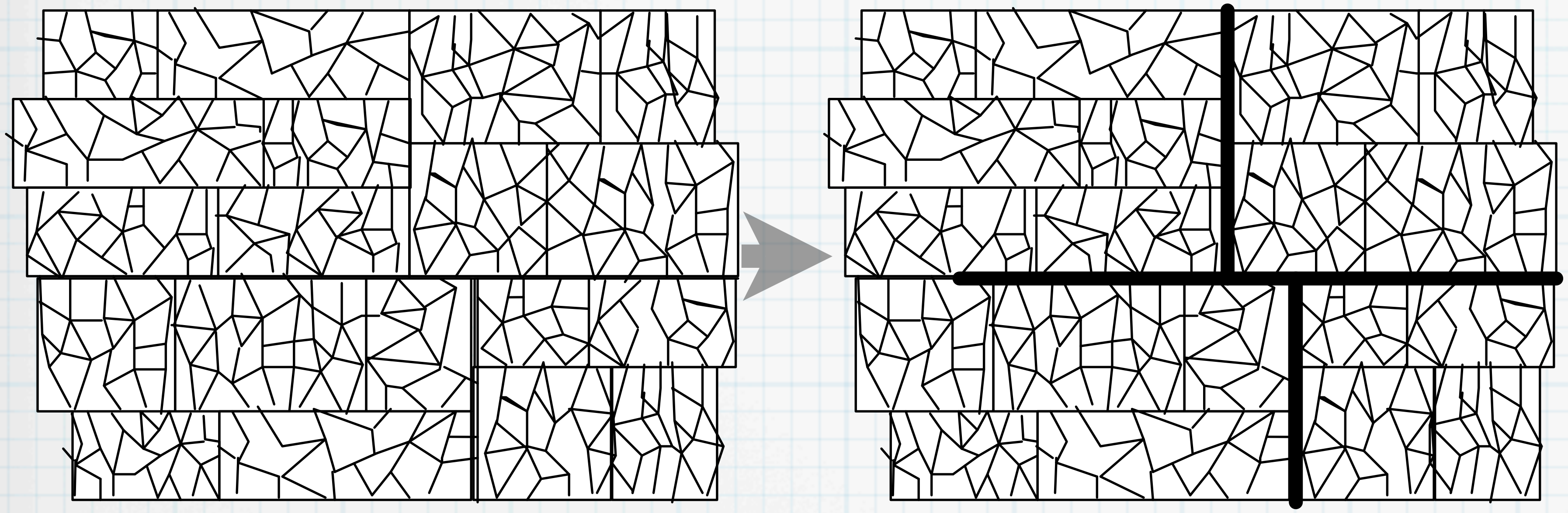

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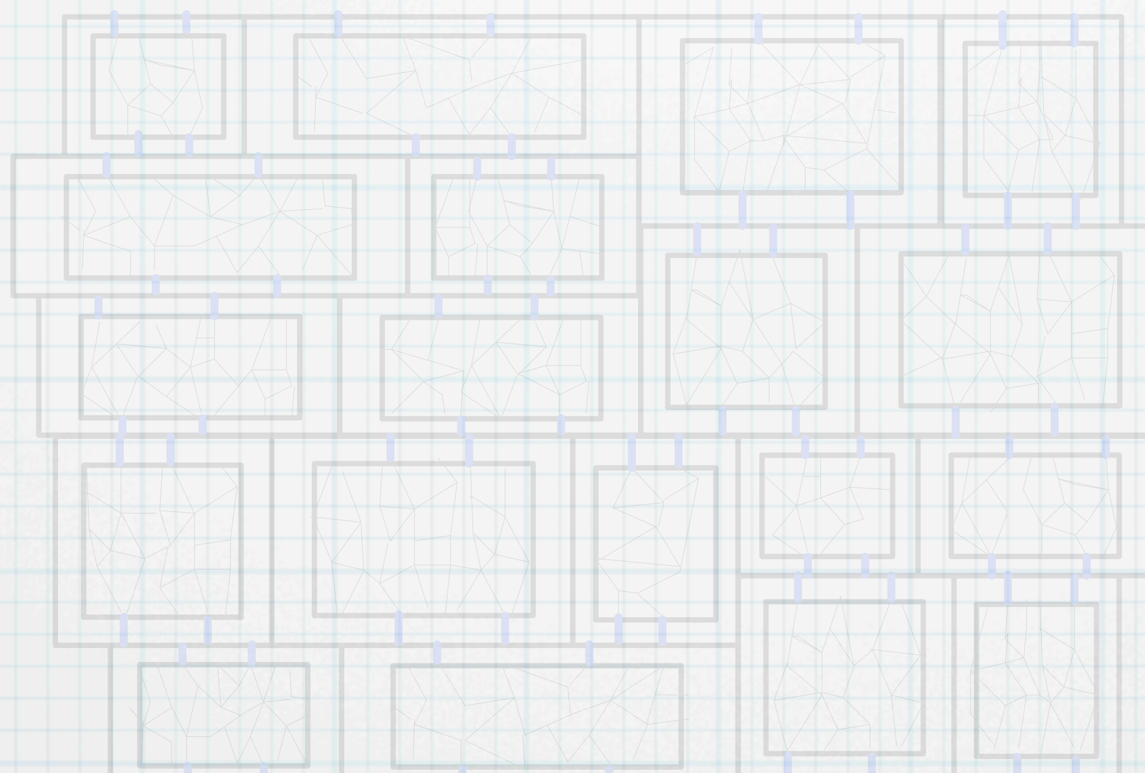

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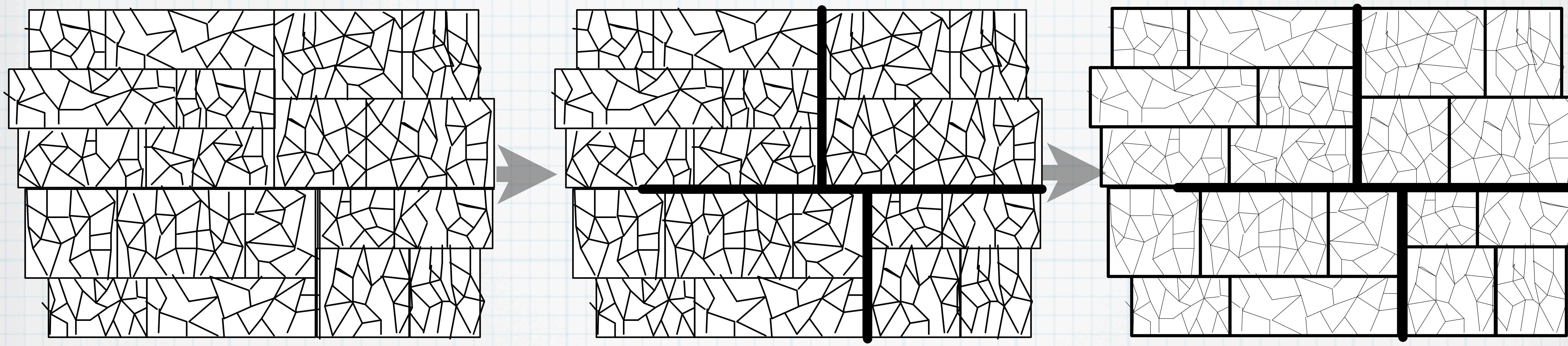

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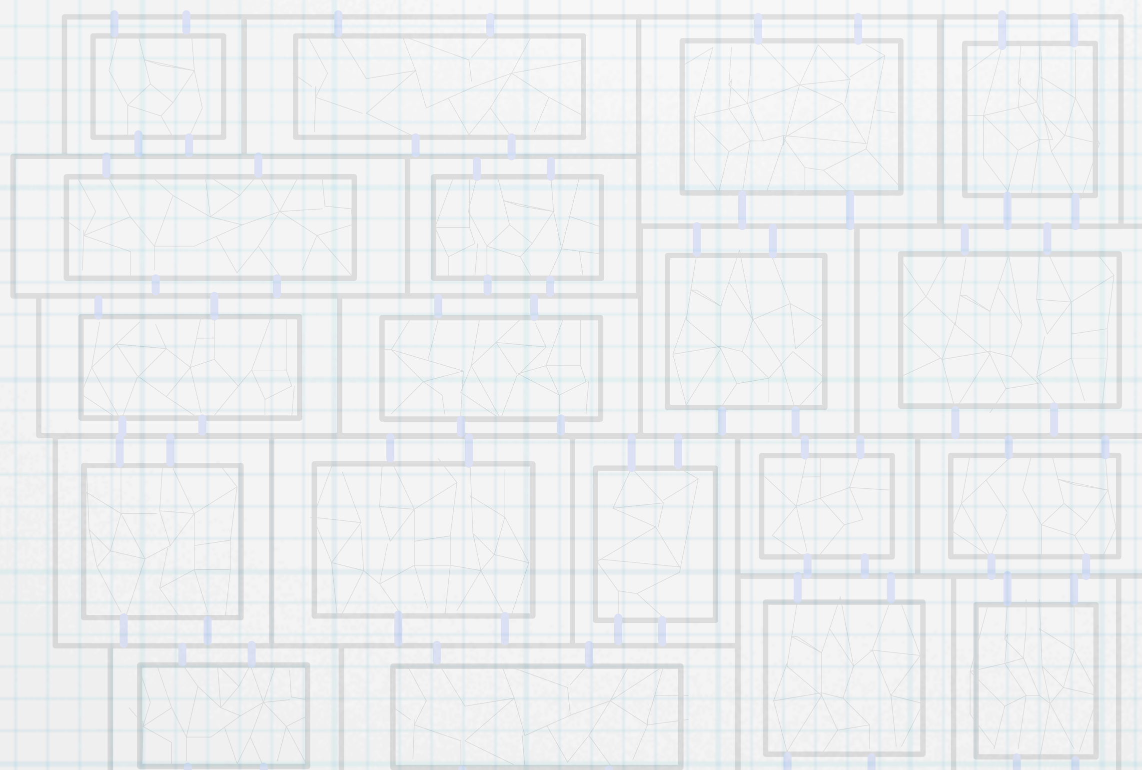

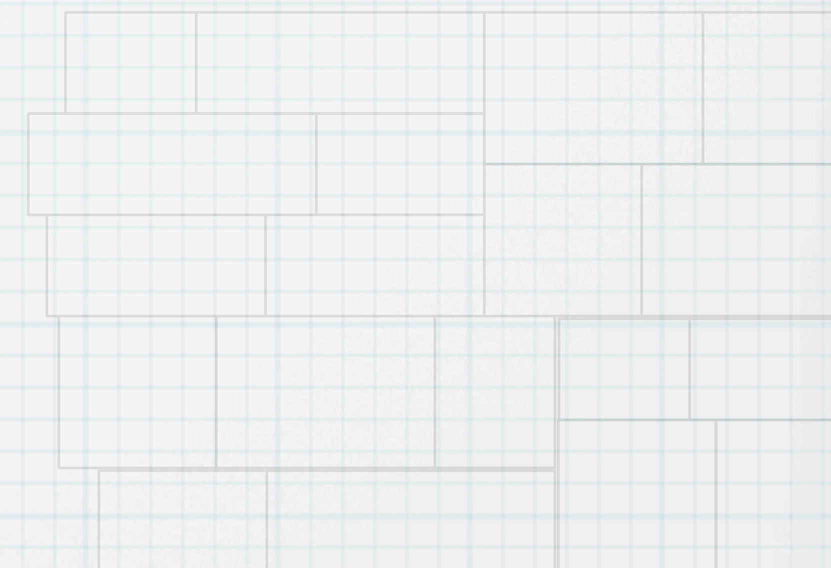

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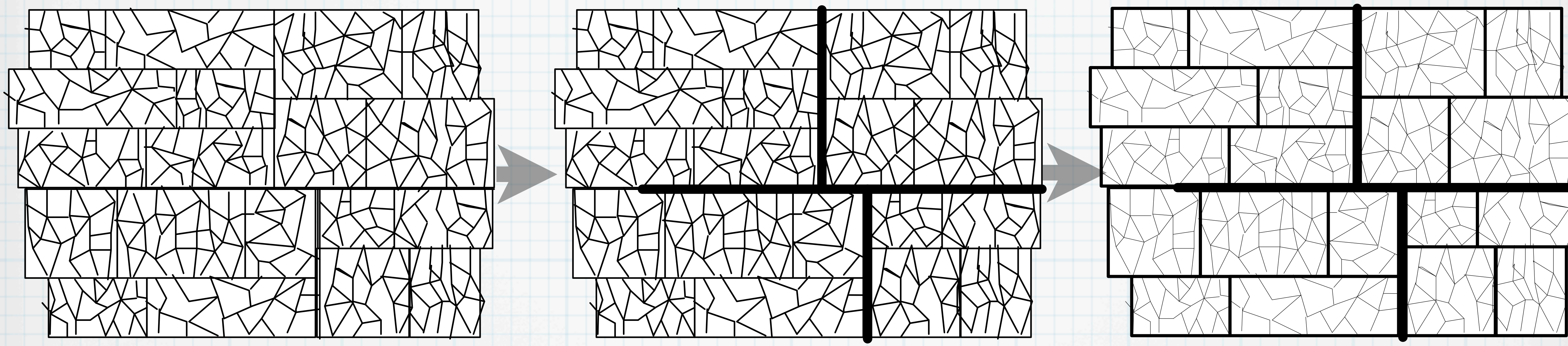

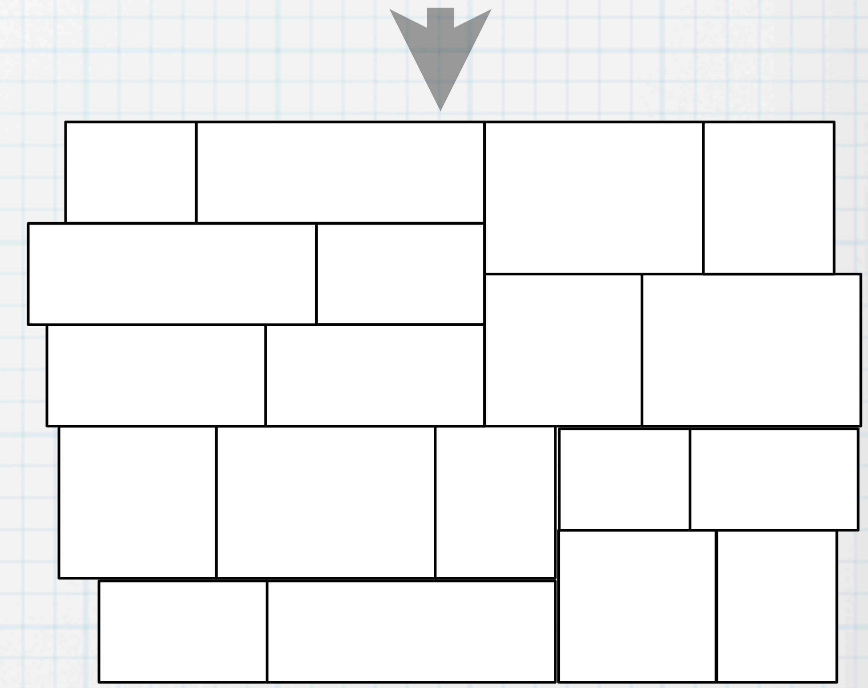

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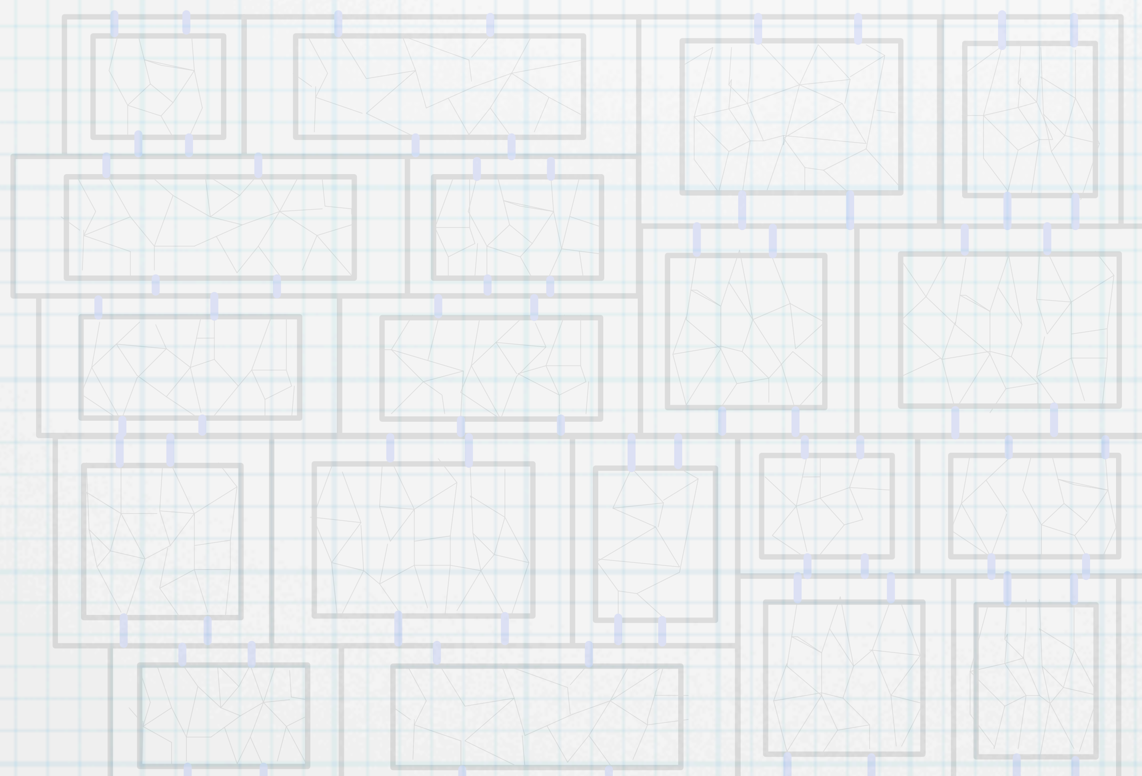

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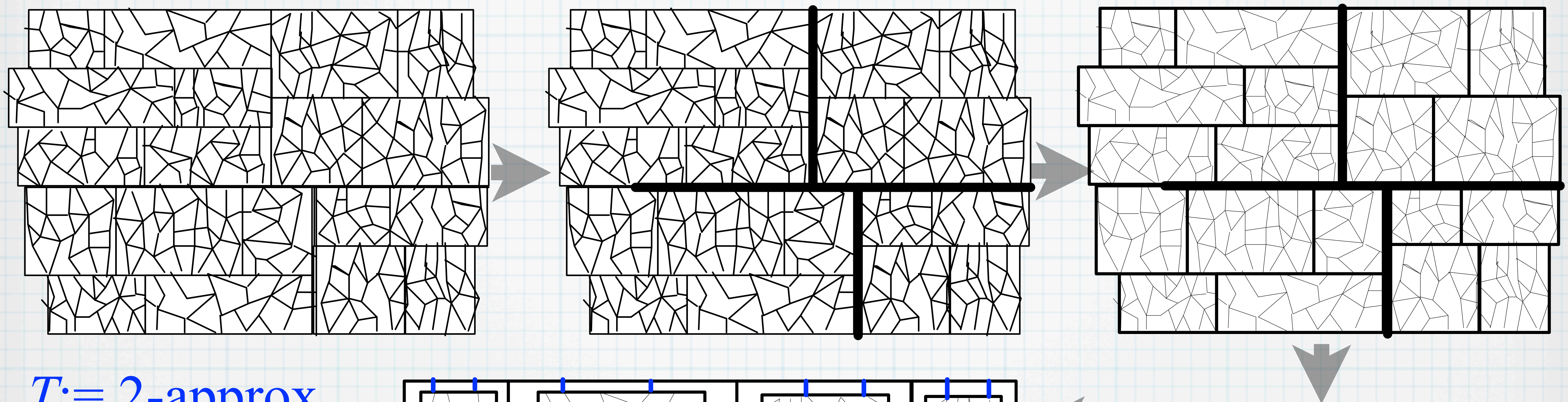

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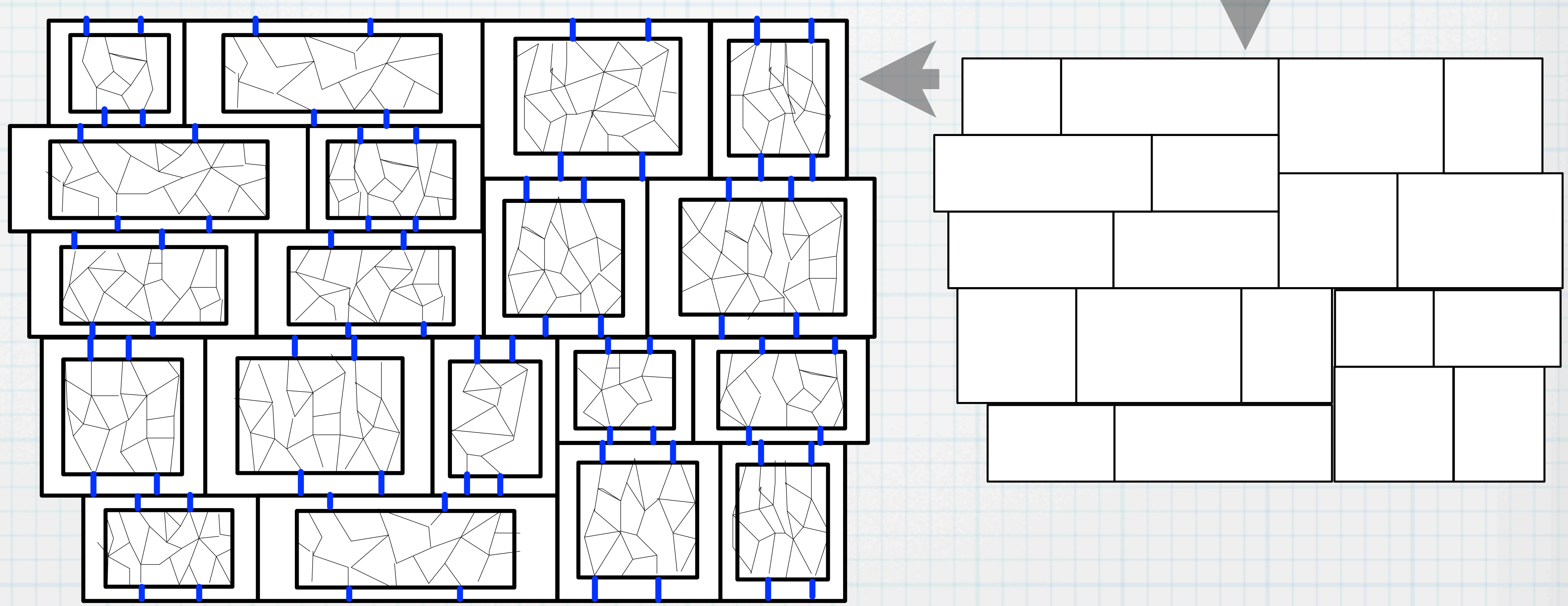

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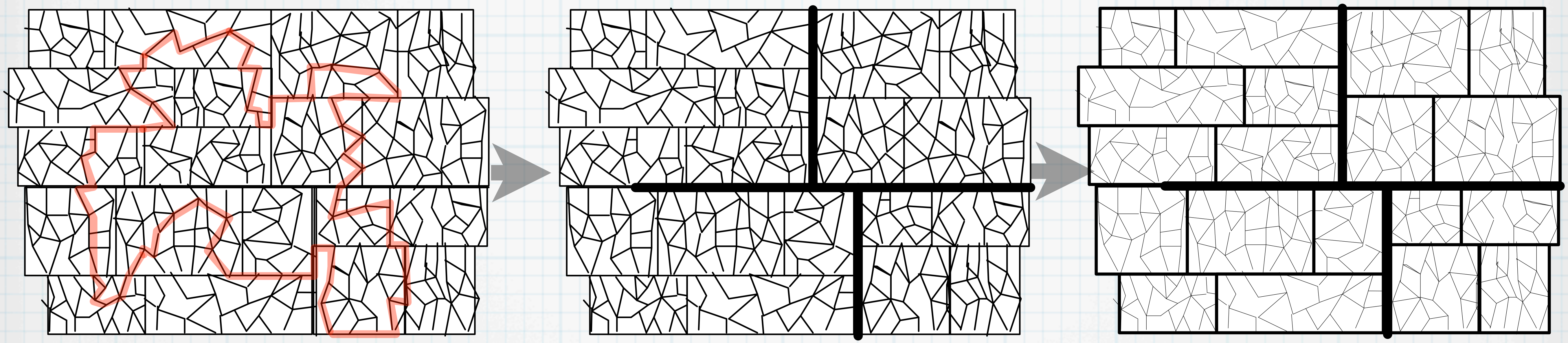

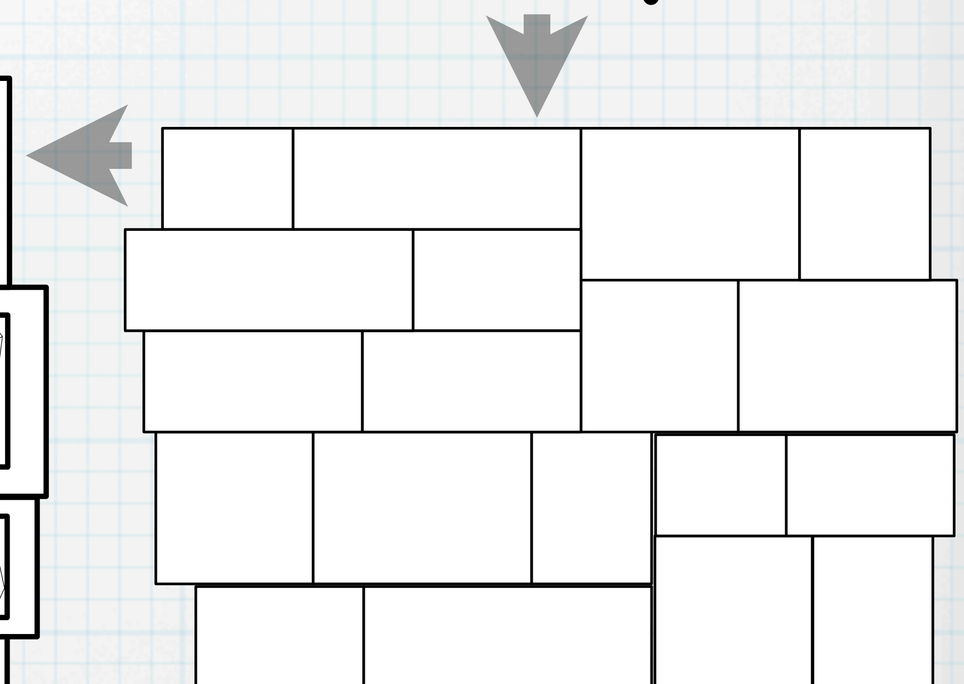

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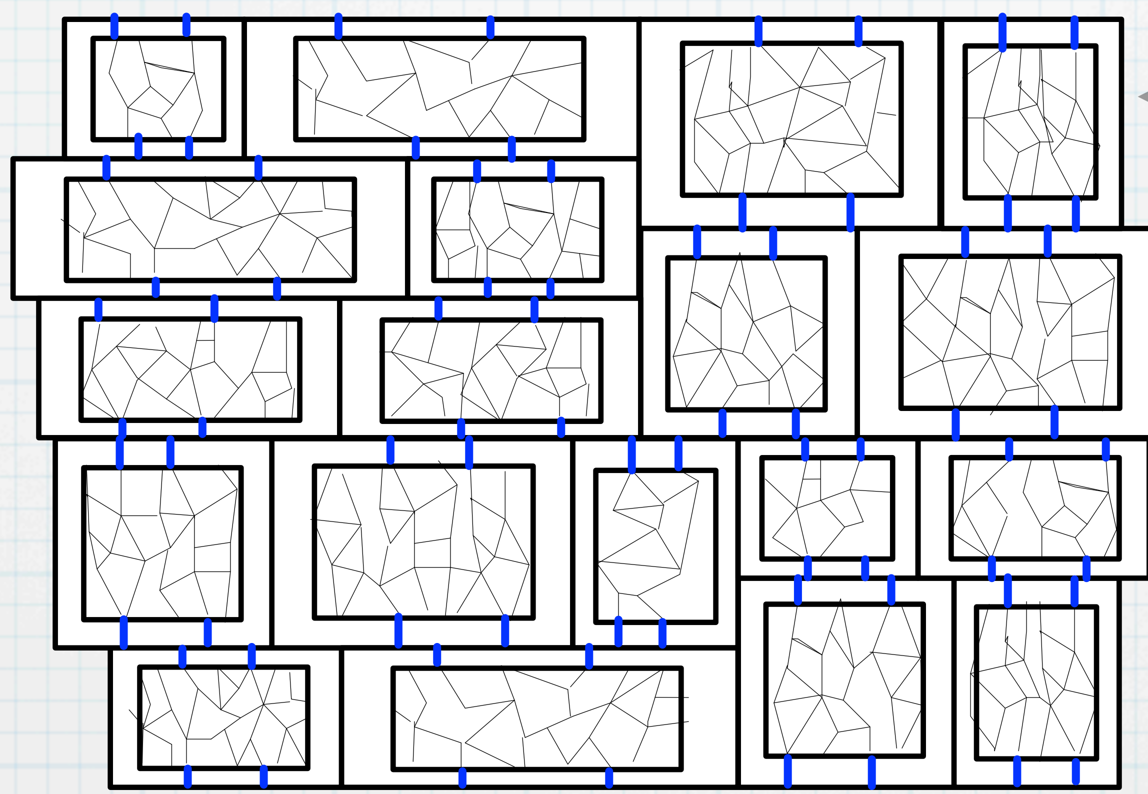

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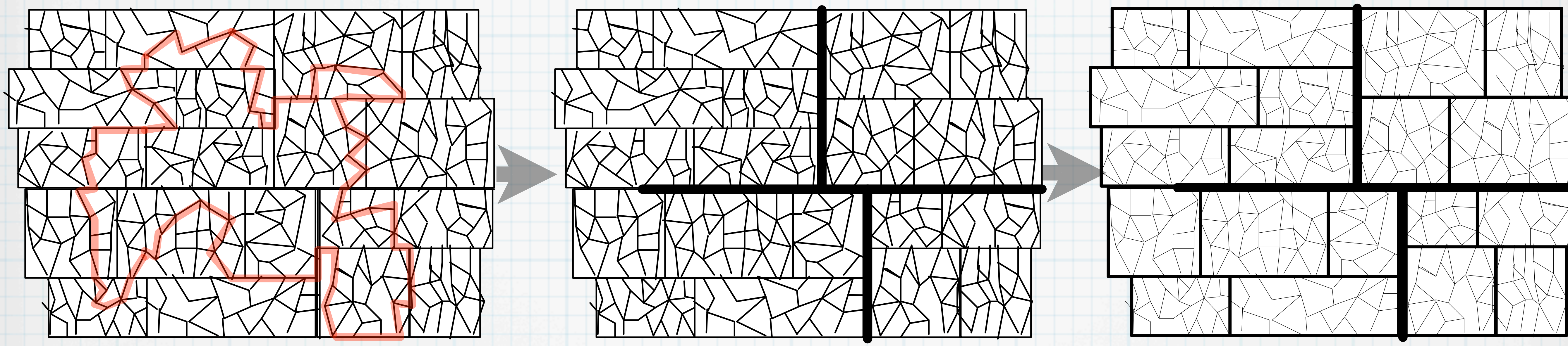

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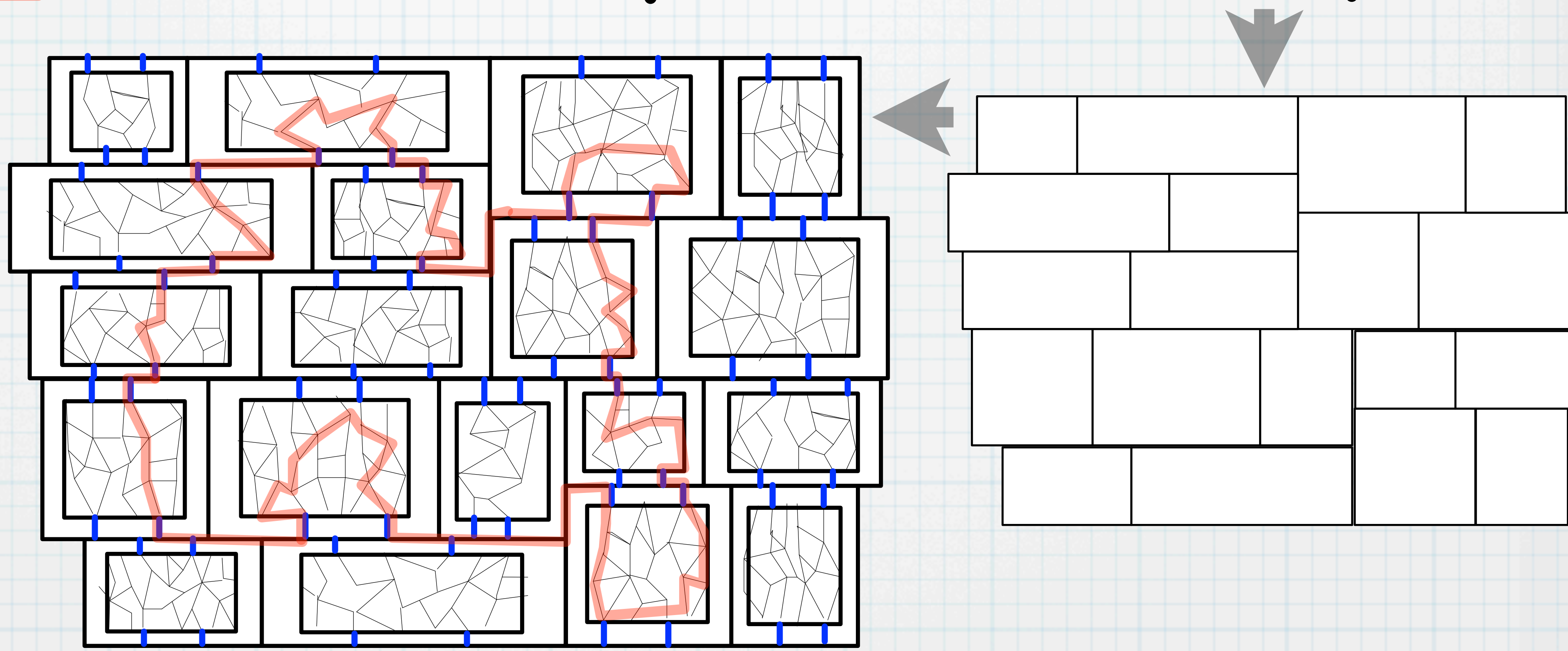

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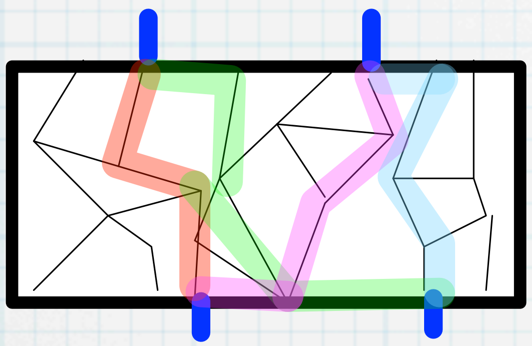
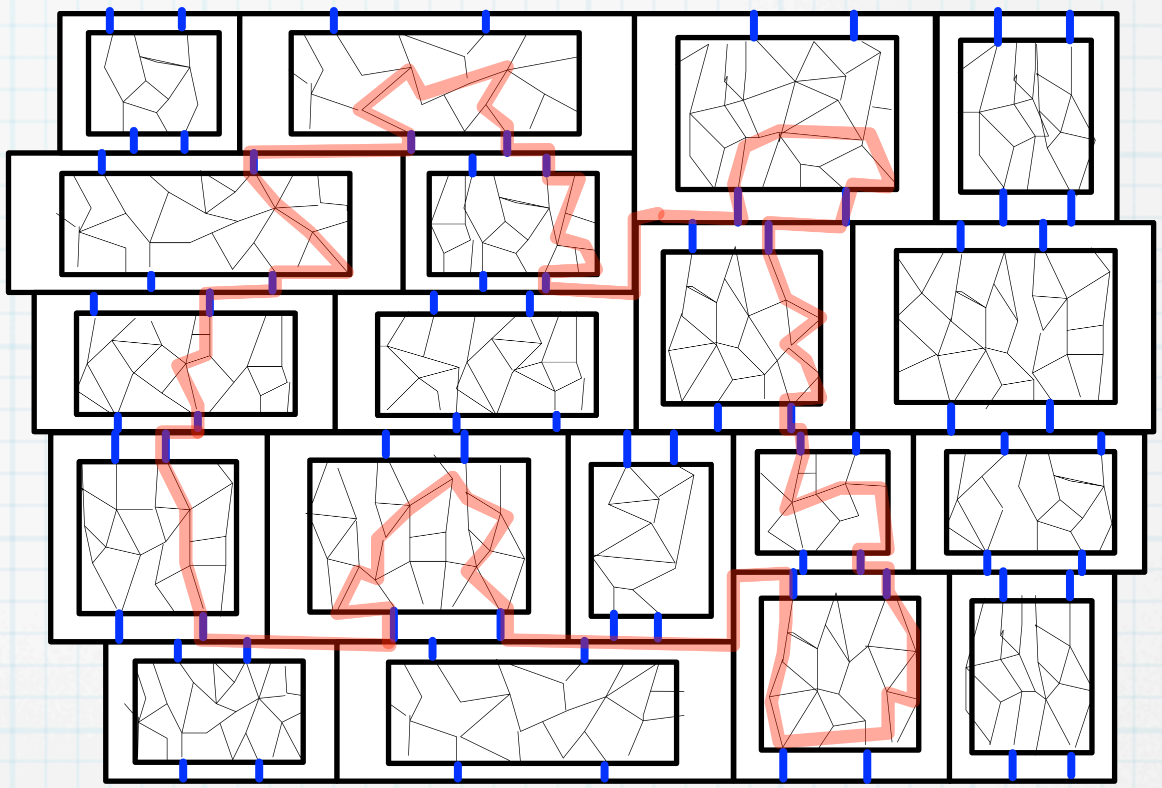
TSP Spanner construction: Include M and, for each brick B, all portal-to-portal shortest-paths within B.





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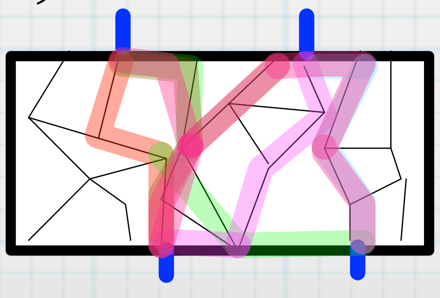

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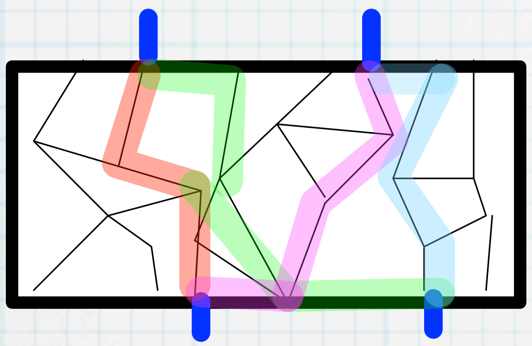

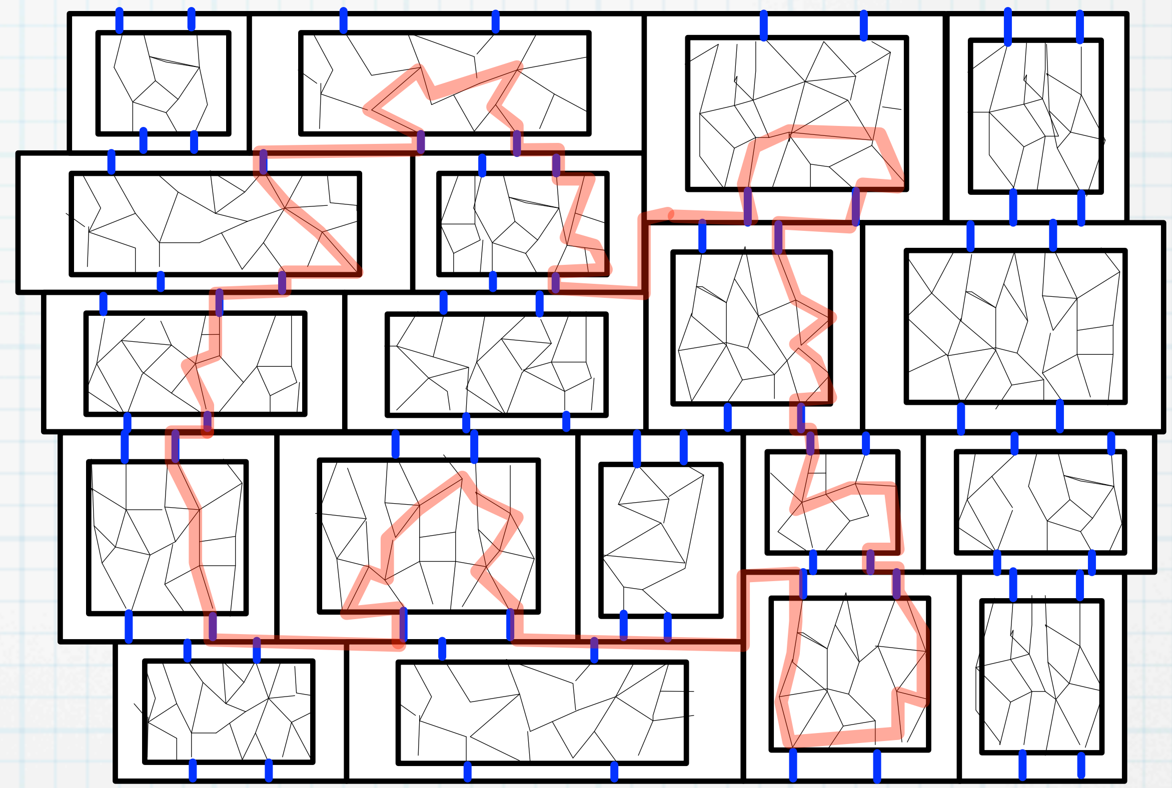

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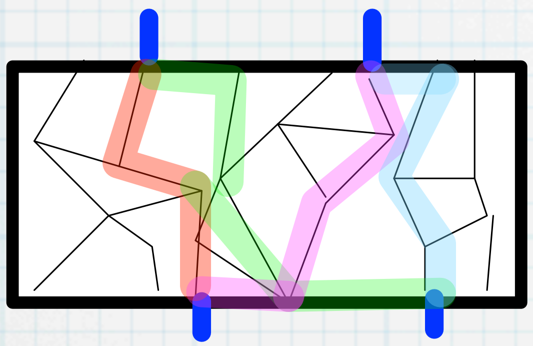

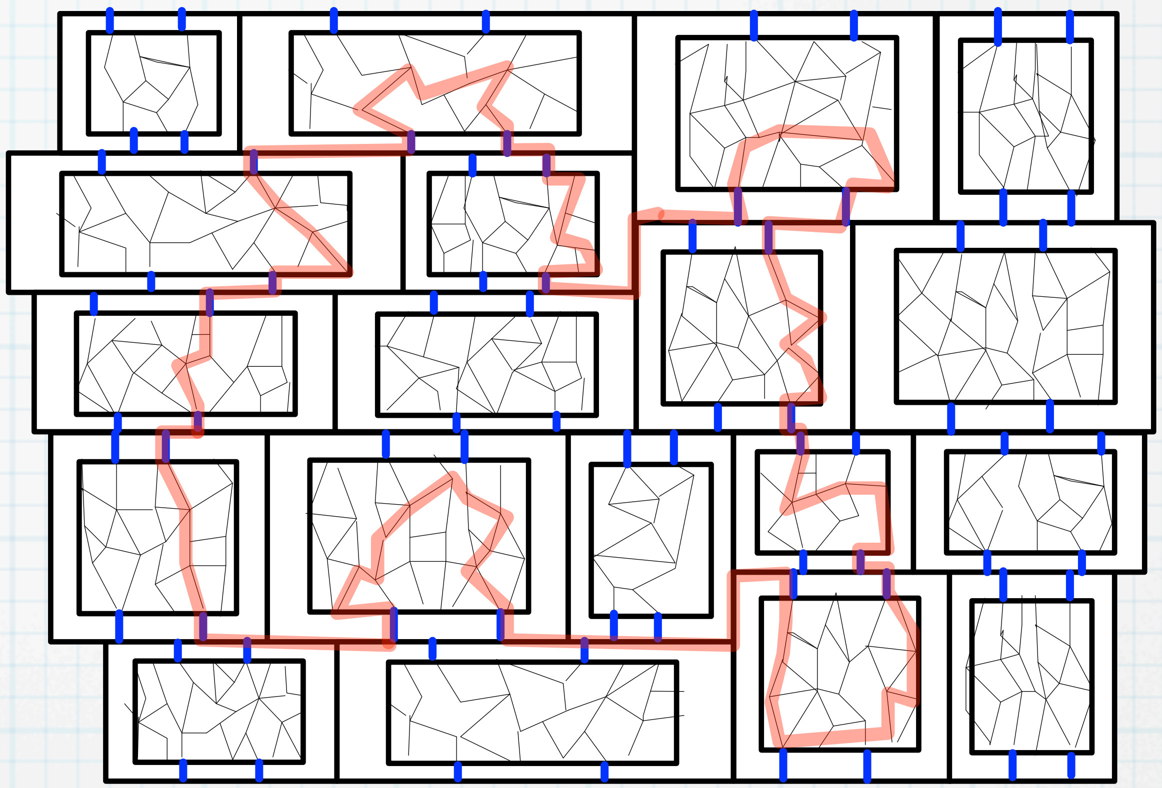

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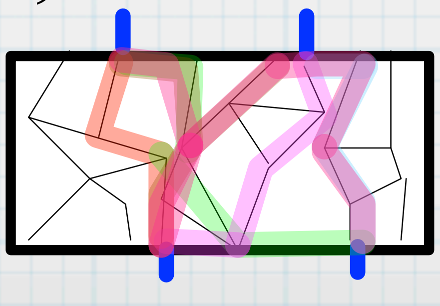

Remarks about spanner methodology

- Brick-decomposition construction takes *O(n log n)* time.

- There's a way to use brick decompositions that avoid some of the overhead of the spanner methodology, leads to better dependence on ε.

Steiner-tree approximation scheme has been implemented!
(Constants tweaked to get a fast algorithm that gets very good solutions.)
[Tazari, Müller-Hannemann, 2009]

- Brick decomposition can start with any *connected* subgraph, not just tree.

- To cope with disconnected subgraphs, *prize-collecting clustering* (MohammadTaghi's talk) has become an essential technique.

Open problems

Lots! We're just gaining steam.

Will need new techniques for these problems....

- Facility location problems
- Vehicle routing problems
- *k*-tree
- vertex-weighted Steiner tree
- directed Steiner tree
- two-edge connected Steiner
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Advertisements:

I'm writing a book about (some) optimization algorithms for planar graphs. Email me if you want to receive a draft.

Also, we are working to develop a library of reference implementations of planar-graph algorithms.