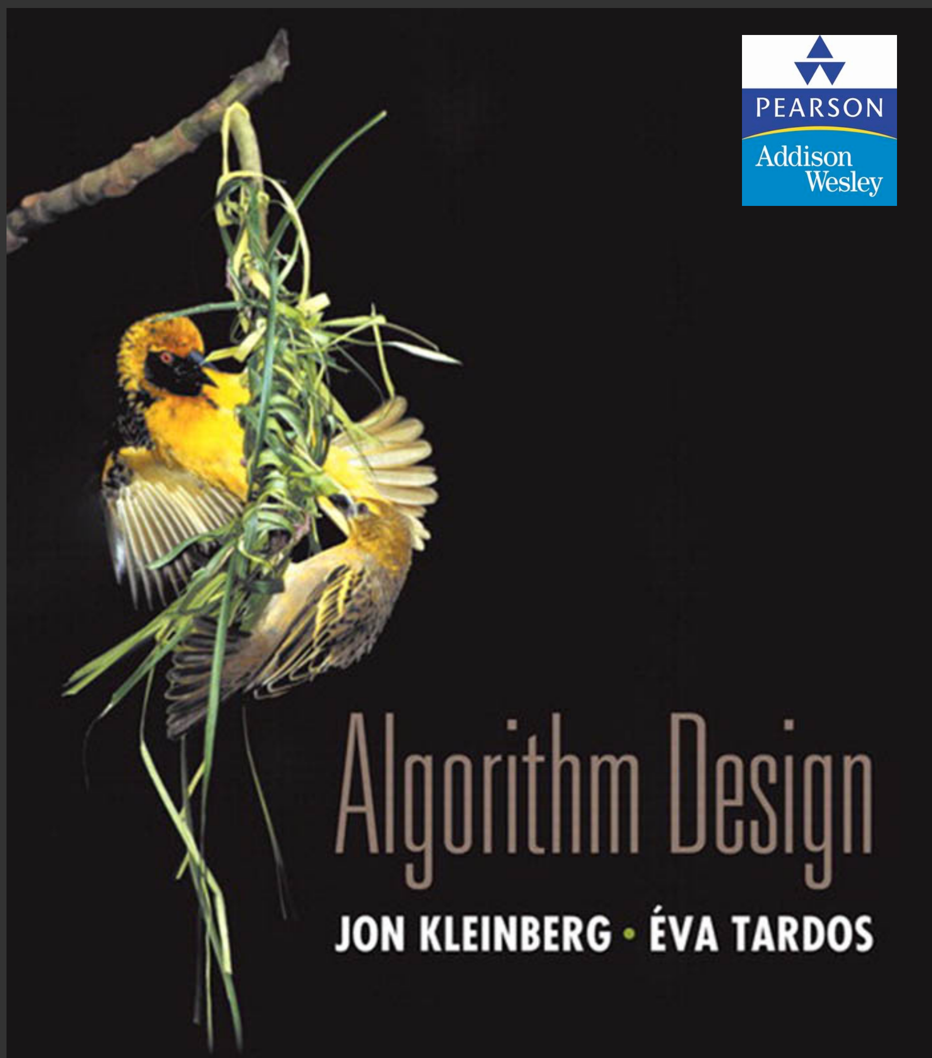


13. Randomized Algorithms

- *contention resolution*
- *global min cut*
- *linearity of expectation*
- *max 3-satisfiability*
- *universal hashing*
- *Chernoff bounds*
- *load balancing*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 1/5/22 12:31 PM

Randomization

Algorithmic design patterns.

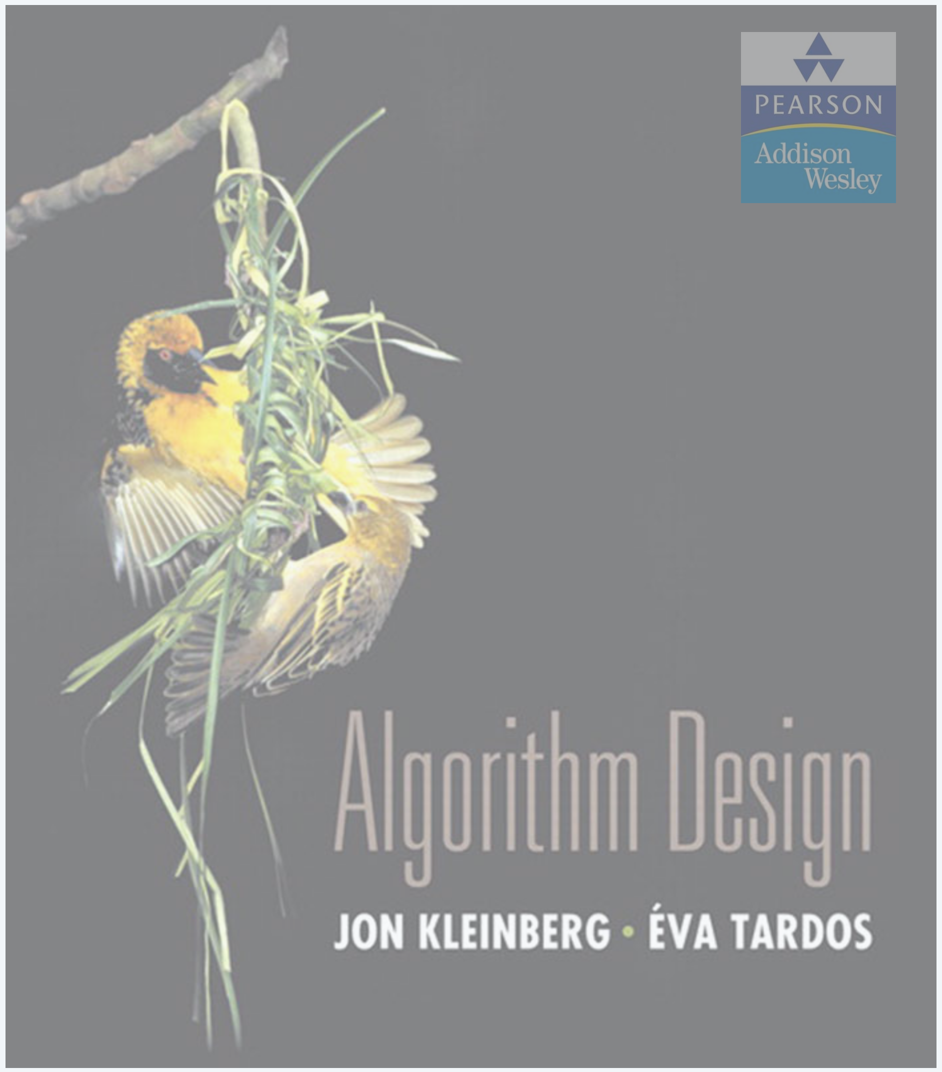
- Greedy.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomization.

in practice, access to a pseudo-random number generator

Randomization. Allow fair coin flip in unit time.

Why randomize? Can lead to simplest, fastest, or only known algorithm for a particular problem.

Ex. Symmetry-breaking protocols, graph algorithms, quicksort, hashing, load balancing, closest pair, Monte Carlo integration, cryptography, ...



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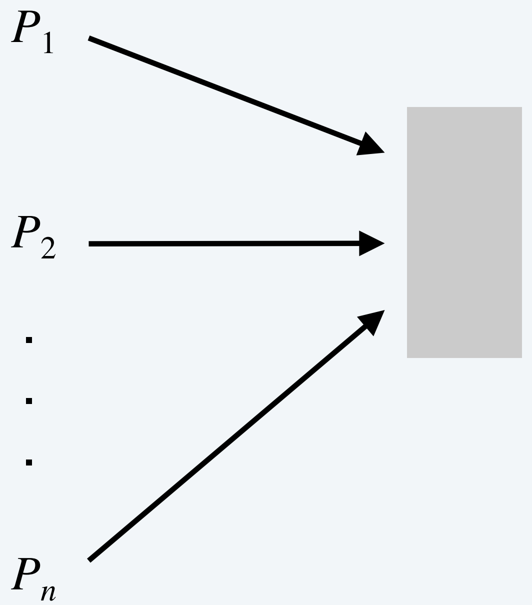
Contention resolution in a distributed system

Contention resolution. Given n processes $P_1, \ldots, P_n$, each competing for access to a shared database. If two or more processes access the database simultaneously, all processes are locked out. Devise protocol to ensure all processes get through on a regular basis.

Restriction. Processes can't communicate.

Challenge. Need symmetry-breaking paradigm.



Contention resolution: randomized protocol

Protocol. Each process requests access to the database at time t with probability $p = 1/n$.

Claim. Let $S[i, t]$ = event that process i succeeds in accessing the database at time t. Then $1 / (e \cdot n) \leq \Pr[S(i, t)] \leq 1/(2n)$.

Pf. By independence, $\Pr[S(i, t)] = p\,(1 - p)^{n-1}$.

process i requests access

none of remaining n-1 processes request access

- Setting $p = 1/n$, we have $\Pr[S(i, t)] = 1/n\ \underbrace{(1 - 1/n)^{n-1}}_{\text{between } 1/e \text{ and } 1/2}$. ▪

value that maximizes Pr[S(i, t)]

Useful facts from calculus. As n increases from 2, the function:

- $(1 - 1/n)^{n}$ converges monotonically from $1/4$ up to $1 / e$.
- $(1 - 1/n)^{n-1}$ converges monotonically from $1/2$ down to $1 / e$.

Contention resolution: randomized protocol

Claim. The probability that process i fails to access the database in en rounds is at most $1 / e$. After $e \cdot n\,(c \ln n)$ rounds, the probability $\leq n^{-c}$.

Pf. Let $F[i, t]$ = event that process i fails to access database in rounds 1 through t. By independence and previous claim, we have $\Pr\,[F[i, t]] \;\leq\; (1 - 1/(en))^{\,t}$.

- Choose $t = \lceil e \cdot n \rceil$: $\Pr[F(i,t)] \;\leq\; \left(1 - \frac{1}{en}\right)^{\lceil en \rceil} \;\leq\; \left(1 - \frac{1}{en}\right)^{en} \;\leq\; \frac{1}{e}$
- Choose $t = \lceil e \cdot n \rceil \lceil c \ln n \rceil$: $\Pr[F(i,t)] \;\leq\; \left(\frac{1}{e}\right)^{c \ln n} \;=\; n^{-c}$

Contention resolution: randomized protocol

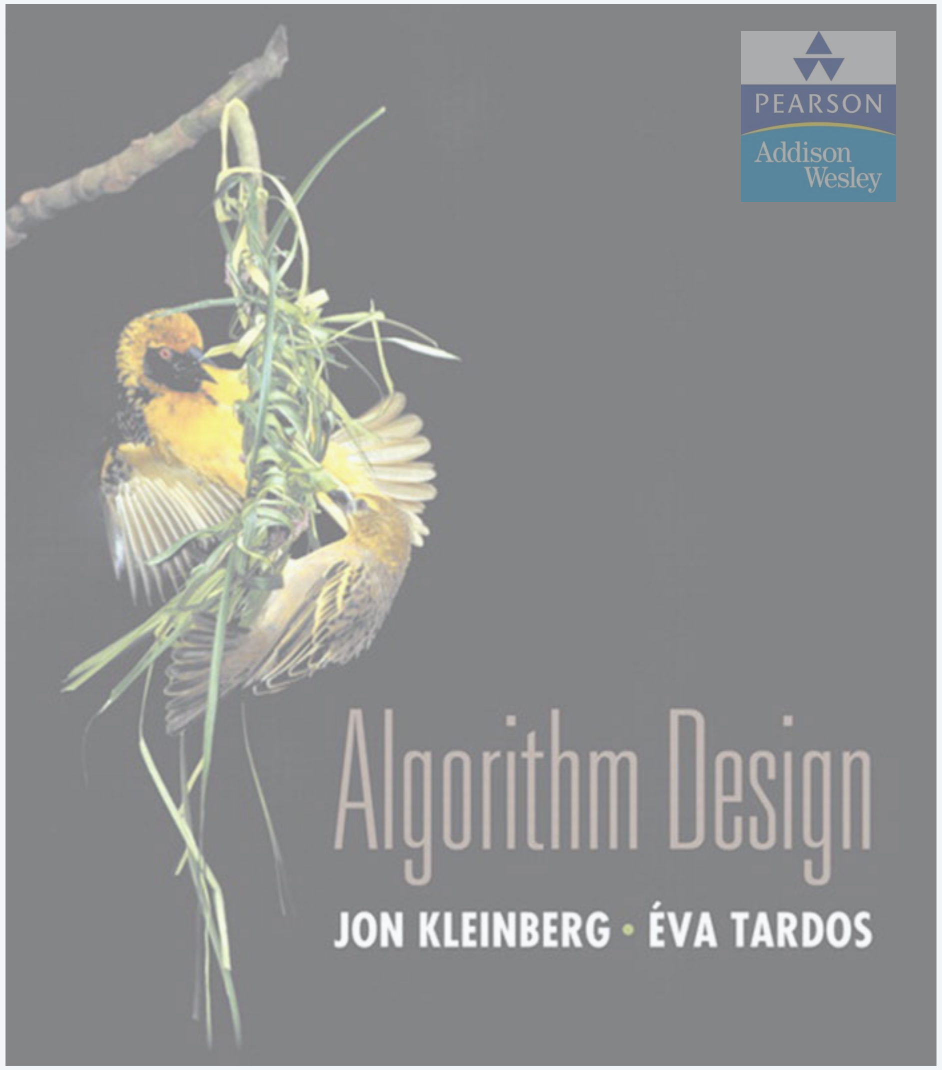
Claim. The probability that all processes succeed within $2e \cdot n \ln n$ rounds is $\geq 1 - 1/n$.

Pf. Let $F[t]$ = event that at least one of the n processes fails to access database in any of the rounds 1 through t.

$$\Pr\big[F[t]\big] \;=\; \Pr\left[\bigcup_{i=1}^{n} F[i,t]\right] \;\underset{\text{union bound}}{\leq}\; \sum_{i=1}^{n} \Pr[F[i,t]] \;\underset{\text{previous slide}}{\leq}\; n\left(1-\tfrac{1}{en}\right)^{t}$$

- Choosing $t = 2\lceil en \rceil \lceil c \ln n \rceil$ yields $\Pr[F[t]] \leq n \cdot n^{-2} = 1/n$. ▪

Union bound. Given events $E_1, \ldots, E_n$, $\quad \Pr\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_{i=1}^{n} \Pr[E_i]$



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Global minimum cut

Global min cut. Given a connected, undirected graph $G = (V, E)$, find a cut (A, B) of minimum cardinality.

Applications. Partitioning items in a database, identify clusters of related documents, network reliability, network design, circuit design, TSP solvers.

Network flow solution.

- Replace every edge (u, v) with two antiparallel edges (u, v) and (v, u).
- Pick some vertex s and compute min s–v cut separating s from each other node $v \in V$.

False intuition. Global min-cut is harder than min s-t cut.

Contraction algorithm

Contraction algorithm. [Karger 1995]

- Pick an edge $e = (u, v)$ uniformly at random.
- Contract edge e.
  - replace u and v by single new super-node w
  - preserve edges, updating endpoints of u and v to w
  - keep parallel edges, but delete self-loops
- Repeat until graph has just two nodes u_1 and v_1.
- Return the cut (all nodes that were contracted to form v_1).

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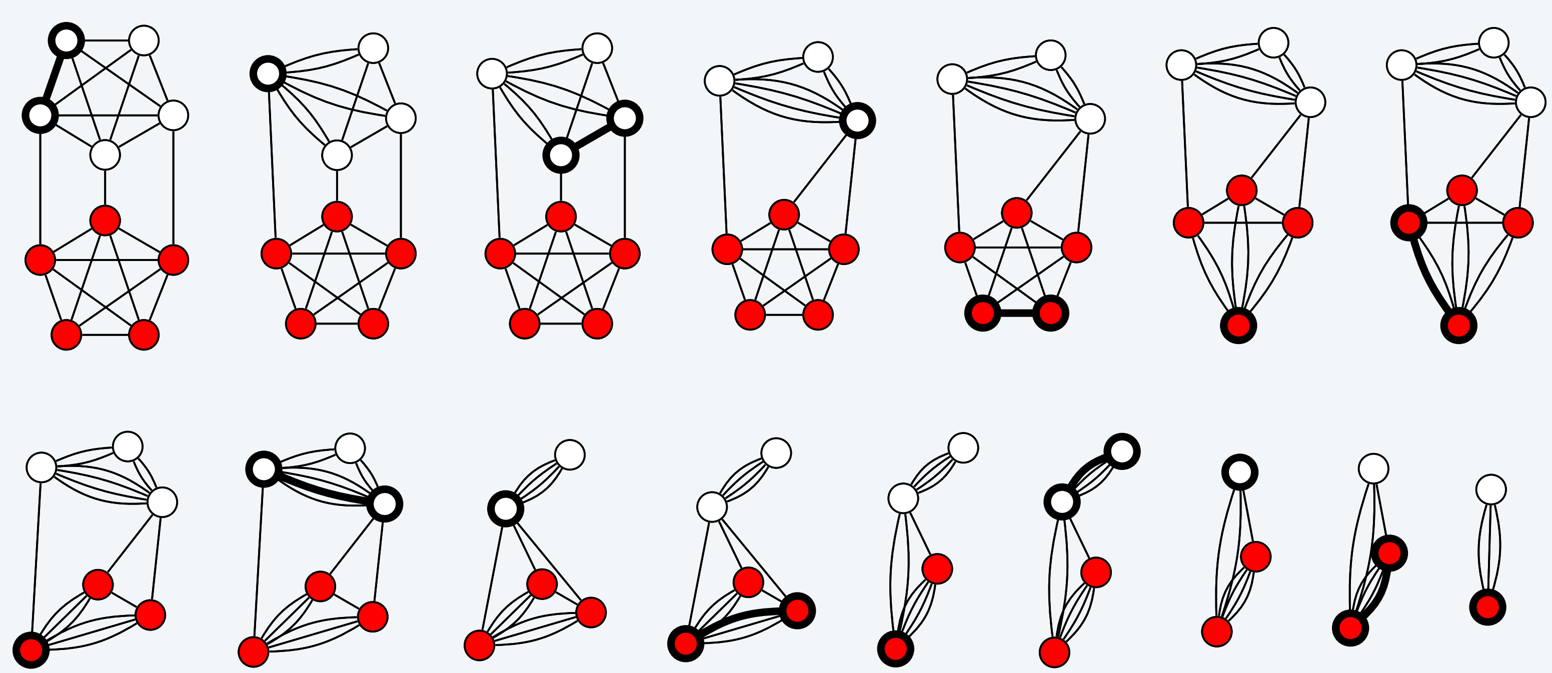


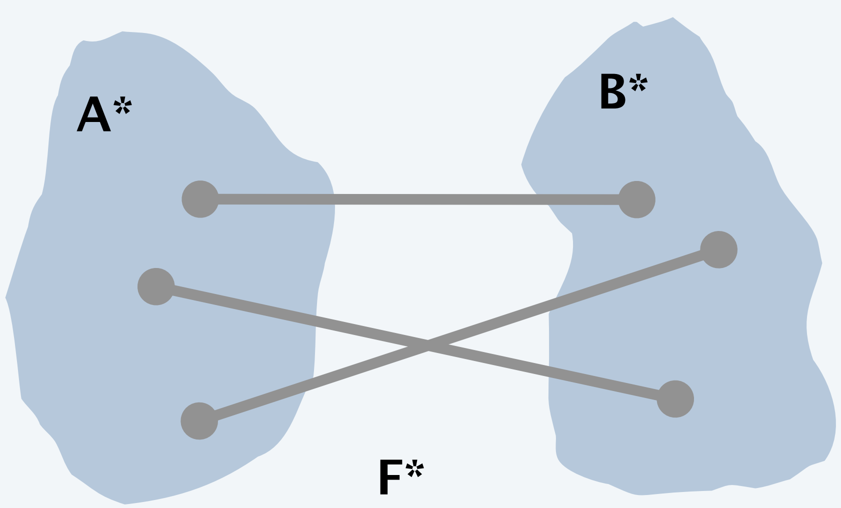
Reference: Thore Husfeldt

Contraction algorithm

Claim. The contraction algorithm returns a min cut with prob $\geq 2 / n^2$.

Pf. Consider a global min-cut (A^*, B^*) of G.

- Let F^* be edges with one endpoint in A^* and the other in B^*.
- Let $k = |F^*|$ = size of min cut.
- In first step, algorithm contracts an edge in F^* probability $k / |E|$.
- Every node has degree $\geq k$ since otherwise (A^*, B^*) would not be a min-cut $\Rightarrow |E| \geq \frac{1}{2} k n \Leftrightarrow k / |E| \leq 2 / n$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2 / n$.



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Pf. Consider a global min-cut (A^*, B^*) of G.

- Let F^* be edges with one endpoint in A^* and the other in B^*.
- Let $k = |F^*|$ = size of min cut.
- Let G' be graph after j iterations. There are $n' = n - j$ supernodes.
- Suppose no edge in F^* has been contracted. The min-cut in G' is still k.
- Since value of min-cut is k, $|E'| \geq \frac{1}{2} k n' \;\Leftrightarrow\; k / |E'| \leq 2 / n'$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2 / n'$.
- Let E_j = event that an edge in F^* is not contracted in iteration j.

$$
\begin{aligned}
\Pr[E_1 \cap E_2 \cdots \cap E_{n-2}] &= \Pr[E_1] \times \Pr[E_2 \mid E_1] \times \cdots \times \Pr[E_{n-2} \mid E_1 \cap E_2 \cdots \cap E_{n-3}] \\
&\geq \left(1 - \tfrac{2}{n}\right)\left(1 - \tfrac{2}{n-1}\right) \cdots \left(1 - \tfrac{2}{4}\right)\left(1 - \tfrac{2}{3}\right) \\
&= \left(\tfrac{n-2}{n}\right)\left(\tfrac{n-3}{n-1}\right) \cdots \left(\tfrac{2}{4}\right)\left(\tfrac{1}{3}\right) \\
&= \frac{2}{n(n-1)} \\
&\geq \frac{2}{n^2}
\end{aligned}
$$

Contraction algorithm

Amplification. To amplify the probability of success, run the contraction algorithm many times.

with independent random choices,

Claim. If we repeat the contraction algorithm $n^2 \ln n$ times, then the probability of failing to find the global min-cut is $\leq 1 / n^2$.

Pf. By independence, the probability of failure is at most

$$\left(1-\frac{2}{n^2}\right)^{n^2 \ln n} = \left[\left(1-\frac{2}{n^2}\right)^{\frac{1}{2}n^2}\right]^{2\ln n} \leq \left(e^{-1}\right)^{2\ln n} = \frac{1}{n^2}$$

$(1 - 1/x)^x \leq 1/e$

Contraction algorithm: example execution

trial 1

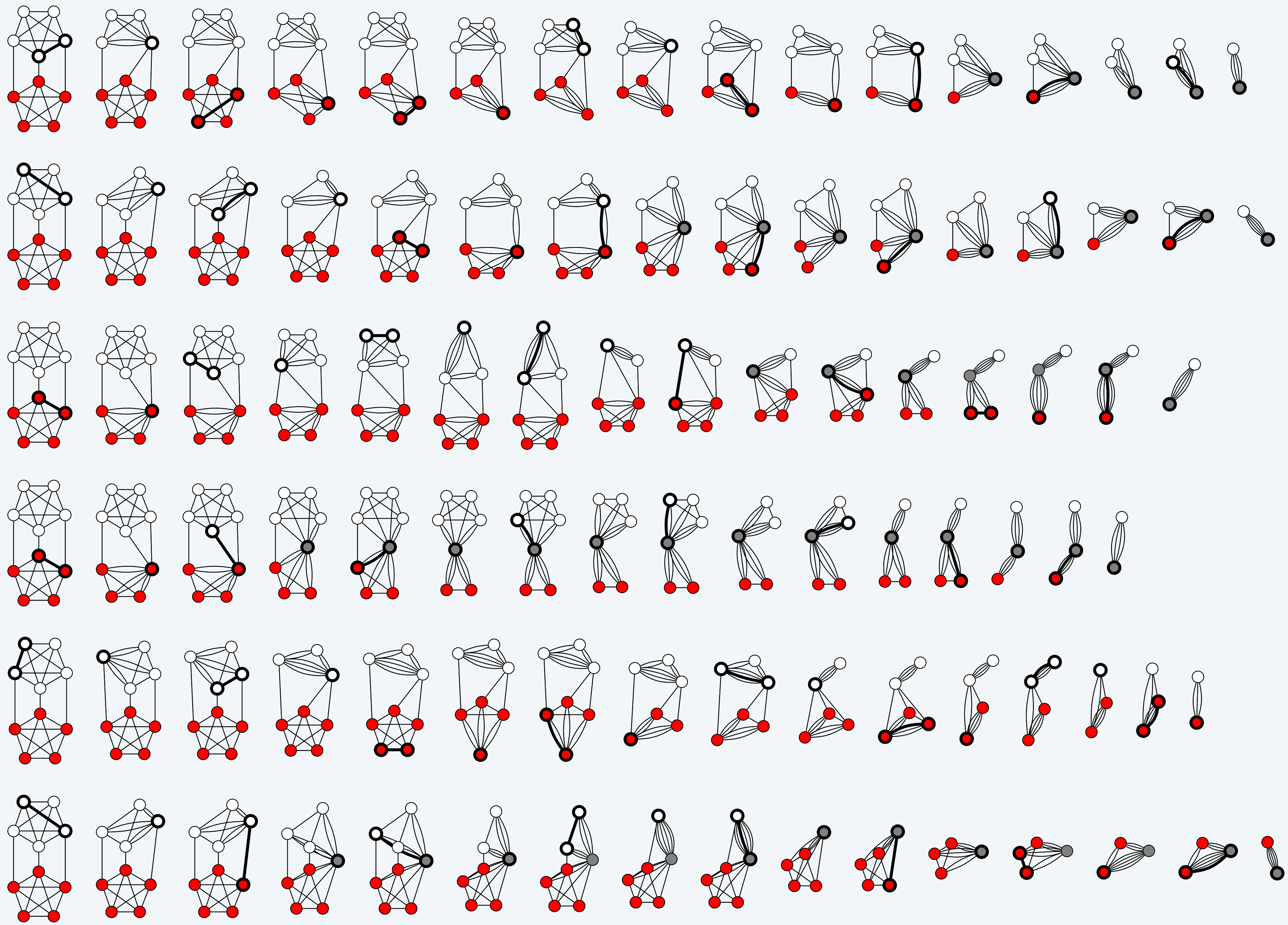
trial 2

trial 3

trial 4

trial 5
(finds min cut)

trial 6



...

Reference: Thore Husfeldt

Global min cut: context

Remark. Overall running time is slow since we perform $\Theta(n^2 \log n)$ iterations and each takes $\Omega(m)$ time.

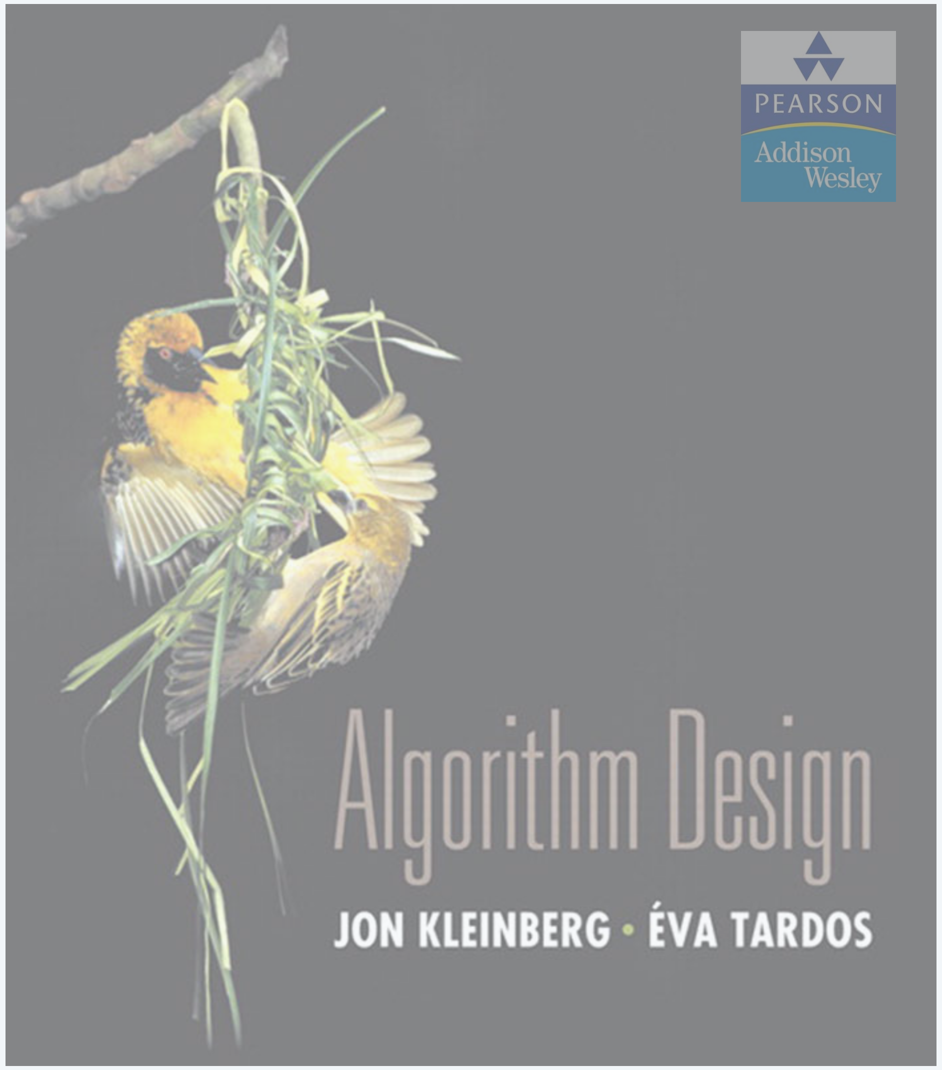
Improvement. [Karger–Stein 1996] $O(n^2 \log^3 n)$.

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm until $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm twice on resulting graph and return best of two cuts.

Extensions. Naturally generalizes to handle positive weights.

Best known. [Karger 2000] $O(m \log^3 n)$.

faster than best known max flow algorithm or deterministic global min cut algorithm



13. RANDOMIZED ALGORITHMS

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- *load balancing*

Expectation

Expectation. Given a discrete random variable X, its expectation $E[X]$ is defined by:

$$E[X] = \sum_{j=0}^{\infty} j \Pr[X = j]$$

Waiting for a first success. Coin is heads with probability p and tails with probability $1-p$. How many independent flips X until first heads?

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X = j] = \sum_{j=0}^{\infty} j \, \underbrace{(1-p)^{j-1}}_{\text{j −1 tails}} \underbrace{p}_{\text{1 head}} = \frac{p}{1-p} \sum_{j=0}^{\infty} j \, (1-p)^{j} = \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \frac{1}{p}$$

$$\sum_{j=0}^{\infty} j \, x^j = \frac{x}{(1-x)^2}$$

Expectation: two properties

Useful property. If X is a 0/1 random variable, $E[X] = \Pr[X = 1]$.

Pf. $$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X = j] = \sum_{j=0}^{1} j \cdot \Pr[X = j] = \Pr[X = 1]$$

not necessarily independent

Linearity of expectation. Given two random variables X and Y defined over the same probability space, $E[X + Y] = E[X] + E[Y]$.

Benefit. Decouples a complex calculation into simpler pieces.

Guessing cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Memoryless guessing. No psychic abilities; can't even remember what's been turned over already. Guess a card from full deck uniformly at random.

Claim. The expected number of correct guesses is 1.

Pf. [surprisingly effortless using linearity of expectation]

- Let $X_i = 1$ if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses $= X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1 / n$.
- $E[X] \underset{\uparrow}{=} E[X_1] + \ldots + E[X_n] = 1 / n + \ldots + 1 / n = 1$. ▪

linearity of expectation

Guessing cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Guessing with memory. Guess a card uniformly at random from cards not yet seen.

Claim. The expected number of correct guesses is $\Theta(\log n)$.

Pf.

- Let $X_i = 1$ if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1 / (n - (i - 1))$.
- $E[X] = E[X_1] + \ldots + E[X_n] = 1/n + \ldots + 1/2 + 1/1 = H(n)$. ▪

linearity of expectation

$\ln(n+1) < H(n) < 1 + \ln n$

Coupon collector

Coupon collector. Each box of cereal contains a coupon. There are n different types of coupons. Assuming all boxes are equally likely to contain each coupon, how many boxes before you have ≥ 1 coupon of each type?

Claim. The expected number of steps is $\Theta(n \log n)$.

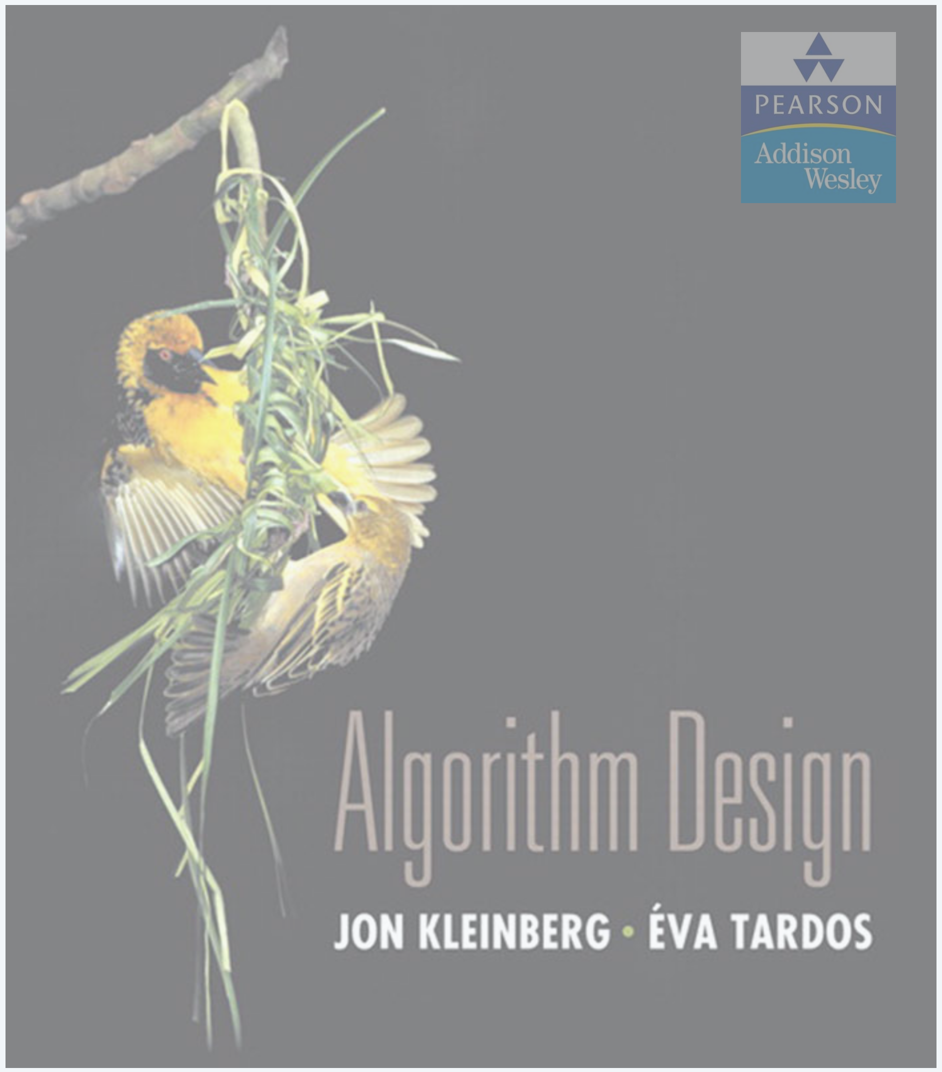
Pf.

- Phase j = time between j and $j+1$ distinct coupons.
- Let X_j = number of steps you spend in phase j.
- Let X = number of steps in total $= X_0 + X_1 + \ldots + X_{n-1}$.

$$E[X] = \sum_{j=0}^{n-1} E[X_j] = \sum_{j=0}^{n-1} \frac{n}{n-j} = n \sum_{i=1}^{n} \frac{1}{i} = n H(n)$$

prob of success = (n – j) / n

⇒ expected waiting time = n / (n – j)



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Maximum 3-satisfiability

exactly 3 literals per clause and
each literal corresponds to a different variable

Maximum 3-satisfiability. Given a 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$\begin{aligned}
C_1 &= x_2 \vee \overline{x_3} \vee \overline{x_4} \\
C_2 &= x_2 \vee x_3 \vee \overline{x_4} \\
C_3 &= \overline{x_1} \vee x_2 \vee x_4 \\
C_4 &= \overline{x_1} \vee \overline{x_2} \vee x_3 \\
C_5 &= x_1 \vee \overline{x_2} \vee \overline{x_4}
\end{aligned}$$

Remark. **NP**-hard optimization problem.

Simple idea. Flip a coin, and set each variable true with probability ½, independently for each variable.

Maximum 3-satisfiability: analysis

Claim. Given a 3-SAT formula with k clauses, the expected number of clauses satisfied by a random assignment is $7k / 8$.

Pf. Consider random variable $Z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

- Let Z = number of clauses satisfied by random assignment.

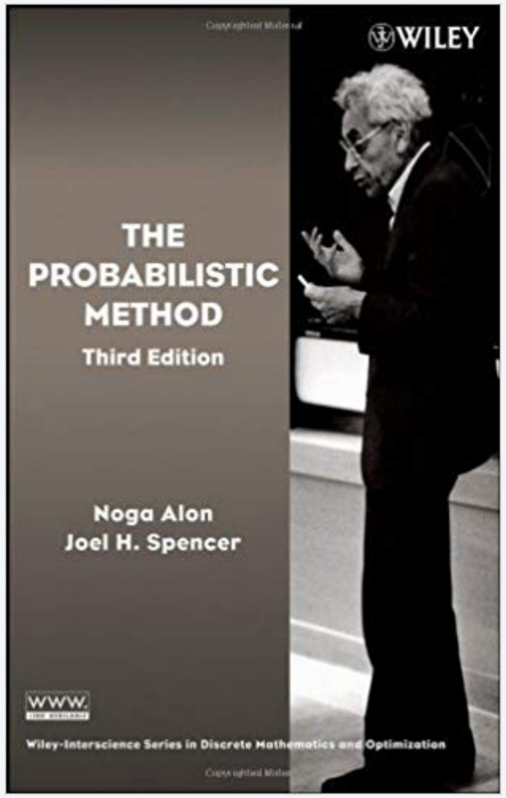
$$
\begin{aligned}
E[Z] &\overset{\text{linearity of expectation}}{=} \sum_{j=1}^{k} E[Z_j] \\
&= \sum_{j=1}^{k} \Pr[\text{clause } C_j \text{ is satisfied}] \\
&= \tfrac{7}{8} k
\end{aligned}
$$

The probabilistic method

Corollary. For any instance of 3-SAT, there exists a truth assignment that satisfies at least a $7/8$ fraction of all clauses.

Pf. Random variable is at least its expectation some of the time. ▪

Probabilistic method. [Paul Erdös] Prove the existence of a non-obvious property by showing that a random construction produces it with positive probability!



Maximum 3-satisfiability: analysis

Q. Can we turn this idea into a $7/8$-approximation algorithm?

A. Yes (but a random variable can almost always be below its mean).

Lemma. The probability that a random assignment satisfies $\geq 7k / 8$ clauses is at least $1 / (8k)$.

Pf. Let p_j be probability that exactly j clauses are satisfied; let p be probability that $\geq 7k / 8$ clauses are satisfied.

$$
\begin{aligned}
\tfrac{7}{8}k \;=\; E[Z] \;&=\; \sum_{j \geq 0} j\, p_j \\
&=\; \sum_{j < 7k/8} j\, p_j \;+\; \sum_{j \geq 7k/8} j\, p_j \\
&\leq\; \left(\tfrac{7k}{8} - \tfrac{1}{8}\right) \sum_{j < 7k/8} p_j \;+\; k \sum_{j \geq 7k/8} p_j \\
&\leq\; \left(\tfrac{7}{8}k - \tfrac{1}{8}\right) \cdot 1 \;+\; k\, p
\end{aligned}
$$

Rearranging terms yields $p \geq 1 / (8k)$. ▪

Maximum 3-satisfiability: analysis

Johnson's algorithm. Repeatedly generate random truth assignments until one of them satisfies $\geq 7k / 8$ clauses.

Theorem. Johnson's algorithm is a $7/8$-approximation algorithm.

Pf. By previous lemma, each iteration succeeds with probability $\geq 1 / (8k)$. By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most $8k$. ▪

Maximum satisfiability

Extensions.

- Allow one, two, or more literals per clause.
- Find max weighted set of satisfied clauses.

Theorem. [Asano–Williamson 2000] There exists a 0.784-approximation algorithm for MAX-SAT.

Theorem. [Karloff–Zwick 1997, Zwick+computer 2002] There exists a $7/8$-approximation algorithm for version of MAX-3-SAT in which each clause has at most 3 literals.

Theorem. [Håstad 1997] Unless **P** = **NP**, no ρ-approximation algorithm for MAX-3-SAT (and hence MAX-SAT) for any $\rho > 7/8$.

very unlikely to improve over simple randomized algorithm for MAX-3-SAT

Monte Carlo vs. Las Vegas algorithms

Monte Carlo. Guaranteed to run in poly-time, likely to find correct answer.
Ex: Contraction algorithm for global min cut.

Las Vegas. Guaranteed to find correct answer, likely to run in poly-time.
Ex: Randomized quicksort, Johnson's MAX-3-SAT algorithm.

stop algorithm
after a certain point

Remark. Can always convert a Las Vegas algorithm into Monte Carlo,
but no known method (in general) to convert the other way.

RP and ZPP

RP. [Monte Carlo] Decision problems solvable with one-sided error in poly-time.

One-sided error.

can decrease probability of false negative to 2^{-100} by 100 independent repetitions

- If the correct answer is *no*, always return *no*.
- If the correct answer is *yes*, return *yes* with probability $\geq ½$.

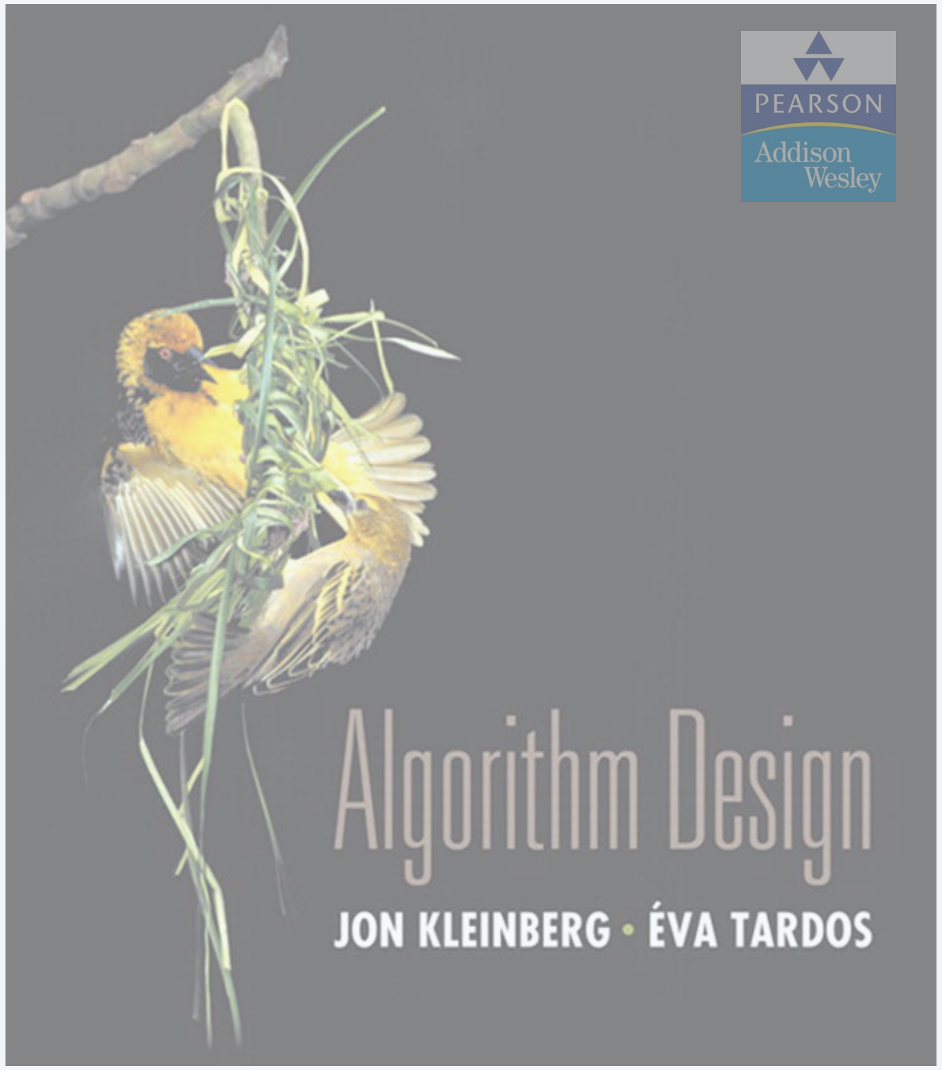
ZPP. [Las Vegas] Decision problems solvable in expected poly-time.

running time can be unbounded, but fast on average

Theorem. $\mathbf{P} \subseteq \mathbf{ZPP} \subseteq \mathbf{RP} \subseteq \mathbf{NP}$.

Fundamental open questions. To what extent does randomization help?

Does $\mathbf{P} = \mathbf{ZPP}$? Does $\mathbf{ZPP} = \mathbf{RP}$? Does $\mathbf{RP} = \mathbf{NP}$?



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Dictionary data type

Dictionary. Given a universe U of possible elements, maintain a subset $S \subseteq U$ so that inserting, deleting, and searching in S is efficient.

Dictionary interface.

- $create()$: initialize a dictionary with $S = \varnothing$.
- $insert(u)$: add element $u \in U$ to S.
- $delete(u)$: delete u from S (if u is currently in S).
- $lookup(u)$: is u in S ?

Challenge. Universe U can be extremely large so defining an array of size $|U|$ is infeasible.

Applications. File systems, databases, Google, compilers, checksums, P2P networks, associative arrays, cryptography, web caching, etc.

Hashing

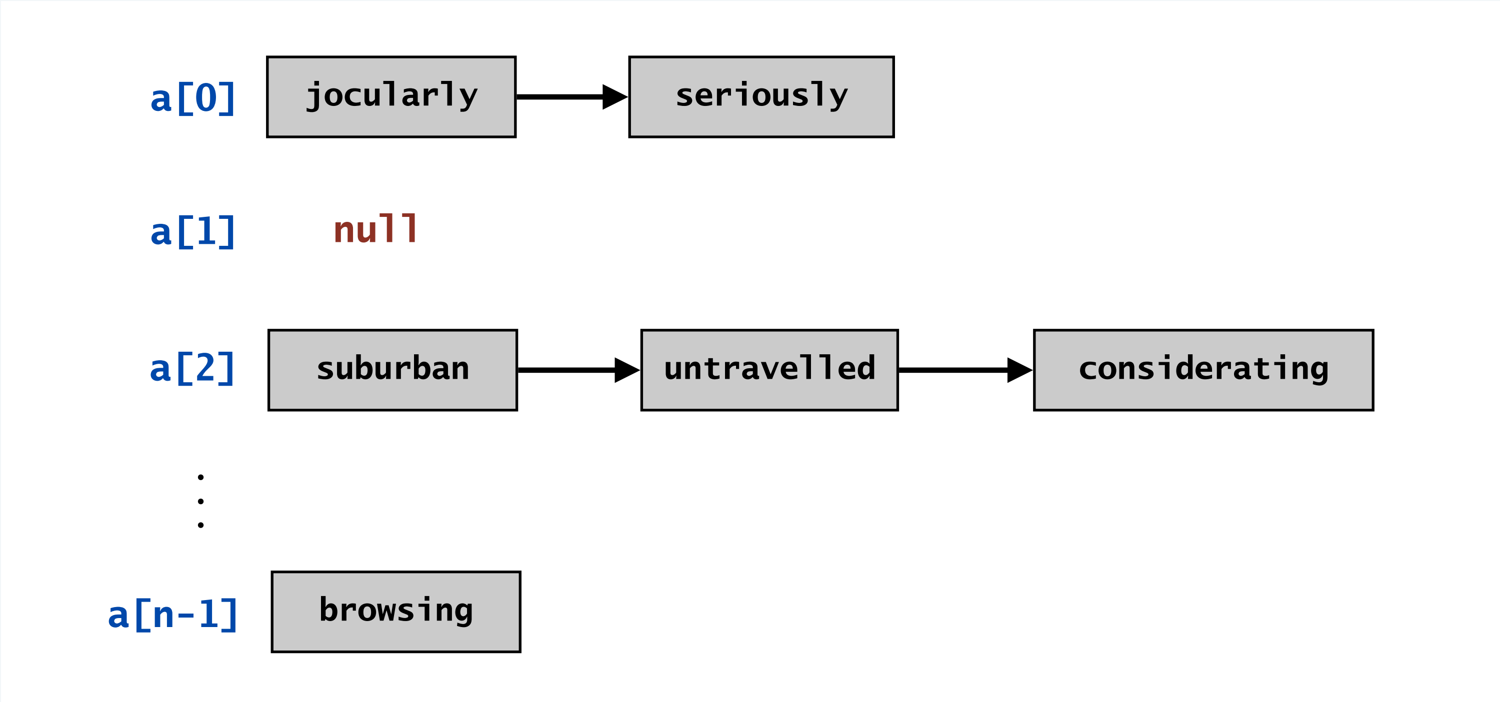
Hash function. $h : U \to \{ 0, 1, \ldots, n-1 \}$.

Hashing. Create an array a of length n. When processing element u, access array element $a[h(u)]$.

birthday paradox

Collision. When $h(u) = h(v)$ but $u \neq v$.

- A collision is expected after $\Theta(\sqrt{n})$ random insertions.
- Separate chaining: $a[i]$ stores linked list of elements u with $h(u) = i$.



Ad-hoc hash function

Ad-hoc hash function.

```
int hash(String s, int n) {
   int hash = 0;
   for (int i = 0; i < s.length(); i++)
      hash = (31 * hash) + s[i];
   return hash % n;
}
```

hash function à la Java string library

Deterministic hashing. If $|U| \geq n^2$, then for any fixed hash function h, there is a subset $S \subseteq U$ of n elements that all hash to same slot. Thus, $\Theta(n)$ time per lookup in worst-case.

Q. But isn't ad-hoc hash function good enough in practice?

Algorithmic complexity attacks

When can't we live with ad-hoc hash function?

- Obvious situations: aircraft control, nuclear reactor, pace maker, ...
- Surprising situations: denial-of-service (DOS) attacks.

malicious adversary learns your ad-hoc hash function (e.g., by reading Java API) and causes a big pile-up in a single slot that grinds performance to a halt

Real world exploits. [Crosby–Wallach 2003]

- Linux 2.4.20 kernel: save files with carefully chosen names.
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem.

Hashing performance

Ideal hash function. Maps m elements uniformly at random to n hash slots.

- Running time depends on length of chains.
- Average length of chain = $\alpha = m / n$.
- Choose $n \approx m \Rightarrow$ expect $O(1)$ per insert, lookup, or delete.

Challenge. Hash function h that achieves $O(1)$ per operation.

Approach. Use randomization for the choice of h.

adversary knows the randomized algorithm you're using,
but doesn't know random choice that the algorithm makes

Universal hashing (Carter–Wegman 1980s)

A universal family of hash functions is a set of hash functions H mapping a universe U to the set $\{ 0, 1, \ldots, n-1 \}$ such that

- For any pair of elements $u \neq v$: $\Pr_{h \in H}\left[h(u) = h(v) \right] \leq 1/n$
- Can select random h efficiently.
- Can compute $h(u)$ efficiently.

chosen uniformly at random

Ex. $U = \{ a, b, c, d, e, f \},\ n = 2.$

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1

$H = \{h_1, h_2\}$

$\Pr_{h \in H}[h(a) = h(b)] = 1/2$

$\Pr_{h \in H}[h(a) = h(c)] = 1$

$\Pr_{h \in H}[h(a) = h(d)] = 0$

...

not universal

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1
$h_3(x)$	0	0	1	0	1	1
$h_4(x)$	1	0	0	1	1	0

$H = \{h_1, h_2, h_3, h_4\}$

$\Pr_{h \in H}[\mathrm{h}(a) = \mathrm{h}(b)] = 1/2$

$\Pr_{h \in H}[\mathrm{h}(a) = \mathrm{h}(c)] = 1/2$

$\Pr_{h \in H}[\mathrm{h}(a) = \mathrm{h}(d)] = 1/2$

$\Pr_{h \in H}[\mathrm{h}(a) = \mathrm{h}(e)] = 1/2$

$\Pr_{h \in H}[\mathrm{h}(a) = \mathrm{h}(f)] = 0$

...

universal

Universal hashing: analysis

Proposition. Let H be a universal family of hash functions mapping a universe U to the set $\{ 0, 1, \ldots, n-1 \}$; let $h \in H$ be chosen uniformly at random from H; let $S \subseteq U$ be a subset of size at most n; and let $u \notin S$. Then, the expected number of items in S that collide with u is at most 1.

Pf. For any $s \in S$, define random variable $X_s = 1$ if $h(s) = h(u)$, and 0 otherwise. Let X be a random variable counting the total number of collisions with u.

$$E_{h \in H}[X] \;=\; E[\textstyle\sum_{s \in S} X_s] \;\underset{\text{linearity of expectation}}{=}\; \textstyle\sum_{s \in S} E[X_s] \;\underset{X_s \text{ is a 0–1 random variable}}{=}\; \textstyle\sum_{s \in S} \Pr[X_s = 1] \;\underset{\text{universal}}{\le}\; \textstyle\sum_{s \in S} \frac{1}{n} \;=\; |S| \frac{1}{n} \;\le\; 1$$

Q. OK, but how do we design a universal class of hash functions?

Designing a universal family of hash functions

Modulus. We will use a prime number p for the size of the hash table.

Integer encoding. Uniquely identify each element $u \in U$ with a base-p integer of r digits: $x = (x_1, x_2, \ldots, x_r)$.

distinct elements have different encodings

Hash function. Let A = set of all r-digit, base-p integers. For each $a = (a_1, a_2, \ldots, a_r)$ where $0 \le a_i < p$, define

$$h_a(x) = \left(\sum_{i=1}^{r} a_i x_i \right) \bmod p$$

maps universe U to set $\{ 0, 1, \ldots, p-1 \}$

Hash function family. $H = \{ h_a : a \in A \}$.

Designing a universal family of hash functions

Theorem. $H = \{ h_a : a \in A \}$ is a universal family of hash functions.

Pf. Let $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ encode two distinct elements of U. We need to show that $\Pr[h_a(x) = h_a(y)] \leq 1 / p$.

- Since $x \neq y$, there exists an integer j such that $x_j \neq y_j$.
- We have $h_a(x) = h_a(y)$ iff

$$a_j \underbrace{(y_j - x_j)}_{z} \equiv \underbrace{\sum_{i \neq j} a_i(x_i - y_i)}_{m} \mod p$$

- Can assume a was chosen uniformly at random by first selecting all coordinates a_i where $i \neq j$, then selecting a_j at random. Thus, we can assume a_i is fixed for all coordinates $i \neq j$.
- Since p is prime, $a_j z \equiv m \mod p$ has at most one solution among p possibilities. ⟵ see lemma on next slide
- Thus $\Pr[h_a(x) = h_a(y)] \leq 1 / p$. ▪

Number theory fact

Fact. Let p be prime, and let $z \not\equiv 0 \bmod p$. Then $\alpha z \equiv m \bmod p$ has at most one solution $0 \leq \alpha < p$.

Pf.

- Suppose $0 \leq \alpha_1 < p$ and $0 \leq \alpha_2 < p$ are two different solutions.
- Then $(\alpha_1 - \alpha_2)\, z \equiv 0 \bmod \mathrm{p}$; hence $(\alpha_1 - \alpha_2)\, z$ is divisible by p.
- Since $z \not\equiv 0 \bmod p$, we know that z is not divisible by p.
- It follows that $(\alpha_1 - \alpha_2)$ is divisible by p.
- This implies $\alpha_1 = \alpha_2$. ▪

here's where we use that p is prime

Bonus fact. Can replace "at most one" with "exactly one" in above fact.

Pf idea. Euclid's algorithm.

Universal hashing: summary

Goal. Given a universe U, maintain a subset $S \subseteq U$ so that insert, delete, and lookup are efficient.

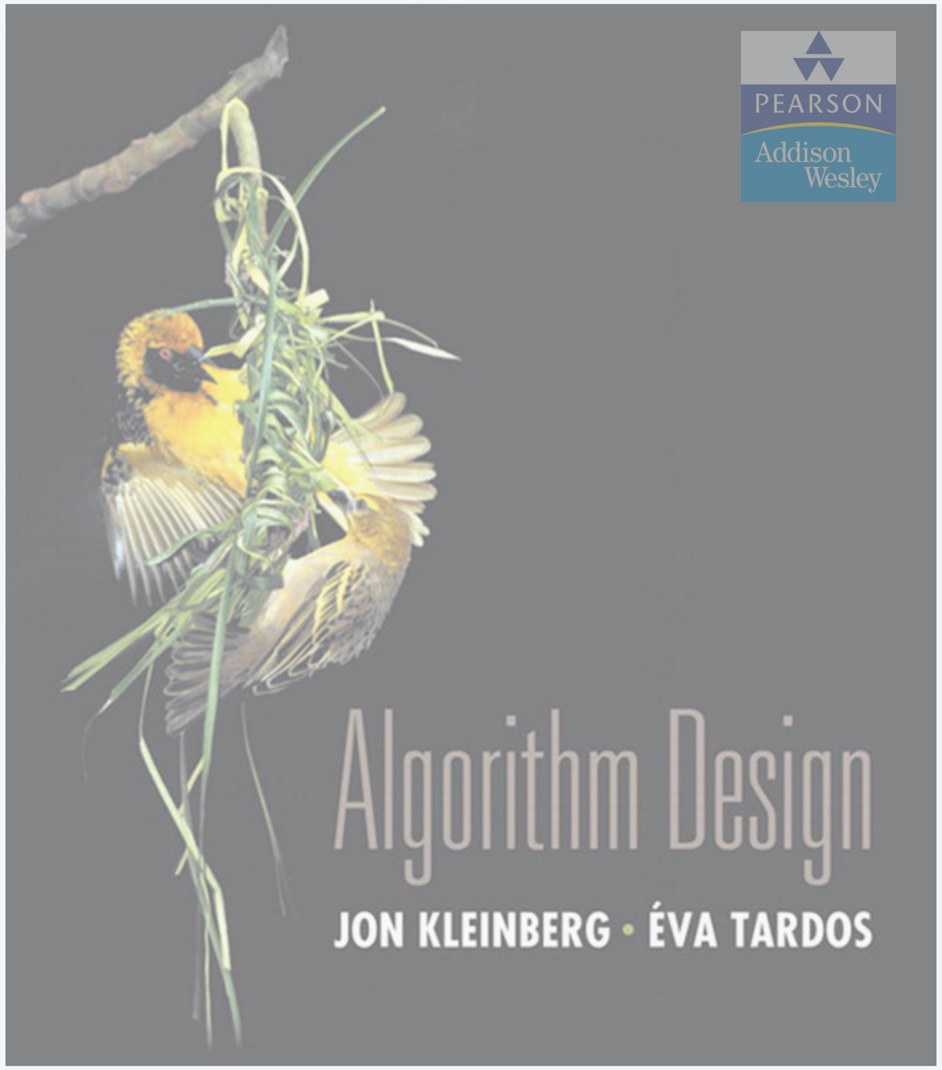
Universal hash function family. $H = \{ h_a : a \in A \}$.

$$h_a(x) = \left(\sum_{i=1}^{r} a_i x_i \right) \bmod p$$

- Choose p prime so that $m \le p \le 2m$, where $m = |S|$.
- Fact: there exists a prime between m and $2m$. ← can find such a prime using another randomized algorithm (!)

Consequence.

- Space used = $\Theta(m)$.
- Expected number of collisions per operation is ≤ 1
  $\Rightarrow$ $O(1)$ time per insert, delete, or lookup.



13. Randomized Algorithms

- *contention resolution*
- *global min cut*
- *linearity of expectation*
- *max 3-satisfiability*
- *universal hashing*
- ***Chernoff bounds***
- *load balancing*

Chernoff Bounds (above mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \geq E[X]$ and for any $\delta > 0$, we have

$$\Pr[X > (1+\delta)\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

sum of independent 0-1 random variables
is tightly centered on the mean

Pf. We apply a number of simple transformations.

- For any $t > 0$,

$$\Pr[X > (1+\delta)\mu] \;=\; \Pr\left[e^{tX} > e^{t(1+\delta)\mu}\right] \;\leq\; e^{-t(1+\delta)\mu} \cdot E[e^{tX}]$$

$f(x) = e^{tx}$ is monotone in x

Markov's inequality: $\Pr[X > a] \leq E[X] / a$

- Now $$E[e^{tX}] \;=\; E[e^{t\sum_i X_i}] \;=\; \prod_i E[e^{tX_i}]$$

definition of X

independence

Chernoff Bounds (above mean)

Pf. [continued]

- Let $p_i = \Pr[X_i = 1]$. Then,

$$E[e^{t X_i}] = p_i e^t + (1 - p_i)e^0 = 1 + p_i(e^t - 1) \underset{\substack{\uparrow \\ \text{for any } \alpha \geq 0,\ 1+\alpha \ \leq\ e^{\alpha}}}{\leq} e^{p_i(e^t - 1)}$$

- Combining everything:

$$\Pr[X > (1+\delta)\mu] \underset{\substack{\uparrow \\ \text{previous slide}}}{\leq} e^{-t(1+\delta)\mu} \prod_i E[e^{t X_i}] \underset{\substack{\uparrow \\ \text{inequality above}}}{\leq} e^{-t(1+\delta)\mu} \prod_i e^{p_i(e^t - 1)} \underset{\substack{\uparrow \\ \sum_i p_i = E[X] \ \leq\ \mu}}{\leq} e^{-t(1+\delta)\mu}\ e^{\mu(e^t - 1)}$$

- Finally, choose $t = \ln(1 + \delta)$. ▪

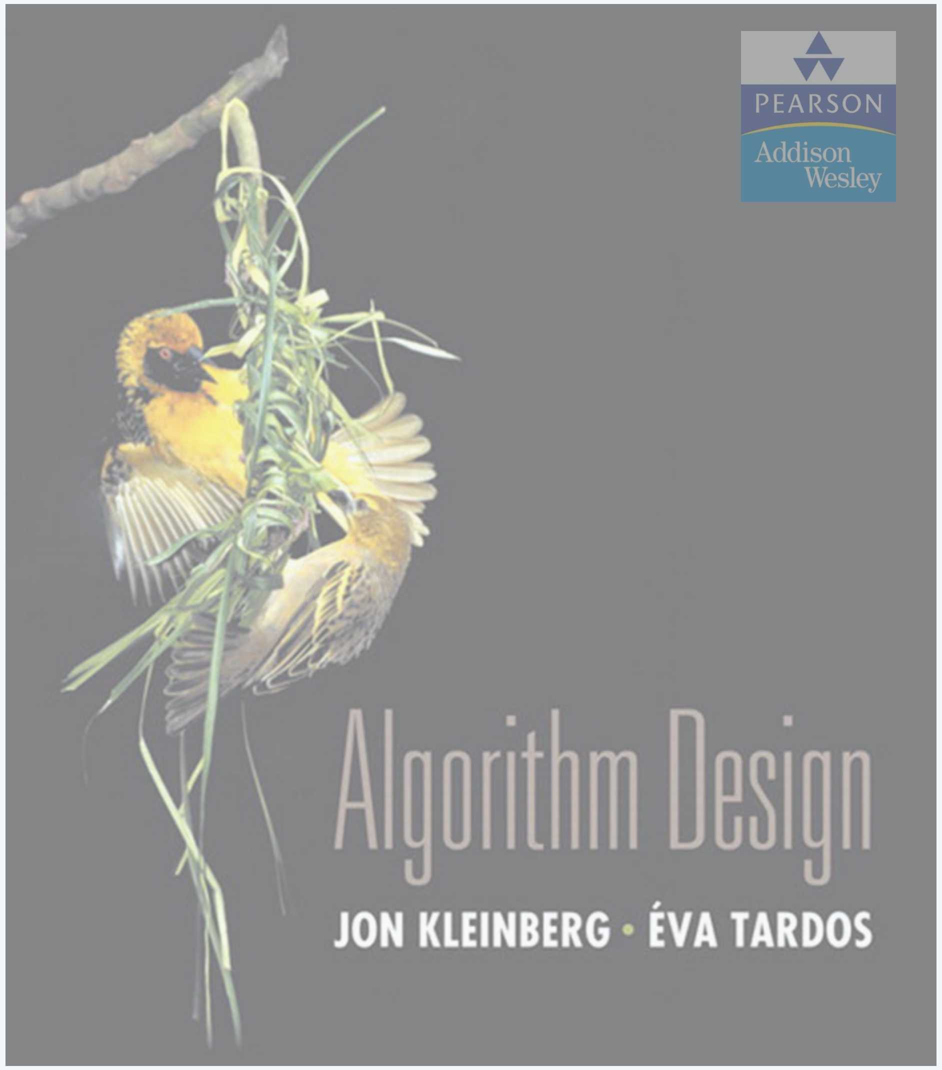
Chernoff Bounds (below mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \le E[X]$ and for any $0 < \delta < 1$, we have

$$\Pr[X < (1-\delta)\mu] \; < \; e^{-\delta^2 \mu / 2}$$

Pf idea. Similar.

Remark. Not quite symmetric since only makes sense to consider $\delta < 1$.



13. Randomized Algorithms

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- *universal hashing*
- *Chernoff bounds*
- *load balancing*

Load balancing

Load balancing. System in which m jobs arrive in a stream and need to be processed immediately on m identical processors. Find an assignment that balances the workload across processors.

Centralized controller. Assign jobs in round-robin manner. Each processor receives at most $\lceil m / n \rceil$ jobs.

Decentralized controller. Assign jobs to processors uniformly at random. How likely is it that some processor is assigned "too many" jobs?

Load balancing

Analysis.

- Let X_i = number of jobs assigned to processor i.
- Let $Y_{ij} = 1$ if job j assigned to processor i, and 0 otherwise.
- We have $\mathrm{E}[Y_{ij}] = 1/\mathrm{n}$.
- Thus, $X_i = \sum_j Y_{ij}$, and $\mu = \mathrm{E}[X_i] = 1$.
- Applying Chernoff bounds with $\delta = \mathrm{c} - 1$ yields $\Pr[X_i > c] < \dfrac{e^{c-1}}{c^c}$
- Let $\gamma(n)$ be number x such that $x^x = n$, and choose $c = e\,\gamma(n)$.

$$\Pr[X_i > c] < \frac{e^{c-1}}{c^c} < \left(\frac{e}{c}\right)^c = \left(\frac{1}{\gamma(n)}\right)^{e\gamma(n)} < \left(\frac{1}{\gamma(n)}\right)^{2\gamma(n)} = \frac{1}{n^2}$$

- Union bound $\Rightarrow$ with probability $\geq 1 - 1/n$ no processor receives more than $e\,\gamma(n) = \Theta(\log n \,/\, \log\log n)$ jobs.

Bonus fact: with high probability,
some processor receives Θ(logn / log log n) jobs

Load balancing: many jobs

Theorem. Suppose the number of jobs $m = 16\, n \ln n$. Then on average, each of the n processors handles $\mu = 16 \ln n$ jobs. With high probability, every processor will have between half and twice the average load.

Pf.

- Let X_i , Y_{ij} be as before.
- Applying Chernoff bounds with $\delta = 1$ yields

$$\Pr[X_i > 2\mu] \;<\; \left(\frac{e}{4}\right)^{16n \ln n} \;<\; \left(\frac{1}{e}\right)^{\ln n} \;=\; \frac{1}{n^2}$$

$$\Pr\left[X_i < \tfrac{1}{2}\mu\right] \;<\; e^{-\frac{1}{2}\left(\frac{1}{2}\right)^2 16n \ln n} \;=\; \frac{1}{n^2}$$

- Union bound $\Rightarrow$ every processor has load between half and twice the average with probability $\geq 1 - 2/n$. ▪