# Generalized Min Cost Circulation

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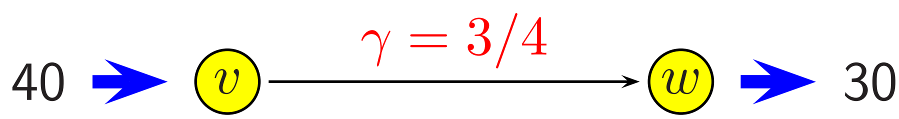
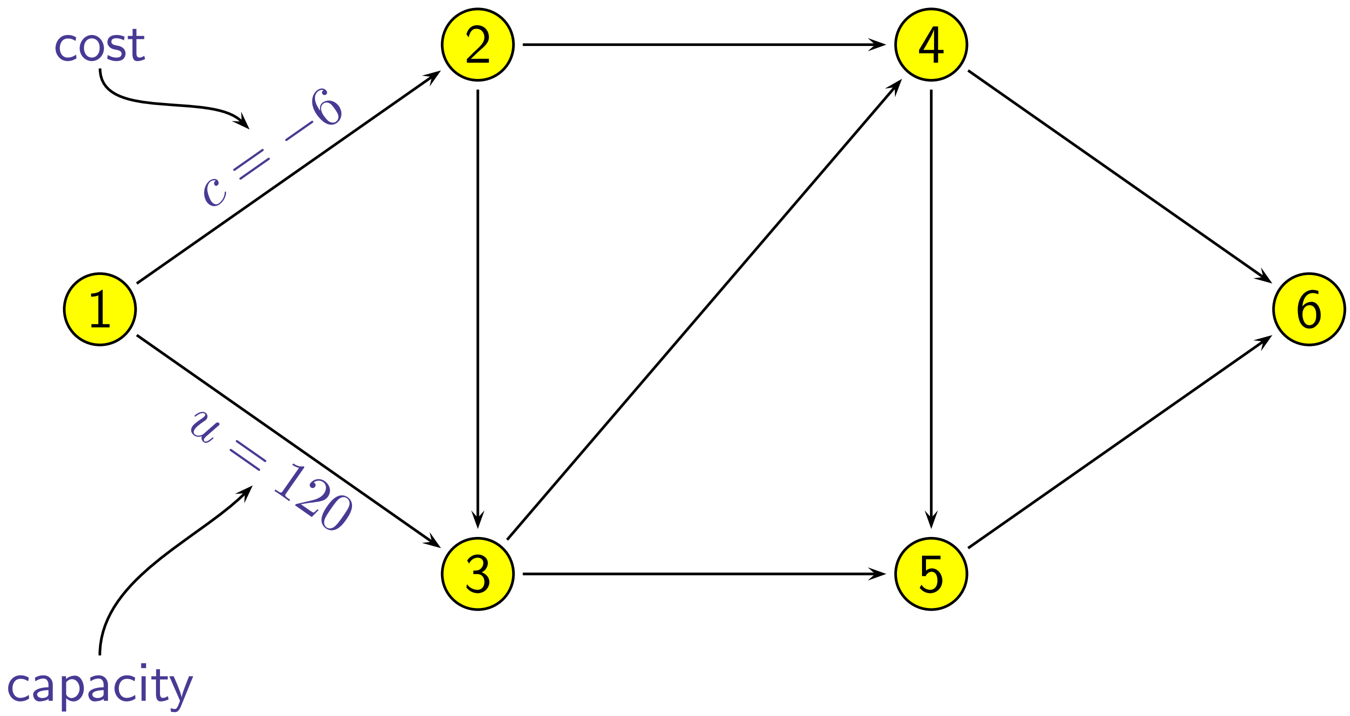


# Outline of Talk

- Problem, applications, and history
- Combinatorial structure
- Algorithms and analyses
- Conclusions

# Min Cost Circulation




Find flow of minimum cost

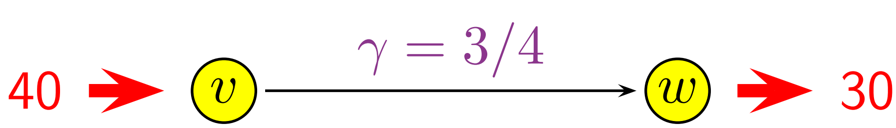
- capacity constraints
- flow conservation constraints

# Generalized

$\gamma = 3/4$

gain/loss factor

(generalized)



# LP Formulation

$$
\begin{array}{lll}
\min & \sum_{vw} c_{vw} g_{vw} & \\
\forall v: & \sum_{w} g_{vw} - \sum_{w} \gamma_{wv} g_{wv} & = \underbrace{b}_{\text{w.l.o.g. } b=0} \\
\forall vw: & 0 \le g_{vw} & \le u_{vw}
\end{array}
$$

| Input Data | |
|---|---|
| $0 < \gamma_{vw} =$ | gain factor |
| $c_{vw} =$ | cost |
| $0 < u_{vw} =$ | capacity |
| $b_v =$ | supply / demand vector |

A circulation is function $g \geq 0$ that satisfies the flow conservation constraints

It is feasible if it also satisfies the capacity constraints

# Applications

*I discovered that a whole range of problems of the most diverse character relating to the scientific organization of production lead to the formulation of a single group of mathematical problems.*

Kantorovich '39 (MS 1960, pp. 366-422)

"do not arise in the economy of a capatilistic society" since "the majority of enterprises work at half capacity. There the choice of output is determined not by the plan but by the interests and profits of individual capatilists."

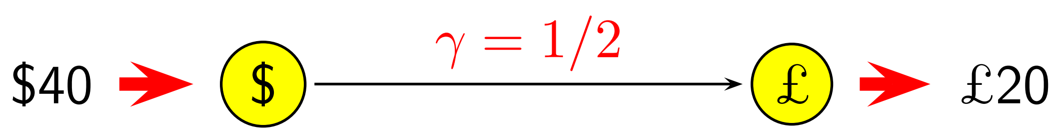
# Applications

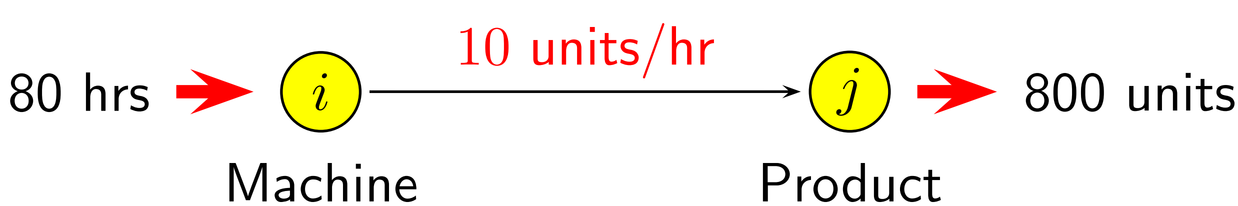
Commodity leaks:

leaky pipes, defects, production yields, evaporation, spoilage, theft, taxes

Transform one commodity into another:

currency conversion, machine loading, fuel utilization, crop management, aircraft fleet assignment

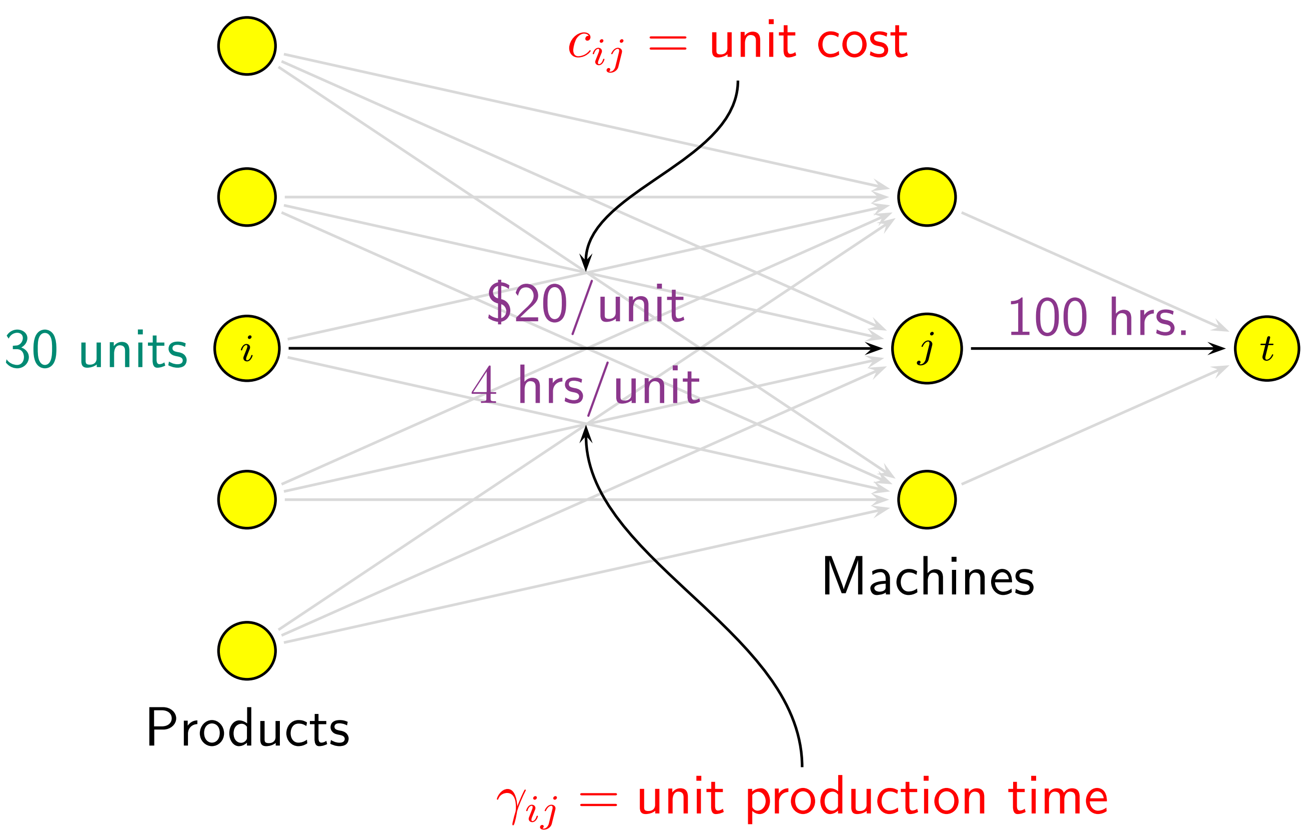


# Machine Loading

Schedule $p$ products on $q$ machines to minimize cost

- machines have limited capacity
- producing 1 unit of product $i$ on machine $j$ takes $\gamma_{ij}$ hours and costs $c_{ij}$ dollars
- find optimal product mix for each machine



# General Approach

- Can be solved by general purpose LP techniques
  - linear algebraic methods
- We take combinatorial (flow-based) approach
  - exploit underlying network structure
  - superior algorithms for many network problems
- Why generalized flows are harder:
  - total supply $\neq$ total demand
  - no max-flow min-cut theorem
  - no integrality theorem

# Problem History

**Kantorovich '39** dual simplex

**Dantzig '62** network simplex

**Jewell '62** primal-dual

**Vaidya '89** interior point – $\mathcal{O}^*(m^{1.5}n^2 \log B)$ time

**No previous polynomial combinatorial algorithm**

**No strongly polynomial (approximation) algorithm**

| | |
|---|---|
| $m = \#$ arcs | $\mathcal{O}^* =$ hides polylog($m$) factors |
| $n = \#$ nodes | $B =$ biggest integer |

# New Results

- Generalized min cost circulation
  - first polynomial combinatorial algorithm
  - version without costs previously solved:

    Goldberg, Plotkin, Tardos '88
    Cohen, Megiddo '92
    Radzik '93a, '93b
    Goldfarb, Jin '95,'96
    Goldfarb, Jin, Orlin '97
    Tardos, W '98
    Fleischer, W '99

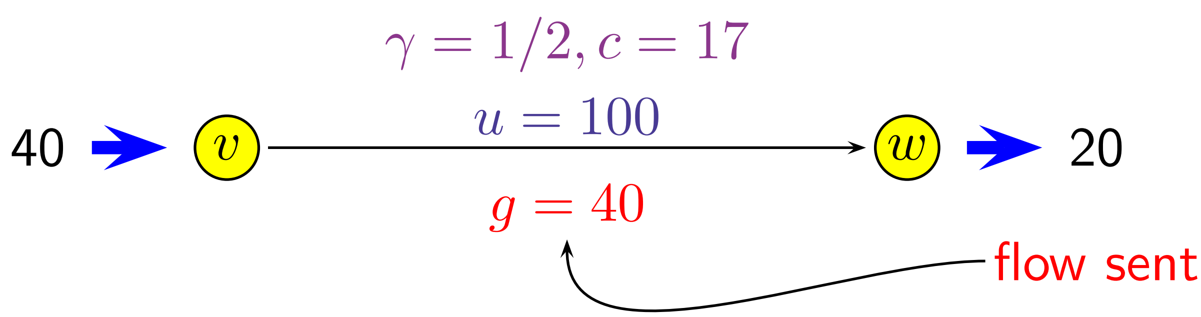
  - first strongly polynomial approximation scheme
  - primal "cycle-canceling" algorithm

- Two variable per inequality linear programs
  - first polynomial combinatorial algorithm
  - feasibility version previously solved:

    Aspvall, Shiloach '80
    Megiddo '83
    Cohen, Megiddo '94
    Hochbaum, Naor '94

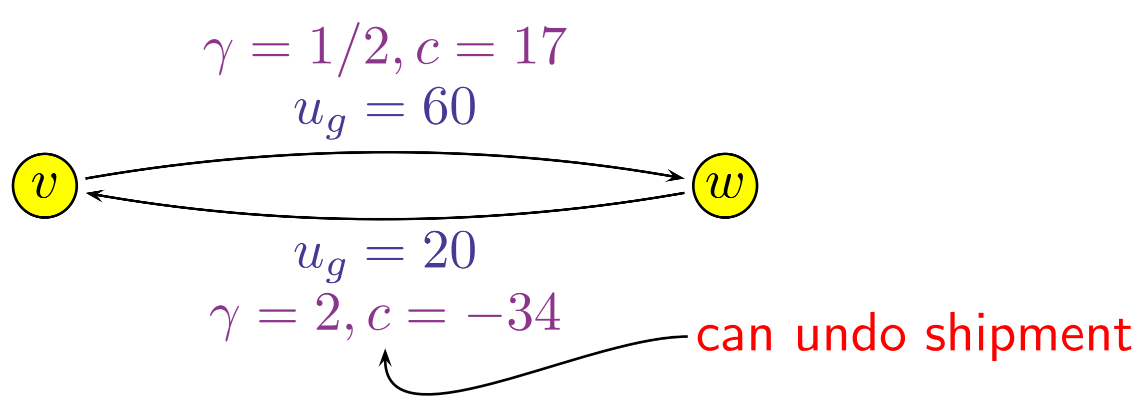
# Residual Network

**Original network:** $G = (V, E, u)$



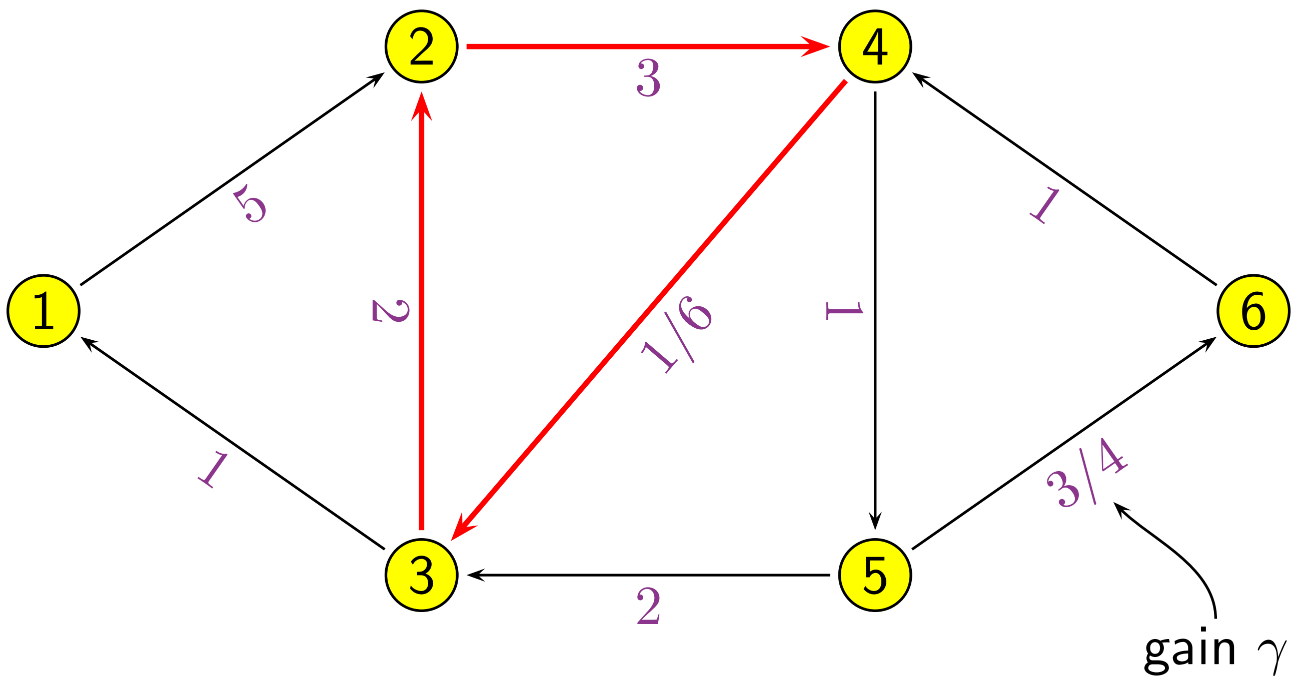

$\gamma =$ gain, $c =$ cost, $u =$ capacity

**Residual network:** $G_g = (V, E_g, u_g)$

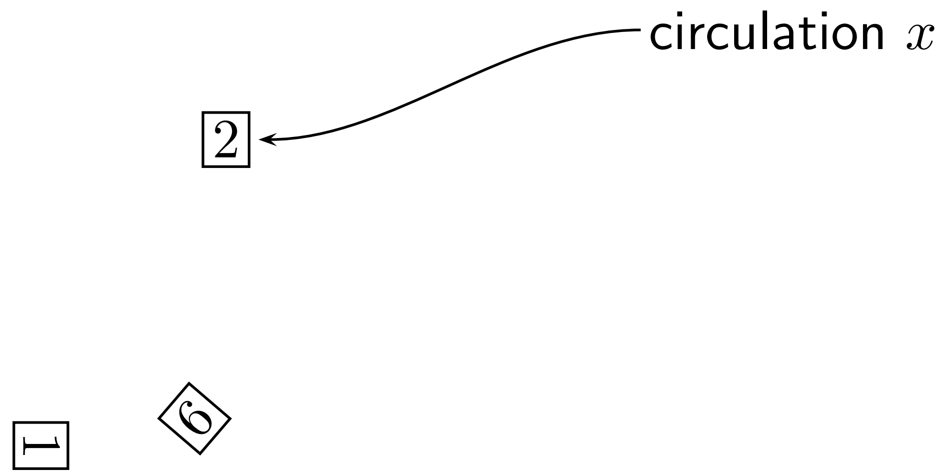


# Unit-Gain Cycles

Residual network $G_g$:




**Unit-gain cycle:** cycle with $\prod_{e \in \Gamma} \gamma(e) = 1$



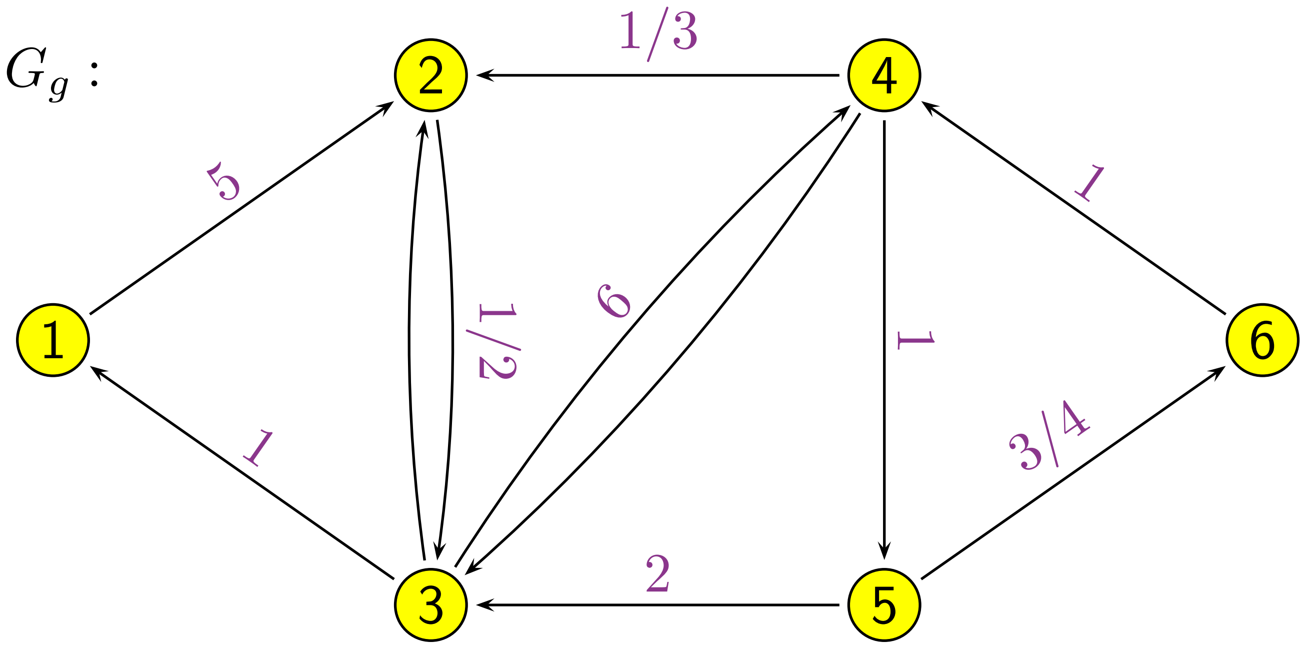

**Corresponding circulation:**

$x_{24} = 2, x_{43} = 6, x_{32} = 1$

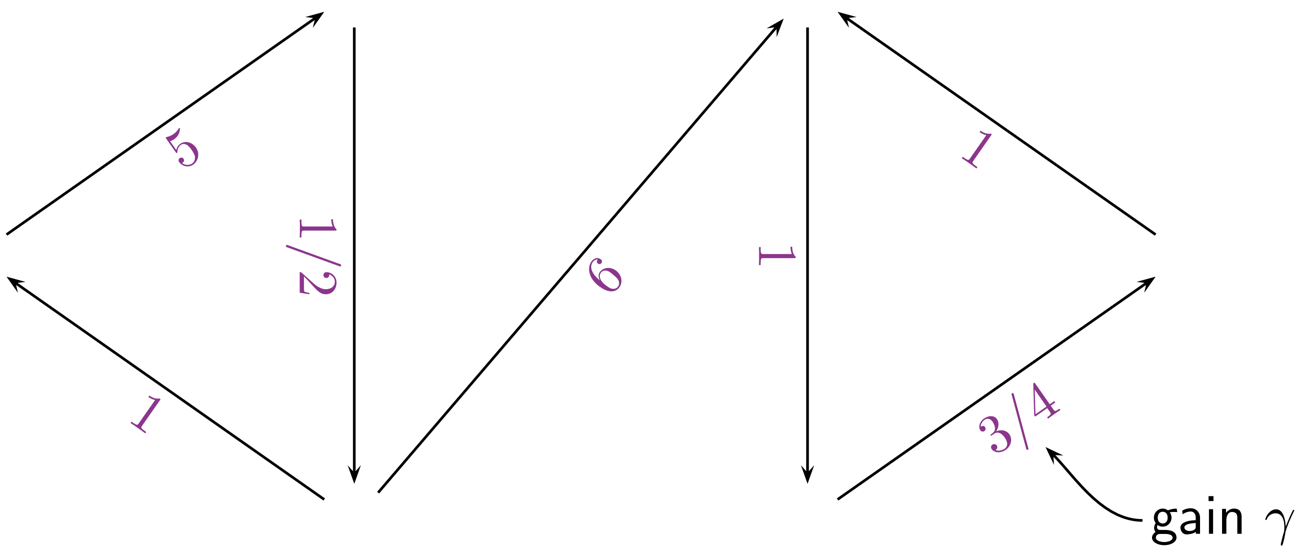
**Negative cost:** if $\sum_{vw} c_{vw} x_{vw} < 0$

# Optimality Conditions

optimal $\iff$ no negative cost unit-gain cycles?



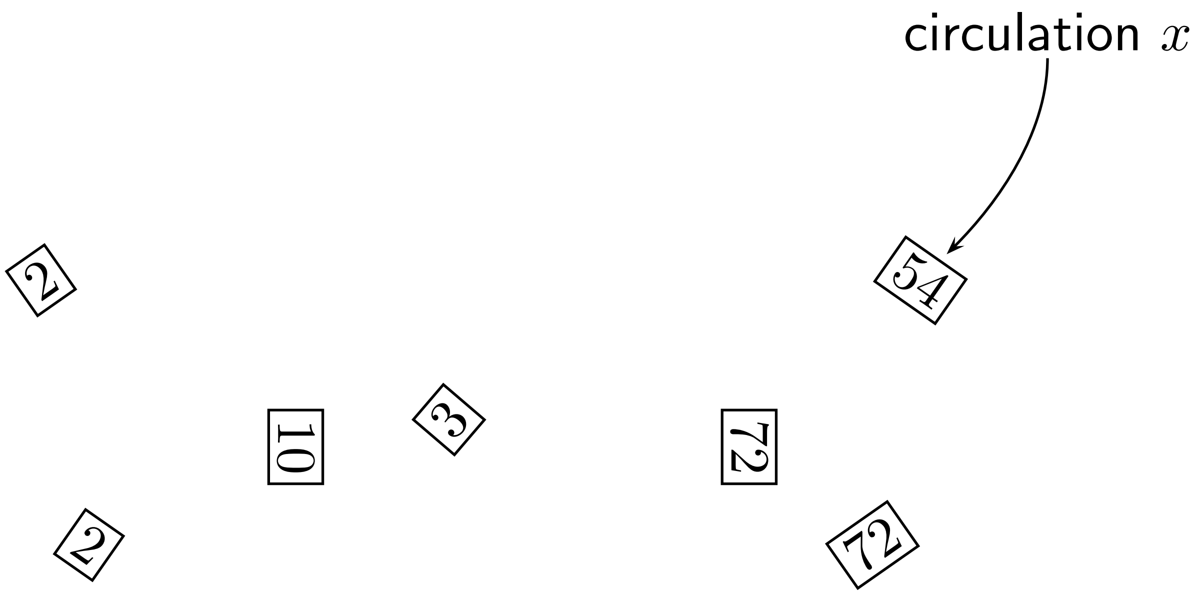
# Bicycles




**Flow-generating cycle:** cycle with $\prod_{e \in \Gamma} \gamma(e) > 1$

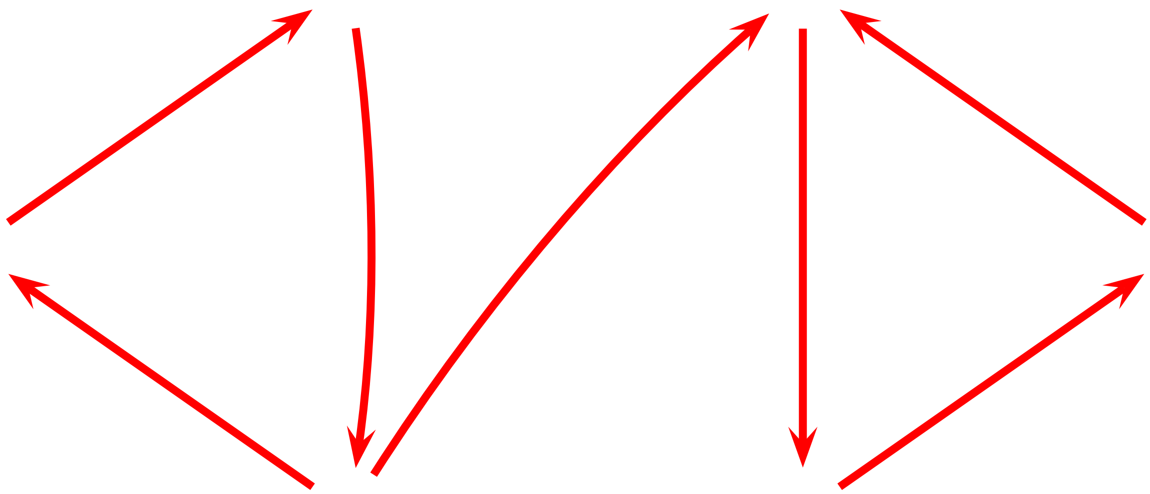
**Flow-absorbing cycle:** cycle with $\prod_{e \in \Gamma} \gamma(e) < 1$

**Bicycle:** flow-generating cycle $\rightsquigarrow$ flow-absorbing cycle




**Negative cost:** if $\sum_{vw} c_{vw} x_{vw} < 0$

No.



**Theorem [1960s]:** optimal $\iff$ no negative cost residual unit-gain cycles or bicycles

# Circuit Canceling Algorithm

We generalize Klein's cycle-canceling algorithm:

> A circuit is a circulation that sends flow only on the arcs of a single unit-gain cycle or bicycle.

Initialize $g \leftarrow 0$
**repeat**
  Cancel a negative cost circuit in $G_g$
  Update $g$
**until** optimal

**Complexity:** Very bad!

**Can we even find a negative cost circuit?**

- NP-hard even to detect a unit-gain cycle

# Min Mean Circuit Canceling Algorithm

We generalize the Goldberg-Tarjan algorithm:

The mean cost of a cycle $\Gamma$ is $\frac{\sum_{e \in \Gamma} c(e)}{|\Gamma|}$

The **mean cost** of a circuit $x$ is $\frac{\sum_e c(e)x(e)}{\sum_e x(e)}$

Initialize $g \leftarrow 0$
**repeat**
  Cancel a min mean cost circuit in $G_g$
  Update $g$
**until** optimal

**Theorem:** Finite complexity

# Min Ratio Cycle Algorithm

Wallacher's traditional min cost flow algorithm:

The ratio of cycle $\Gamma$ is $\dfrac{\sum_{e \in \Gamma} c(e)}{\sum_{e \in \Gamma} \frac{1}{u(e)}}$

Initialize $f \leftarrow 0$
**repeat**
  Cancel a min ratio cycle in $G_f$
  Update $f$
**until** optimal

## Related work:

- Wallacher, Zimmerman '97 submodular flow
- Goldberg, Rao '98 adaptive length function, max flow
- McCormick, Shioura '99 unimodular linear programs
- Schulz, Weismantel '99 special class of integer programs

# Min Ratio Circuit Algorithm

We generalize Wallacher's algorithm:

The ratio of cycle $\Gamma$ is $\dfrac{\sum_{e \in \Gamma} c(e)}{\sum_{e \in \Gamma} \frac{1}{u(e)}}$

The ratio of a circuit $x$ is $\dfrac{\sum_e c(e)x(e)}{\sum_e \frac{x(e)}{u(e)}}$

Initialize $g \leftarrow 0$
**repeat**
  Cancel a min ratio circuit in $G_g$
  Update $g$
**until** $\epsilon$-optimal

# Overview of Analysis

**Fact:** Canceling circuit with ratio $\mu$ improves objective by at least $\mu$

**Key Lemma:** There exists a circuit with good ratio

- captures at least $1/m$ fraction of remaining profit
- $m \log(1/\epsilon)$ iterations

**Lemma:** Can find min ratio circuit in $\mathcal{O}^*(mn^3)$ time

**Theorem:** PTAS, can find $\epsilon$-optimal circulation in $\mathcal{O}^*(m^2 n^3 \log(1/\epsilon))$ time

**Lemma:** $\epsilon = B^{-4m} \Rightarrow$ can "round" to optimal vertex

# Analysis

**Fact:** Canceling circuit $x$ with ratio $\mu$ improves objective by at least $\mu$

$$\mu \;=\; \sum_e c(e)x(e) \;/\; \underbrace{\sum_e \frac{x(e)}{u(e)}}_{\geq 1}$$

$x$ canceled $\Rightarrow$ rescaled so that some arc is saturated

$$\underbrace{\sum_e c(e)x(e)}_{\text{improvement}} \;\leq\; \mu \;\leq\; 0$$

# Analysis

**Lemma:** There exists a good circuit

- geometric improvement
- captures at least $1/m$ fraction of remaining profit

$x$ = min ratio circuit, $\mu$ = value

$x^*$ = optimal circulation

$$\mu \;\leq\; \underbrace{\sum_e c(e)x^*(e)}_{\text{OPT}} \;/\; \underbrace{\sum_e \frac{x^*(e)}{u(e)}}_{\leq m}$$

# Finding a Negative Cost Circuit

- Suffices to find negative cost circulation

$$\sum_{vw} c_{vw} x_{vw} < 0, \quad x \text{ circulation} \qquad (I)$$

$$\forall vw \ : \quad \underbrace{c_{vw} + \pi_v - \gamma_{vw} \pi_w}_{\text{reduced cost}} \ \geq 0 \qquad (II)$$

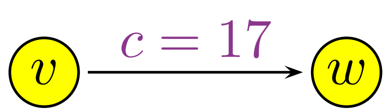
- Farkas: $\exists$ negative cost circuit iff $(II)$ infeasible
- Can detect feasibility of $(II)$ using Bellman-Ford updates since it has two variables per inequality....

# Detecting Feasibility of System $(II)$

Build intuition from familiar TVPI system

**Shortest path:** Bellman-Ford-Moore '58

- node labels $\pi_v$ = cost of cheapest $s$-$v$ path
- inequalities: $\forall\ vw: \quad \pi_w \le \pi_v + c_{vw}$




- $\pi_s = 0$, update $\pi_w \leftarrow \min_{vw}\{\pi_w,\ \pi_v + c_{vw}\}$
- negative cost cycles identify infeasibilities

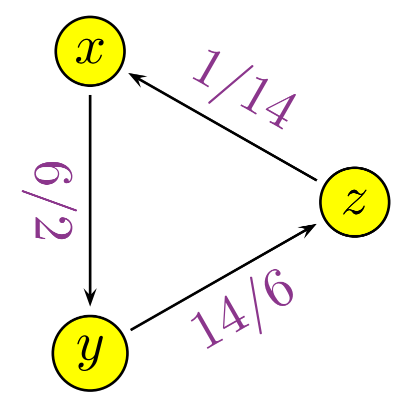
# Detecting Feasibility of System $(II)$

Idea: extend shortest path algorithm + binary search

**TVPI feasibility:** Shostak '81

- general TVPI: $\pi_w \leq \gamma_{vw}\pi_v + c_{vw}$
- guess $\pi_s \leq 17$ and binary search
- update $\pi_w \leftarrow \min_{vw}\{\pi_w,\ \gamma_{vw}\pi_v + c_{vw}\}$
- negative cost circuits identify infeasibilities
- what about non unit-gain cycles?

$$\begin{array}{rcrl} 2\pi_x & \leq & 6\pi_y & + \ \ 5 \\ 6\pi_y & \leq & 14\pi_z & + 16 \\ 14\pi_z & \leq & \pi_x & \\ \hline 2\pi_x & \leq & \pi_x & + 21 \end{array}$$



# Finding a Min Ratio Circuit

- Consider parametric system of inequalities

$$\sum_e c(e)x(e) - \mu \sum_e \frac{x(e)}{u(e)} < 0, \;\; x \text{ circuit} \qquad (I_\mu)$$

- $(I_\mu)$ feasible iff $\mu > \mu^*$

- Test feasibility of $(I_\mu)$ in $\mathcal{O}^*(mn^2)$ time since dual system is TVPI

- Binary search for $\mu^*$ in $\mathcal{O}(m \log B)$ iterations

- Megiddo's parametric search $\Rightarrow \mathcal{O}^*(mn^3)$ time

# Rounding to a Vertex

**Goal:** Given feasible circulation $g$, "round" to a vertex without increasing objective value.

**Notation:** $S(g) := \{e \in E \,:\, 0 < g(e) < u(e)\}$

**Lemma:** $g$ is a vertex $\Leftrightarrow$ $S(g)$ has no bicycles or unit-gain cycles.

Input: feasible circulation $g$
**repeat**
Cancel a bicycle or unit-gain cycle in subgraph $S(g)$ in direction that does not increase objective value, and update $g$.
**until** $S(g)$ has no bicycles or unit-gain cycles

**Complexity:** $\mathcal{O}(m^2 n)$.

- At most $m$ iterations, since $|S(g)|$ strictly decreases.
- Can find a bicycle or unit-gain cycle in $\mathcal{O}(mn)$ time using 2 shortest path computations.

# A Faster Scaling Version

**Idea:** Cancel approximately min ratio circuits

**Improvement:** cancel negative cost circuits instead of min ratio circuits (factor $n$ speedup)

Initialize $g \leftarrow 0,\ \mu \leftarrow$ min ratio circuit value
$\bar{c}(e) \leftarrow c(e) - \frac{\mu}{2u_g(e)}$
**repeat**
  **while** $\exists$ negative $\bar{c}$-cost circuit **do**
    Cancel such a circuit and update $g, \bar{c}$
  $\mu \leftarrow \mu/2$
**until** $\epsilon$-optimal

**Theorem:** $\mathcal{O}^*(m^2 n^2 \log(1/\epsilon))$ approximation scheme
$\mathcal{O}^*(m^3 n^2 \log B)$ exact algorithm

# $\underbrace{\text{TVPI}}_{\text{two variables per inequality}}$ Linear Programming

Dual of generalized flow problem with costs is monotone TVPI linear program:

$$\begin{array}{lll} \max & \sum_v b_v \pi_v & \\ \forall\, vw: & c_{vw} - \pi_v + \gamma_{vw}\pi_w \geq 0 & \end{array} \qquad (D)$$

Allow negative gain factors $\Rightarrow$ general TVPI

**Theorem:** $\mathcal{O}^*(m^3 n^2 \log B)$ combinatorial algorithm for optimizing TVPI liner programs

# Closing Remarks

## Conclusions

- Generalized min cost circulation problem
  - first combinatorial polynomial algorithm
  - first strongly polynomial approximation scheme
  - primal method

- TVPI linear programming
  - first combinatorial polynomial algorithm

## Open Problems

- dualize
- polynomial simplex variant
- strongly polynomial complexity
- generalized multicommodity flow

# Optimization is as Easy as Feasibility

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = 0 \\ & l \le x \le u \end{array} \qquad (P)$$

$$\begin{array}{l} Ax = 0 \\ l' \le x \le u' \end{array} \qquad (F)$$

**Theorem [McCormick, W]:** Can solve $(P)$ using a polynomial number of calls to $(F)$

**Idea:** dualize min-ratio algorithm and generalize to LP