# Max Flow, Min Cut
COS 521

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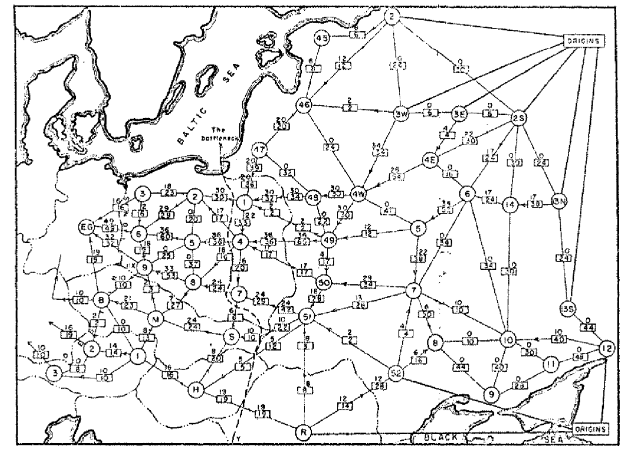
Fall 2005

## Soviet Rail Network, 1955



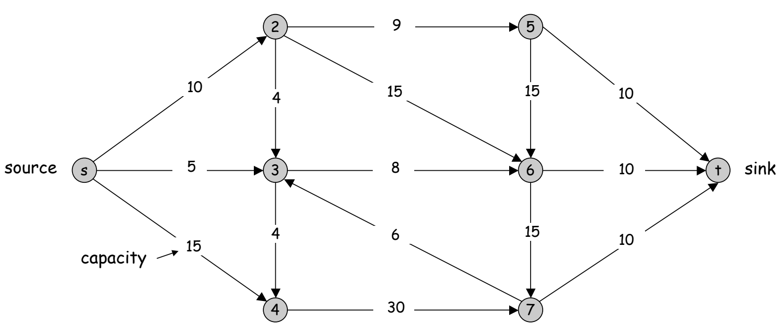

Reference: *On the history of the transportation and maximum flow problems.*
Alexander Schrijver in Math Programming, 91: 3, 2002.



## Minimum Cut Problem

Flow network.

- Digraph G = (V, E), nonnegative edge capacities c(e).
- Two distinguished nodes: s = source, t = sink.
- Assumptions: no parallel edges, no edges entering s or leaving t.

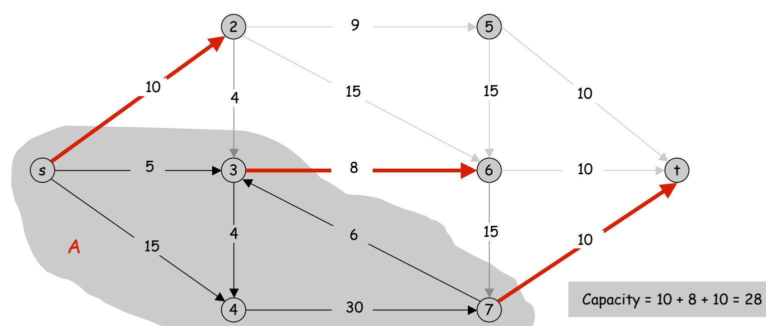





## Cuts

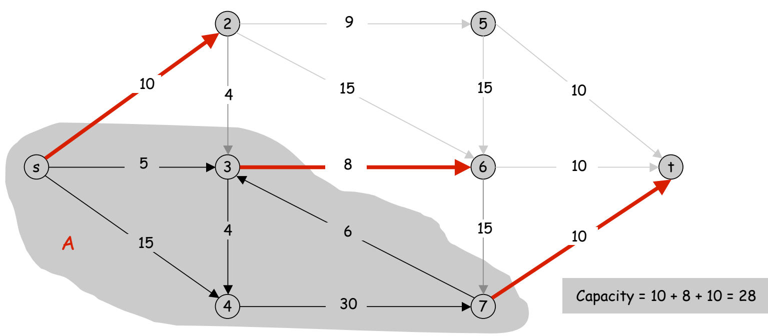
Def. An s-t cut is a partition (A, B) of V with s ∈ A and t ∈ B.

Def. The capacity of a cut (A, B) is: $cap(A, B) \;=\; \sum_{e \text{ out of } A} c(e)$



## Minimum Cut Problem

Min s-t cut problem. Find an s-t cut of minimum capacity.



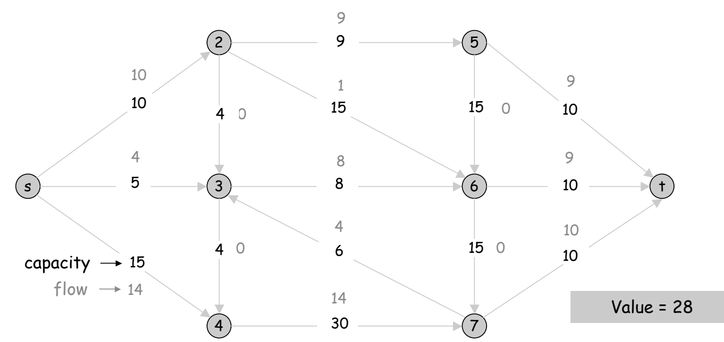



## Flows

Def. An s-t flow is a function that satisfies:

- For each $e \in E$: $0 \le f(e) \le c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

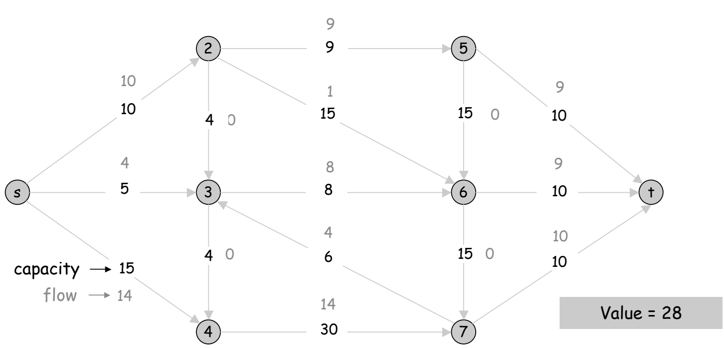
Def. The value of a flow f is: $val(f) = \sum_{e \text{ out of } s} f(e)$ .






## Maximum Flow Problem

Max flow problem. Find s-t flow of maximum value.

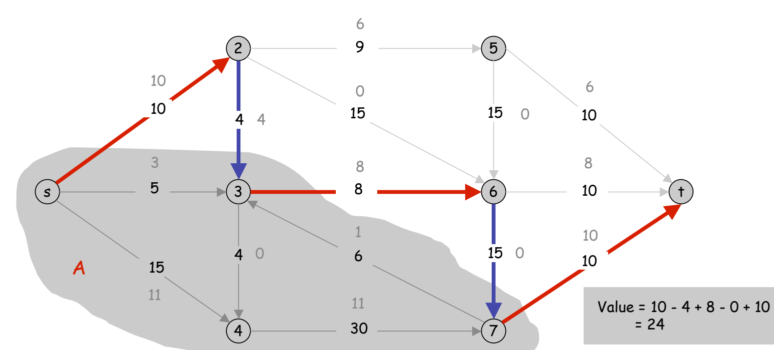





## Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = val(f)$$



## Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = val(f).$$

Pf.

$$\begin{aligned} val(f) &= \sum_{e \text{ out of } s} f(e) \\ &= \sum_{v \in A} \left( \sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right) \\ &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e). \end{aligned}$$

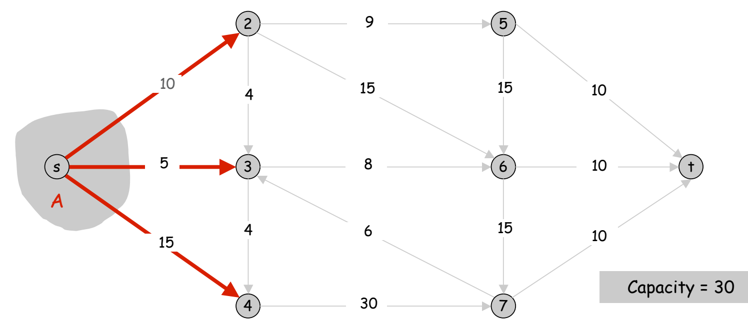
by flow conservation, all terms except v = s are 0 $\longrightarrow$



## Flows and Cuts

Weak duality. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 $\Rightarrow$ Flow value $\leq$ 30






## Flows and Cuts

Weak duality. Let f be any flow. Then, for any s-t cut (A, B) we have val(f) $\leq$ cap(A, B).

Pf.

$$\begin{aligned} val(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\leq \sum_{e \text{ out of } A} f(e) \\ &\leq \sum_{e \text{ out of } A} c(e) \\ &= \text{cap}(A, B) \end{aligned}$$

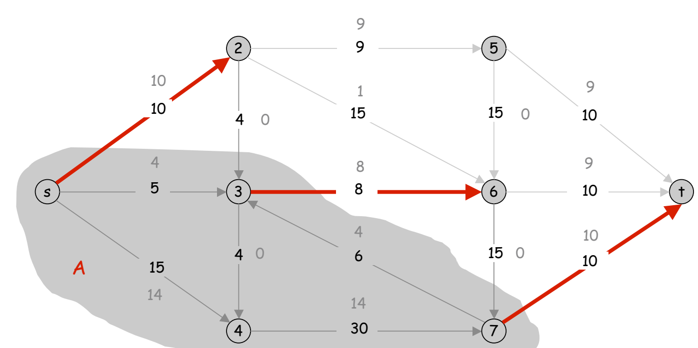
▪



## Certificate of Optimality

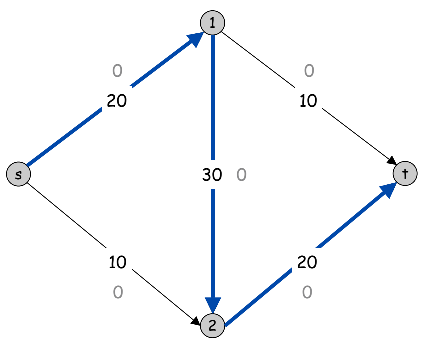
Corollary. Let f be any flow, and let (A, B) be any cut.
If val(f) = cap(A, B), then f is a max flow and (A, B) is a min cut.

Value of flow = 28
Cut capacity = 28 $\Rightarrow$ Flow value $\leq$ 28



## Towards a Max Flow Algorithm

Greedy algorithm.
- Start with f(e) = 0 for all edge $e \in E$.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.




Flow value = 0



## Towards a Max Flow Algorithm

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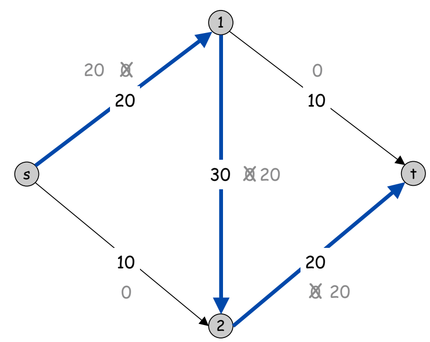


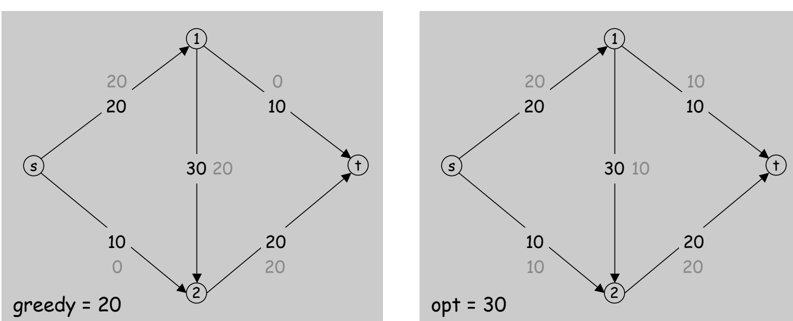

Flow value = 20



## Towards a Max Flow Algorithm

Greedy algorithm.
- Start with f(e) = 0 for all edge $e \in E$.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.

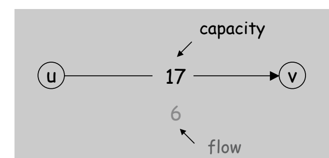
locally optimality $\not\Rightarrow$ global optimality





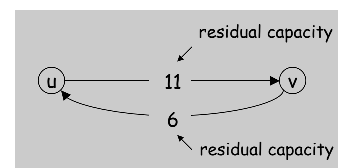

## Residual Graph

Original edge: e = (u, v) $\in$ E.
- Flow f(e), capacity c(e).



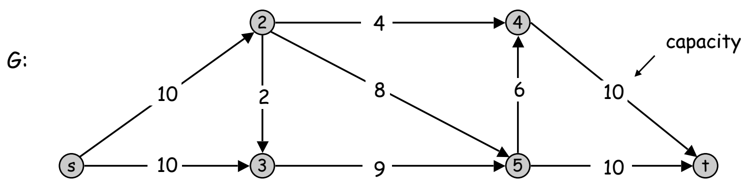

Residual edge.
- "Undo" flow sent.
- e = (u, v) and $e^R$ = (v, u).
- Residual capacity:

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$




Residual graph: $G_f = (V, E_f)$.
- Residual edges with positive residual capacity.
- $E_f = \{e : f(e) < c(e)\} \cup \{e^R : c(e) > 0\}$.

## Ford-Fulkerson Algorithm






## Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Elias-Feinstein-Shannon 1956, Ford-Fulkerson 1956]
The value of the max flow is equal to the value of the min cut.

Pf. Let f be a flow. Then TFAE:

(i) There exists a cut (A, B) such that val(f) = cap(A, B).
(ii) Flow f is a max flow.
(iii) There is no augmenting path relative to f.

(i) $\Rightarrow$ (ii) This was the corollary to weak duality lemma.

(ii) $\Rightarrow$ (iii) We show contrapositive.

- Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

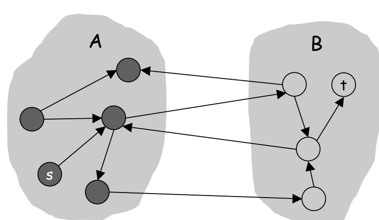


## Proof of Max-Flow Min-Cut Theorem

(iii) $\Rightarrow$ (i)

- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
- By definition of A, $s \in A$.
- By definition of f, $t \notin A$.

$$\begin{aligned} val(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\ &= \sum_{e \text{ out of } A} c(e) \\ &= cap(A,\ B) \ \blacksquare \end{aligned}$$




original network



## Analysis

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value f(e) and every residual capacities $c_f(e)$ remains an integer throughout the algorithm.

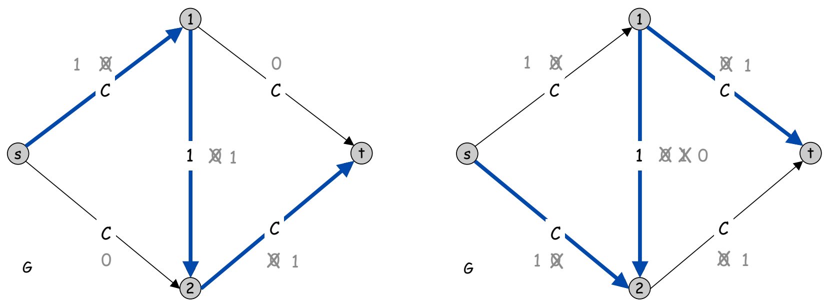
Theorem. The algorithm terminates in at most $val(f^*) \le nC$ iterations. It can be implemented in O(mnC) time.
Pf. Each augmentation increase value by at least 1. ▪

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.
Pf. Since algorithm terminates, theorem follows from invariant. ▪

## Ford-Fulkerson: An Exponential Input

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

m, n, and log C





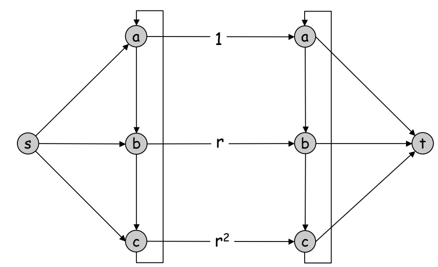
## Ford-Fulkerson: A Pathological Input

Q. Is Ford-Fulkerson algorithm finite?

Let $r = \frac{-1 + \sqrt{5}}{2} \approx 0.618...$ $[\ r^{n+2} = r^n - r^{n+1}\ ]$

Max flow $= 1 + r + r^2$.

Augmentations: first augment 1 unit, then repeatedly choose path with lowest capacity.






## Choosing Good Augmenting Paths

Goal: choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]

- Max bottleneck capacity.
- Sufficiently large bottleneck capacity.
- Fewest number of edges.



## Shortest Augmenting Path: Overview of Analysis

L1. The length of the shortest augmenting path never decreases.

L2. After at most m augmentations, the length of the shortest augmenting path strictly increases.

Theorem. The shortest augmenting path algorithm performs at most O(mn) augmentations. It can be implemented in $O(m^2n)$ time.

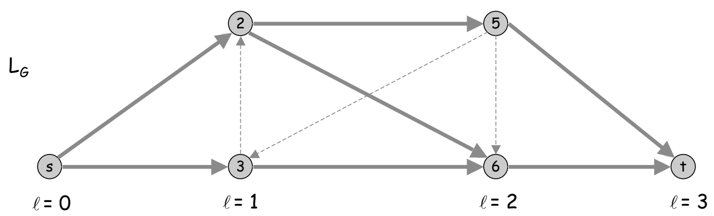
- O(m) time to find shortest augmenting path via BFS.
- O(m) augmentations for paths of exactly k edges. ▪

k < n

## Shortest Augmenting Path: Analysis

Level graph.
number of edges

- Define $\ell(v)$ = length of shortest s-v path in G.
- $L_G$ = (V, F) is subgraph of G that contains only those edges $(u, v) \in E$ with $\ell(v) = \ell(u) + 1$.
- Compute $L_G$ in O(m+n) time using BFS, deleting back and side edges.
- P is a shortest s-u path in G iff it is an s-u path $L_G$.

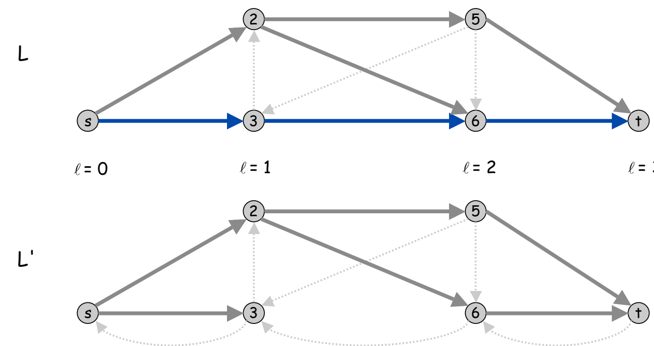





## Shortest Augmenting Path: Analysis

L1. The length of the shortest augmenting path never decreases.

- Let f and f' be flow before and after a shortest path augmentation.
- Let L and L' be level graphs of $G_f$ and $G_{f'}$.
- Only back edges added to $G_{f'}$.
- Path with back edge has length greater than previous length.

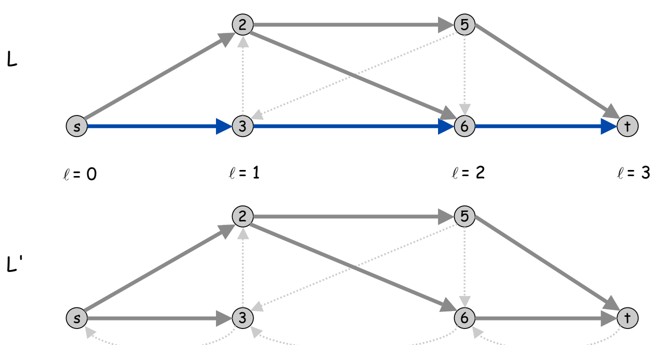





## Shortest Augmenting Path: Analysis

L2. After at most m augmentations, the length of the shortest augmenting path strictly increases.

- At least one edge (the bottleneck edge) is deleted from L after each augmentation.
- No new edges added to L until length of shortest path strictly increases.






## Shortest Augmenting Path: Review of Analysis

L1. The length of the shortest augmenting path never decreases.

L2. After at most m augmentations, the length of the shortest augmenting path strictly increases.

Theorem. The shortest augmenting path algorithm performs at most O(mn) augmentations. It can be implemented in $O(m^2n)$ time.

Note: $\Theta(mn)$ augmentations necessary on some networks.

- Try to decrease time per augmentation instead.
- Dynamic trees $\Rightarrow$ O(mn log n) [Sleator-Tarjan, 1983]
- Simple idea $\Rightarrow$ $O(mn^2)$

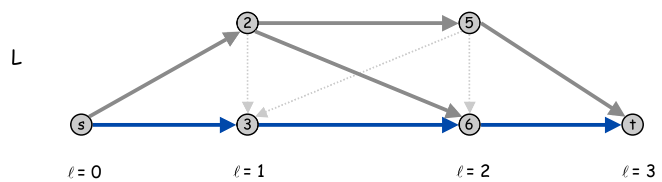
## Shortest Augmenting Path: Improved Version

Two types of augmentations.

- Normal augmentation: length of shortest path doesn't change.
- Special augmentation: length of shortest path strictly increases.

L3. Group of normal augmentations takes O(mn) time.

- Explicitly maintain level graph - it changes by at most 2n edges after each normal augmentation.
- Start at s, advance along an edge in L until reach t or get stuck.
  - if reach t, augment and delete at least one edge
  - if get stuck, delete node



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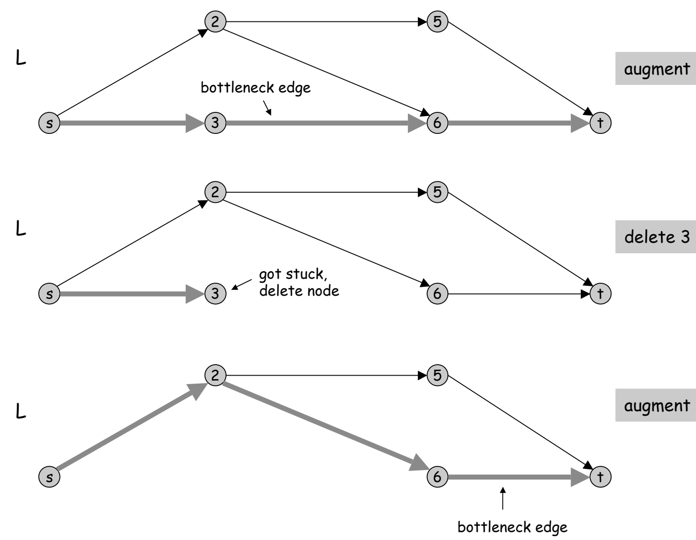
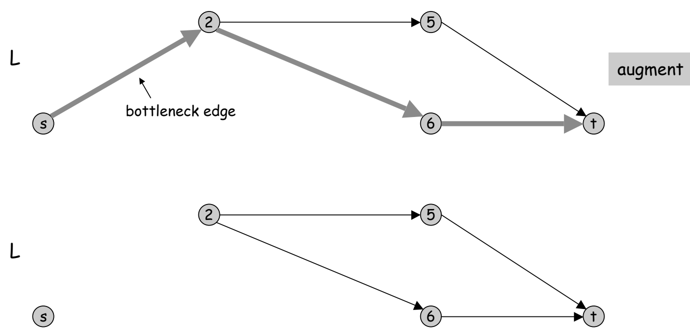





## Shortest Augmenting Path: Improved Version




Stop: length of shortest path must have strictly increased.



## Shortest Augmenting Path: Improved Version

Two types of augmentations.

- Normal augmentation: length of shortest path doesn't change.
- Special augmentation: length of shortest path strictly increases.

L3. Group of normal augmentations takes O(mn) time.

- At most n advance steps before you either
  - get stuck: delete a node from level graph
  - reach t: augment and delete an edge from level graph

Theorem. Algorithm runs in $O(mn^2)$ time.

- O(mn) time between special augmentations.
- At most n special augmentations.

## History of Worst-Case Running Times

| Year | Discoverer | Method | Asymptotic Time |
|---|---|---|---|
| 1951 | Dantzig | Simplex | $m\, n^2\, C$ † |
| 1955 | Ford, Fulkerson | Augmenting path | $m\, n\, C$ † |
| 1970 | Edmonds-Karp | Shortest path | $m^2\, n$ |
| 1970 | Edmonds-Karp | Fattest path | $m \log C\, (m \log n)$ † |
| 1970 | Dinitz | Improved shortest path | $m\, n^2$ |
| 1972 | Edmonds-Karp, Dinitz | Capacity scaling | $m^2 \log C$ † |
| 1973 | Dinitz-Gabow | Improved capacity scaling | $m\, n \log C$ † |
| 1974 | Karzanov | Preflow-push | $n^3$ |
| 1983 | Sleator-Tarjan | Dynamic trees | $m\, n \log n$ |
| 1986 | Goldberg-Tarjan | FIFO preflow-push | $m\, n \log (n^2 / m)$ |
| . . . | . . . | . . . | . . . |
| 1997 | Goldberg-Rao | Length function | $m^{3/2} \log (n^2 / m) \log C$ † <br> $mn^{2/3} \log (n^2 / m) \log C$ † |

† Edge capacities are between 1 and $C$.

next time



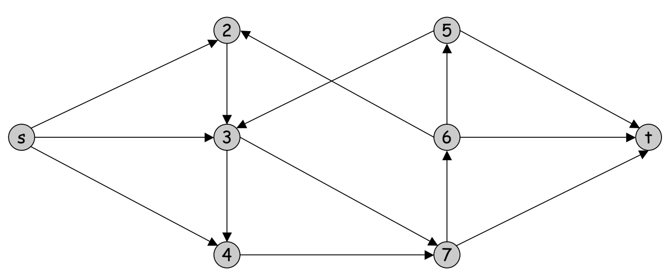
# Disjoint Paths

## Edge Disjoint Paths

Disjoint path problem. Given a digraph $G = (V, E)$ and two nodes $s$ and $t$, find the max number of edge-disjoint s-t paths.

Def. Two paths are edge-disjoint if they have no edge in common.

Ex: communication networks.



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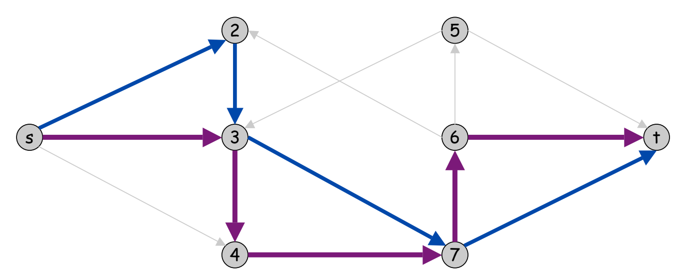
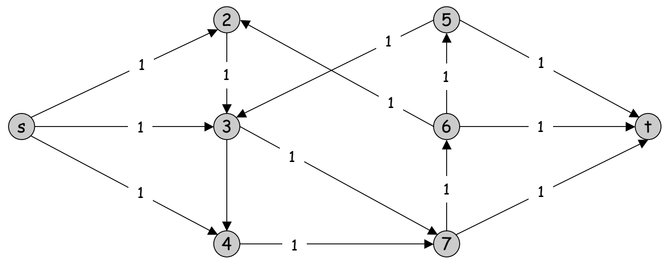


## Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



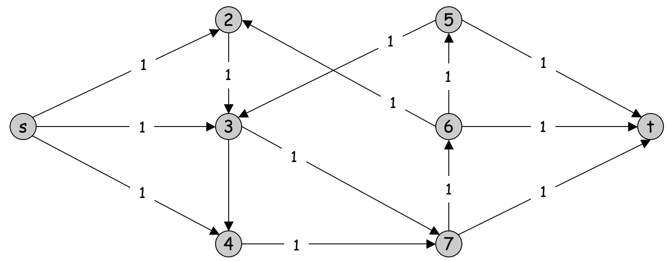



## Unit Capacity Networks

Unit capacity network.

- Every edge capacity is one.
- If G is unit capacity, so is $G_f$, assuming f is 0-1 flow.

Ex: disjoint paths, bipartite matching.

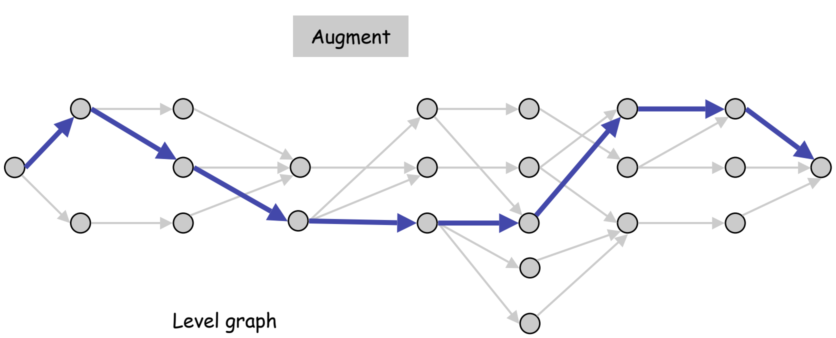





## Unit Capacity Networks

Lemma 1. Phase of normal augmentations takes O(m) time.

- Start at s, advance along an edge in L until reach t or get stuck.
  - if reach t, augment and delete all edges on path
  - if get stuck, delete node and retreat to previous node



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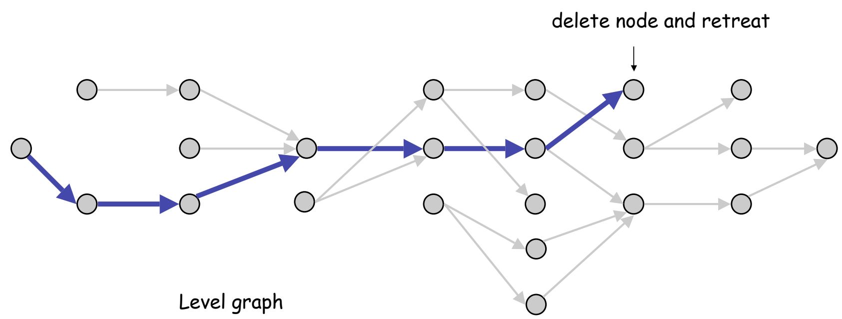

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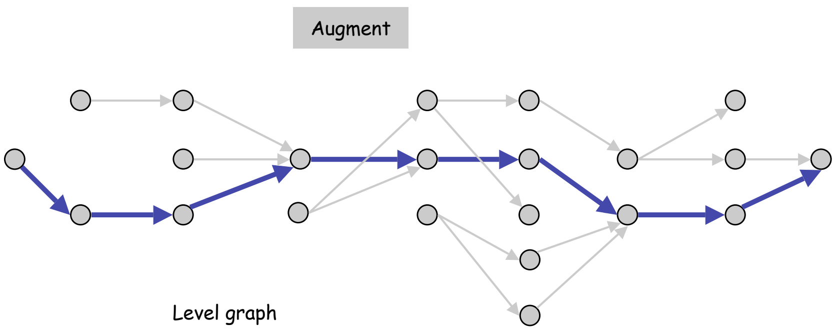




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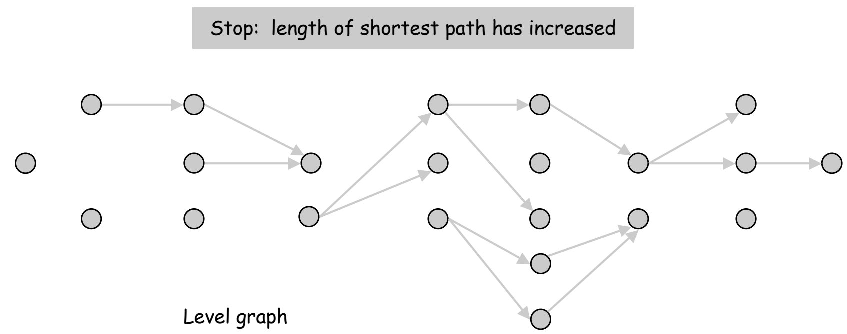




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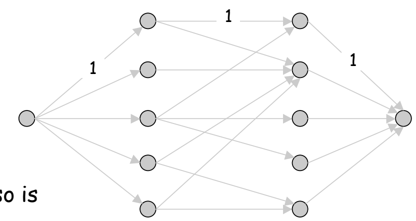
- O(m) running time.
  - O(m) to create level graph
  - O(1) per edge, since each edge traversed at most once
  - O(1) per node deletion



# Unit Capacity Simple Networks

## Unit Capacity Simple Networks

**Unit capacity simple network.**
- Every edge capacity is one.
- Every node has either:
  - (i) at most one incoming edge, or
  - (ii) at most one outgoing edge.
- If $G$ is simple unit capacity, then so is $G_f$, assuming f is 0-1 flow.

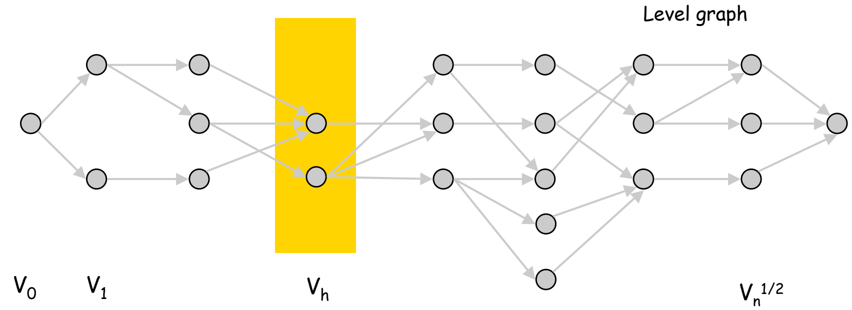



**Theorem.** Shortest augmenting path algorithm runs in $O(m\, n^{1/2})$ time.
- L1. Each phase of normal augmentations takes $O(m)$ time.
- L2. After at most $n^{1/2}$ phases, $val(f) \geq val(f^*) - n^{1/2}$.
- L3. After at most $n^{1/2}$ additional augmentations, flow is optimal.



## Unit Capacity Simple Networks

**Lemma 2.** After at most $n^{1/2}$ phases, $val(f) \geq val(f^*) - n^{1/2}$.
- After $n^{1/2}$ phases, length of shortest augmenting path is $> n^{1/2}$.
- Level graph has more than $n^{1/2}$ levels.
- Let $1 \leq h \leq n^{1/2}$ be layer with min number of nodes: $|V_h| \leq n^{1/2}$.



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- Level graph has more than $n^{1/2}$ levels.
- Let $1 \leq h \leq n^{1/2}$ be layer with min number of nodes: $|V_h| \leq n^{1/2}$.
- $A := \{v : \ell(v) < h\} \cup \{v : \ell(v) = h$ and $v$ has $\leq 1$ outgoing residual edge$\}$.
- $cap_f(A, B) \leq |V_h| \leq n^{1/2} \Rightarrow val(f) \geq val(f^*) - n^{1/2}$.