Chapter 12
Local Search
Coping With NP-Hardness

Q. Suppose I need to solve an NP-hard problem. What should I do?
A. Theory says you're unlikely to find poly-time algorithm.

Must sacrifice one of three desired features.
- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve arbitrary instances of the problem.
12.1 Landscape of an Optimization Problem
Gradient Descent: Vertex Cover

**VERTEX-COVER.** Given a graph \( G = (V, E) \), find a subset of nodes \( S \) of minimal cardinality such that for each \( u-v \) in \( E \), either \( u \) or \( v \) (or both) are in \( S \).

**Neighbor relation.** \( S \sim S' \) if \( S' \) can be obtained from \( S \) by adding or deleting a single node. Each vertex cover \( S \) has at most \( n \) neighbors.

**Gradient descent.** Start with \( S = V \). If there is a neighbor \( S' \) that is a vertex cover and has lower cardinality, replace \( S \) with \( S' \).

**Remark.** Algorithm terminates after at most \( n \) steps since each update decreases the size of the cover by one.
Gradient Descent: Vertex Cover

**Local optimum.** No neighbor is strictly better.

- Optimum = center node only
  - Local optimum = all other nodes

- Optimum = all nodes on left side
  - Local optimum = all nodes on right side

- Optimum = even nodes
  - Local optimum = omit every third node
Local Search

Local search. Algorithm that explores the space of possible solutions in sequential fashion, moving from a current solution to a "nearby" one.

Neighbor relation. Let \( S \sim S' \) be a neighbor relation for the problem.

Gradient descent. Let \( S \) denote current solution. If there is a neighbor \( S' \) of \( S \) with strictly lower cost, replace \( S \) with the neighbor whose cost is as small as possible. Otherwise, terminate the algorithm.

A funnel

A jagged funnel
12.2 Metropolis Algorithm
Metropolis Algorithm

**Metropolis algorithm.** [Metropolis, Rosenbluth, Rosenbluth, Teller, Teller 1953]
- Simulate behavior of a physical system according to principles of statistical mechanics.
- Globally biased toward "downhill" steps, but occasionally makes "uphill" steps to break out of local minima.

**Gibbs-Boltzmann function.** The probability of finding a physical system in a state with energy $E$ is proportional to $e^{-E/(kT)}$, where $T > 0$ is temperature and $k$ is a constant.
- For any temperature $T > 0$, function is monotone decreasing function of energy $E$.
- System more likely to be in a lower energy state than higher one.
  - $T$ large: high and low energy states have roughly same probability
  - $T$ small: low energy states are much more probable
Metropolis Algorithm

Metropolis algorithm.

- Given a fixed temperature $T$, maintain current state $S$.
- Randomly perturb current state $S$ to new state $S' \in N(S)$.
- If $E(S') \leq E(S)$, update current state to $S'$
  Otherwise, update current state to $S'$ with probability $e^{-\Delta E / (kT)}$, where $\Delta E = E(S') - E(S) > 0$.

Theorem. Let $f_S(t)$ be fraction of first $t$ steps in which simulation is in state $S$. Then, assuming some technical conditions, with probability 1:

$$\lim_{t \to \infty} f_S(t) = \frac{1}{Z} e^{-E(S)/(kT)},$$

where $Z = \sum_{S \in N(S)} e^{-E(S)/(kT)}$.

Intuition. Simulation spends roughly the right amount of time in each state, according to Gibbs-Boltzmann equation.
Simulated Annealing

**Simulated annealing.**
- $T$ large $\Rightarrow$ probability of accepting an uphill move is large.
- $T$ small $\Rightarrow$ uphill moves are almost never accepted.
- Idea: turn knob to control $T$.
- Cooling schedule: $T = T(i)$ at iteration $i$.

**Physical analog.**
- Take solid and raise it to high temperature, we do not expect it to maintain a nice crystal structure.
- Take a molten solid and freeze it very abruptly, we do not expect to get a perfect crystal either.
- Annealing: cool material gradually from high temperature, allowing it to reach equilibrium at succession of intermediate lower temperatures.
12.3 Hopfield Neural Networks
Hopfield Neural Networks

Hopfield networks. Simple model of an associative memory, in which a large collection of units are connected by an underlying network, and neighboring units try to correlate their states.

Input: Graph $G = (V, E)$ with integer edge weights $w$.

Configuration. Node assignment $s_u = \pm 1$.

Intuition. If $w_{uv} < 0$, then $u$ and $v$ want to have the same state; if $w_{uv} > 0$ then $u$ and $v$ want different states.

Note. In general, no configuration respects all constraints.
Def. With respect to a configuration $S$, edge $e = (u, v)$ is **good** if $w_e s_u s_v < 0$. That is, if $w_e < 0$ then $s_u = s_v$; if $w_e > 0$, $s_u \neq s_v$.

Def. With respect to a configuration $S$, a node $u$ is **satisfied** if the weight of incident good edges $\geq$ weight of incident bad edges.

$$\sum_{v: e=(u,v) \in E} w_e s_u s_v \leq 0$$

Def. A configuration is **stable** if all nodes are satisfied.

Goal. Find a stable configuration, if such a configuration exists.
Hopfield Neural Networks

**Goal.** Find a stable configuration, if such a configuration exists.

**State-flipping algorithm.** Repeated flip state of an unsatisfied node.

```
Hopfield-Flip(G, w) {
    S ← arbitrary configuration

    while (current configuration is not stable) {
        u ← unsatisfied node
        s_u = -s_u
    }

    return S
}
```
State Flipping Algorithm

unsatisfied node
10 - 8 > 0

unsatisfied node
8 - 4 - 1 - 1 > 0

stable

(a)
(b)
(c)
(d)
(e)
(f)
Claim. State-flipping algorithm terminates with a stable configuration after at most \( W = \sum_{e} |w_e| \) iterations.

Pf attempt. Consider measure of progress \( \Phi(S) = \# \text{ satisfied nodes} \).
Hopfield Neural Networks

**Claim.** State-flipping algorithm terminates with a stable configuration after at most \( W = \sum_{e} |w_e| \) iterations.

**Pf.** Consider measure of progress \( \Phi(S) = \sum_{e \text{ good}} |w_e| \).

- Clearly \( 0 \leq \Phi(S) \leq W \).
- We show \( \Phi(S) \) increases by at least 1 after each flip.
  
  **When u flips state:**
  - all good edges incident to u become bad
  - all bad edges incident to u become good
  - all other edges remain the same

\[
\Phi(S') = \Phi(S) - \sum_{e: e = (u, v) \in E \text{ and } e \text{ is bad}} |w_e| + \sum_{e: e = (u, v) \in E \text{ and } e \text{ is good}} |w_e| \geq \Phi(S) + 1
\]

\( u \) is unsatisfied
Complexity of Hopfield Neural Network

Hopfield network search problem. Given a weighted graph, find a stable configuration if one exists.

Hopfield network decision problem. Given a weighted graph, does there exist a stable configuration?

Remark. The decision problem is trivially solvable (always yes), but there is no known poly-time algorithm for the search problem.

\[ \text{polynomial in } n \text{ and } \log W \]
12.4 Maximum Cut
Maximum Cut

**Maximum cut.** Given an undirected graph $G = (V, E)$ with positive integer edge weights $w_e$, find a node partition $(A, B)$ such that the total weight of edges crossing the cut is maximized.

$$w(A, B) := \sum_{u \in A, v \in B} w_{uv}$$

**Toy application.**

- n activities, m people.
- Each person wants to participate in two of the activities.
- Schedule each activity in the morning or afternoon to maximize number of people that can enjoy both activities.

**Real applications.** Circuit layout, statistical physics.
Maximum Cut

**Single-flip neighborhood.** Given a partition \((A, B)\), move one node from \(A\) to \(B\), or one from \(B\) to \(A\) if it improves the solution.

**Greedy algorithm.**

```python
Max-Cut-Local (G, w) {
    Pick a random node partition \((A, B)\)

    while (∃ improving node \(v\)) {
        if \((v\) is in \(A\)) move \(v\) to \(B\)
        else move \(v\) to \(A\)
    }

    return \((A, B)\)
}
```
Theorem. Let \((A, B)\) be a locally optimal partition and let \((A^*, B^*)\) be optimal partition. Then \(w(A, B) \geq \frac{1}{2} \sum_e w_e \geq \frac{1}{2} w(A^*, B^*)\).

Pf.

- Local optimality implies that for all \(u \in A\): \(\sum_{v \in A} w_{uv} \leq \sum_{v \in B} w_{uv}\)
  
  Adding up all these inequalities yields:

  \[
  2 \sum_{\{u,v\} \subseteq A} w_{uv} \leq \sum_{u \in A, v \in B} w_{uv} = w(A,B)
  \]

- Similarly

  \[
  2 \sum_{\{u,v\} \subseteq B} w_{uv} \leq \sum_{u \in A, v \in B} w_{uv} = w(A,B)
  \]

- Now,

  \[
  \sum_{e \in E} w_e = \sum_{\{u,v\} \subseteq A} w_{uv} + \sum_{u \in A, v \in B} w_{uv} + \sum_{\{u,v\} \subseteq A} w_{uv} \leq 2w(A,B)
  \]

weights are nonnegative
Maximum Cut: Big Improvement Flips

Local search. Within a factor of 2 for MAX-CUT, but not poly-time!

Big-improvement-flip algorithm. Only choose a node which, when flipped, increases the cut value by at least $\frac{2\epsilon}{n} w(A, B)$

Claim. Upon termination, big-improvement-flip algorithm returns a cut $(A, B)$ with $(2+\epsilon) w(A, B) \geq w(A^*, B^*)$.

Pf idea. Add $\frac{2\epsilon}{n} w(A, B)$ to each inequality in original proof.

Claim. Big-improvement-flip algorithm terminates after $O(\epsilon^{-1} n \log W)$ flips, where $W = \Sigma_e w_e$.

- Each flip improves cut value by at least a factor of $(1 + \epsilon/n)$.
- After $n/\epsilon$ iterations the cut value improves by a factor of 2.
- Cut value can be doubled at most $\log W$ times. $\uparrow$
  
  if $x \geq 1$, $(1 + 1/x)^x \geq 2$
Maximum Cut: Context

**Theorem.** [Sahni-Gonzales 1976] There exists a $\frac{1}{2}$-approximation algorithm for MAX-CUT.

**Theorem.** [Goemans-Williamson 1995] There exists an 0.878567-approximation algorithm for MAX-CUT.

$$\min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos \theta}$$

**Theorem.** [Håstad 1997] Unless $P = NP$, no $16/17$ approximation algorithm for MAX-CUT.

$$0.941176$$
12.5 Neighbor Relations
Neighbor Relations for Max Cut

1-flip neighborhood. \( (A, B) \) and \( (A', B') \) differ in exactly one node.

k-flip neighborhood. \( (A, B) \) and \( (A', B') \) differ in at most \( k \) nodes.
- \( \Theta(n^k) \) neighbors.

KL-neighborhood. [Kernighan-Lin 1970]
- To form neighborhood of \( (A, B) \):
  - Iteration 1: flip node from \( (A, B) \) that results in best cut value \( (A_1, B_1) \), and mark that node.
  - Iteration \( i \): flip node from \( (A_{i-1}, B_{i-1}) \) that results in best cut value \( (A_i, B_i) \) among all nodes not yet marked.
- Neighborhood of \( (A, B) = (A_1, B_1), \ldots, (A_{n-1}, B_{n-1}) \).
- Neighborhood includes some very long sequences of flips, but without the computational overhead of a k-flip neighborhood.
- Practice: powerful and useful framework.
- Theory: explain and understand its success in practice.

\( \text{cut value of } (A_1, B_1) \text{ may be worse than } (A, B) \)
12.7 Nash Equilibria
Multicast Routing

Multicast routing. Given a directed graph $G = (V, E)$ with edge costs $c_e \geq 0$, a source node $s$, and $k$ agents located at terminal nodes $t_1, \ldots, t_k$. Agent $j$ must construct a path $P_j$ from node $s$ to its terminal $t_j$.

Fair share. If $x$ agents use edge $e$, they each pay $c_e / x$.

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<th>1 pays</th>
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<tr>
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<td>outer</td>
<td>4</td>
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<td>outer</td>
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<td>middle</td>
<td>outer</td>
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<td>middle</td>
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<td>$5/2 + 1$</td>
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Best response dynamics. Each agent is continually prepared to improve its solution in response to changes made by other agents.

Nash equilibrium. Solution where no agent has an incentive to switch.

Fundamental question. When do Nash equilibria exist?

Ex:
- Two agents start with outer paths.
- Agent 1 has no incentive to switch paths (since $4 < 5 + 1$), but agent 2 does (since $8 > 5 + 1$).
- Once this happens, agent 1 prefers middle path (since $4 > 5/2 + 1$).
- Both agents using middle path is a Nash equilibrium.
Local search algorithm. Each agent is continually prepared to improve its solution in response to changes made by other agents.

Analogies.
- Nash equilibrium : local search.
- Best response dynamics : local search algorithm.
- Unilateral move by single agent : local neighborhood.

Contrast. Best-response dynamics need not terminate since no single objective function is being optimized.
Socially Optimum

Social optimum. Minimizes total cost to all agent.

Observation. In general, there can be many Nash equilibria. Even when its unique, it does not necessarily equal the social optimum.

Social optimum = 1 + ε
Nash equilibrium A = 1 + ε
Nash equilibrium B = k

Social optimum = 7
Unique Nash equilibrium = 8
Price of Stability

**Price of stability.** Ratio of best Nash equilibrium to social optimum.

**Fundamental question.** What is price of stability?

**Ex:** Price of stability = $\Theta(\log k)$.

**Social optimum.** Everyone takes bottom paths.

**Unique Nash equilibrium.** Everyone takes top paths.

**Price of stability.** $H(k) / (1 + \varepsilon)$.

$$1 + 1/2 + \ldots + 1/k$$

Diagram:

- Nodes: $s$, $t_1$, $t_2$, $t_3$, ..., $t_k$, $s$
- Edges: $s \rightarrow t_1$, $s \rightarrow t_2$, $s \rightarrow t_3$, ..., $s \rightarrow t_k$, $t_1 \rightarrow s$, $t_2 \rightarrow s$, $t_3 \rightarrow s$, ..., $t_k \rightarrow s$
Finding a Nash Equilibrium

**Theorem.** The following algorithm terminates with a Nash equilibrium.

```
Best-Response-Dynamics(G, c) {
    Pick a path for each agent

    while (not a Nash equilibrium) {
        Pick an agent i who can improve by switching paths
        Switch path of agent i
    }
}
```

**Pf.** Consider a set of paths $P_1, ..., P_k$.

- Let $x_e$ denote the number of paths that use edge $e$.
- Let $\Phi(P_1, ..., P_k) = \sum_{e \in E} c_e \cdot H(x_e)$ be a potential function.
- Since there are only finitely many sets of paths, it suffices to show that $\Phi$ strictly decreases in each step.
Finding a Nash Equilibrium

**Pf. (continued)**

- Consider agent $j$ switching from path $P_j$ to path $P_j'$. 
- Agent $j$ switches because

$$\sum_{f \in P_j' - P_j} \frac{c_f}{x_f + 1} < \sum_{e \in P_j - P_j'} \frac{c_e}{x_e}$$

  newly incurred cost

  cost saved

- $\Phi$ increases by

$$\sum_{f \in P_j' - P_j} c_f \left[ H(x_f + 1) - H(x_f) \right] = \sum_{f \in P_j' - P_j} \frac{c_f}{x_f + 1}$$

- $\Phi$ decreases by

$$\sum_{e \in P_j - P_j'} c_e \left[ H(x_e) - H(x_e - 1) \right] = \sum_{e \in P_j - P_j'} \frac{c_e}{x_e}$$

- Thus, net change in $\Phi$ is negative.
Bounding the Price of Stability

**Claim.** Let $C(P_1, ..., P_k)$ denote the total cost of selecting paths $P_1, ..., P_k$. For any set of paths $P_1, ..., P_k$, we have

$$C(P_1, ..., P_k) \leq \Phi(P_1, ..., P_k) \leq H(k) \cdot C(P_1, ..., P_k)$$

**Pf.** Let $x_e$ denote the number of paths containing edge $e$.

- Let $E^+$ denote set of edges that belong to at least one of the paths.

$$C(P_1, ..., P_k) = \sum_{e \in E^+} c_e \leq \sum_{e \in E^+} c_e H(x_e) \leq \sum_{e \in E^+} c_e H(k) = H(k) \cdot C(P_1, ..., P_k)$$
Bounding the Price of Stability

**Theorem.** There is a Nash equilibrium for which the total cost to all agents exceeds that of the social optimum by at most a factor of $H(k)$.

**Pf.**
- Let $(P_1^*, ..., P_k^*)$ denote set of socially optimal paths.
- Run best-response dynamics algorithm starting from $P^*$.
- Since $\Phi$ is monotone decreasing $\Phi(P_1, ..., P_k) \leq \Phi(P_1^*, ..., P_k^*)$.

\[
C(P_1, ..., P_k) \leq \Phi(P_1, ..., P_k) \leq \Phi(P_1^*, ..., P_k^*) \leq H(k) \cdot C(P_1^*, ..., P_k^*)
\]

\[\uparrow\]
previous claim applied to $P$

\[\uparrow\]
previous claim applied to $P^*$
Summary

Existence. Nash equilibria always exist for $k$-agent multicast routing with fair sharing.

Price of stability. Best Nash equilibrium is never more than a factor of $H(k)$ worse than the social optimum.