11. Approximation Algorithms

- load balancing
- center selection
- pricing method: weighted vertex cover
- LP rounding: weighted vertex cover
- generalized load balancing
- knapsack problem
Coping with NP-completeness

Q. Suppose I need to solve an NP-hard problem. What should I do?

A. Sacrifice one of three desired features.
   i. Solve arbitrary instances of the problem.
   ii. Solve problem to optimality.
   iii. Solve problem in polynomial time.

\(\rho\)-approximation algorithm.
   • Guaranteed to run in poly-time.
   • Guaranteed to solve arbitrary instance of the problem
   • Guaranteed to find solution within ratio \(\rho\) of true optimum.

Challenge. Need to prove a solution’s value is close to optimum, without even knowing what optimum value is
11. Approximation Algorithms

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Load balancing

**Input.** $m$ identical machines; $n$ jobs, job $j$ has processing time $t_j$.
- Job $j$ must run contiguously on one machine.
- A machine can process at most one job at a time.

**Def.** Let $S[i]$ be the subset of jobs assigned to machine $i$. The load of machine $i$ is $L[i] = \sum_{j \in S[i]} t_j$.

**Def.** The makespan is the maximum load on any machine $L = \max_i L[i]$.

**Load balancing.** Assign each job to a machine to minimize makespan.
Load balancing on 2 machines is NP-hard

Claim. Load balancing is hard even if $m = 2$ machines.

Pf. $\textsc{Partition} \leq_p \textsc{Load-Balance}$.

NP-complete by Exercise 8.26

machine 1

| a | d | f |

machine 2

| b | c | e | g |

length of job $f$
Load balancing: list scheduling

**List-scheduling algorithm.**
- Consider $n$ jobs in some fixed order.
- Assign job $j$ to machine $i$ whose load is smallest so far.

**LIST-SCHEDULING** $(m, n, t_1, t_2, \ldots, t_n)$

FOR $i = 1$ TO $m$
- $L[i] \leftarrow 0$.  \hspace{1cm} \text{load on machine } i$
- $S[i] \leftarrow \emptyset$.  \hspace{1cm} \text{jobs assigned to machine } i$

FOR $j = 1$ TO $n$
- $i \leftarrow \text{argmin}_k L[k]$.  \hspace{1cm} \text{machine } i \text{ has smallest load}$
- $S[i] \leftarrow S[i] \cup \{j\}$.  \hspace{1cm} \text{assign job } j \text{ to machine } i$
- $L[i] \leftarrow L[i] + t_j$.  \hspace{1cm} \text{update load of machine } i$

RETURN $S[1], S[2], \ldots, S[m]$.

**Implementation.** $O(n \log m)$ using a priority queue for loads $L[k]$. 
Load balancing: list scheduling analysis

**Theorem.** [Graham 1966] Greedy algorithm is a 2-approximation.
- First worst-case analysis of an approximation algorithm.
- Need to compare resulting solution with optimal makespan $L^*$. 

**Lemma 1.** The optimal makespan $L^* \geq \max_j t_j$.

**Pf.** Some machine must process the most time-consuming job. □

**Lemma 2.** The optimal makespan $L^* \geq \frac{1}{m} \sum_j t_j$.

**Pf.**
- The total processing time is $\sum_j t_j$.
- One of $m$ machines must do at least a $1/m$ fraction of total work. □
**Theorem.** Greedy algorithm is a 2-approximation.

**Pf.** Consider load $L[i]$ of bottleneck machine $i$.

- Let $j$ be last job scheduled on machine $i$.
- When job $j$ assigned to machine $i$, $i$ had smallest load. Its load before assignment is $L[i] - t_j \Rightarrow L[i] - t_j \leq L[k]$ for all $1 \leq k \leq m$. 

---

![Diagram](image-url)
Load balancing: list scheduling analysis

**Theorem.** Greedy algorithm is a 2-approximation.

**Pf.** Consider load $L[i]$ of bottleneck machine $i$.

- Let $j$ be last job scheduled on machine $i$.
- When job $j$ assigned to machine $i$, $i$ had smallest load.
  Its load before assignment is $L[i] - t_j \Rightarrow L[i] - t_j \leq L[k]$ for all $1 \leq k \leq m$.
- Sum inequalities over all $k$ and divide by $m$:

\[
L[i] - t_j \leq \frac{1}{m} \sum_{k} L[k] = \frac{1}{m} \sum_{k} t_k
\]

Lemma 2 $\leq L^*$. 

- Now, $L = L[i] = (L[i] - t_j) + t_j \leq 2L^*$ .
Load balancing: list scheduling analysis

Q. Is our analysis tight?
A. Essentially yes.

Ex: \( m \) machines, \( m(m - 1) \) jobs length 1 jobs, one job of length \( m \).

List scheduling makespan = 19 = 2m – 1

\( m = 10 \)

machine 2 idle
machine 3 idle
machine 4 idle
machine 5 idle
machine 6 idle
machine 7 idle
machine 8 idle
machine 9 idle
machine 10 idle
Load balancing: list scheduling analysis

Q. Is our analysis tight?
A. Essentially yes.

Ex: \( m \) machines, \( m(m - 1) \) jobs length 1 jobs, one job of length \( m \).

optimal makespan = 10 = m
Load balancing: LPT rule

Longest processing time (LPT). Sort \( n \) jobs in decreasing order of processing times; then run list scheduling algorithm.

**LPT-LIST-SCHEDULING** \((m, n, t_1, t_2, \ldots, t_n)\)

1. **SORT** jobs and renumber so that \( t_1 \geq t_2 \geq \ldots \geq t_n \).
2. **FOR** \( i = 1 \) TO \( m \)
   - \( L[i] \leftarrow 0 \).  \( \text{load on machine } i \)
   - \( S[i] \leftarrow \emptyset \).  \( \text{jobs assigned to machine } i \)
3. **FOR** \( j = 1 \) TO \( n \)
   - \( i \leftarrow \text{argmin}_k L[k] \).  \( \text{machine } i \text{ has smallest load} \)
   - \( S[i] \leftarrow S[i] \cup \{ j \} \).  \( \text{assign job } j \text{ to machine } i \)
   - \( L[i] \leftarrow L[i] + t_j \).  \( \text{update load of machine } i \)
4. **RETURN** \( S[1], S[2], \ldots, S[m] \).
Load balancing: LPT rule

Observation. If bottleneck machine $i$ has only 1 job, then optimal.
Pf. Any solution must schedule that job. ■

Lemma 3. If there are more than $m$ jobs, $L^* \geq 2t_{m+1}$.
Pf.
• Consider processing times of first $m+1$ jobs $t_1 \geq t_2 \geq \ldots \geq t_{m+1}$.
• Each takes at least $t_{m+1}$ time.
• There are $m+1$ jobs and $m$ machines, so by pigeonhole principle, at least one machine gets two jobs. ■

Theorem. LPT rule is a 3/2-approximation algorithm.
Pf. [ similar to proof for list scheduling ]
• Consider load $L[i]$ of bottleneck machine $i$.
• Let $j$ be last job scheduled on machine $i$. \[ L = L[i] = (L[i] - t_j) + t_j \leq \frac{3}{2} L^* \]
  assuming machine $i$ has at least 2 jobs, we have $j > m$
  as before $\leq L^*$ $\leq \frac{1}{2} L^*$ Lemma 3 (since $t_j \leq t_{m+1}$)
Load balancing: LPT rule

Q. Is our 3/2 analysis tight?
A. No.

Theorem. [Graham 1969] LPT rule is a 4/3-approximation.
Pf. More sophisticated analysis of same algorithm.

Q. Is Graham’s 4/3 analysis tight?
A. Essentially yes.

Ex.
- $m$ machines
- $n = 2m + 1$ jobs
- 2 jobs of length $m, m+1, \ldots, 2m-1$ and one more job of length $m$.
- Then, $\frac{L}{L^*} = \frac{4m-1}{3m}$
Believe it or not

A RACE IN WHICH LOSING IS AKIN TO DEATH

THE PALIO, a horse race held each summer around the main square of Siena, Italy, traditionally ends with the winners holding a MOCK FUNERAL FOR THE LOSERS

RONALD GRAHAM
head of Bell Laboratories mathematical Studies Center in Murray Hill, N.J., is one of the world’s foremost mathematicians, publishes more than 12 math papers a year and is on the editorial boards of 20 math journals — yet is a highly skilled trampolinist and juggler, and has been elected president of the International Jugglers Association
11. **Approximation Algorithms**

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**Center selection problem**

**Input.** Set of $n$ sites $s_1, \ldots, s_n$ and an integer $k > 0$.

**Center selection problem.** Select set of $k$ centers $C$ so that maximum distance $r(C)$ from a site to nearest center is minimized.
Center selection problem

**Input.** Set of $n$ sites $s_1, \ldots, s_n$ and an integer $k > 0$.

**Center selection problem.** Select set of $k$ centers $C$ so that maximum distance $r(C)$ from a site to nearest center is minimized.

**Notation.**
- $\text{dist}(x, y) =$ distance between sites $x$ and $y$.
- $\text{dist}(s_i, C) = \min_{c \in C} \text{dist}(s_i, c) =$ distance from $s_i$ to closest center.
- $r(C) = \max_i \text{dist}(s_i, C) =$ smallest covering radius.

**Goal.** Find set of centers $C$ that minimizes $r(C)$, subject to $|C| = k$.

**Distance function properties.**
- $\text{dist}(x, x) = 0$ [ identity ]
- $\text{dist}(x, y) = \text{dist}(y, x)$ [ symmetry ]
- $\text{dist}(x, y) \leq \text{dist}(x, z) + \text{dist}(z, y)$ [ triangle inequality ]
Center selection example

**Ex:** each site is a point in the plane, a center can be any point in the plane, \( \text{dist}(x, y) = \text{Euclidean distance} \).

**Remark:** search can be infinite!
Greedy algorithm: a false start

**Greedy algorithm.** Put the first center at the best possible location for a single center, and then keep adding centers so as to reduce the covering radius each time by as much as possible.

**Remark:** arbitrarily bad!
Center selection: greedy algorithm

Repeatedly choose next center to be site farthest from any existing center.

\[ \text{GREEDY-CENTER-SELECTION} (k, n, s_1, s_2, \ldots, s_n) \]

\[
\begin{align*}
C & \leftarrow \emptyset. \\
\text{REPEAT} \ k \ \text{times} & \\
& \quad \text{Select a site } s_i \text{ with maximum distance } \text{dist}(s_i, C). \\
& \quad C \leftarrow C \cup s_i. \\
\text{RETURN } C.
\end{align*}
\]

Property. Upon termination, all centers in \( C \) are pairwise at least \( r(C) \) apart.

Pf. By construction of algorithm.
Lemma. Let $C^*$ be an optimal set of centers. Then $r(C) \leq 2r(C^*)$.

Pf. [by contradiction] Assume $r(C^*) < \frac{1}{2} r(C)$.

- For each site $c_i \in C$, consider ball of radius $\frac{1}{2} r(C)$ around it.
- Exactly one $c_i^*$ in each ball; let $c_i$ be the site paired with $c_i^*$.
- Consider any site $s$ and its closest center $c_i^* \in C^*$.
- $\text{dist}(s, C) \leq \text{dist}(s, c_i) \leq \text{dist}(s, c_i^*) + \text{dist}(c_i^*, c_i) \leq 2r(C^*)$.
- Thus, $r(C) \leq 2r(C^*)$. □

\[ \Delta \text{-inequality} \leq r(C^*) \text{ since } c_i^* \text{ is closest center} \]
Center selection

Lemma. Let $C^*$ be an optimal set of centers. Then $r(C) \leq 2r(C^*)$.

Theorem. Greedy algorithm is a 2-approximation for center selection problem.

Remark. Greedy algorithm always places centers at sites, but is still within a factor of 2 of best solution that is allowed to place centers anywhere.

Question. Is there hope of a $3/2$-approximation? $4/3$?
Dominating set reduces to center selection

**Theorem.** Unless $P = NP$, there no $\rho$-approximation for center selection problem for any $\rho < 2$.

**Pf.** We show how we could use a $(2 - \varepsilon)$ approximation algorithm for CENTER SELECTION selection to solve DOMINATING SET in poly-time.

- Let $G = (V, E)$, $k$ be an instance of DOMINATING SET.
- Construct instance $G'$ of CENTER SELECTION with sites $V$ and distances
  - $dist(u, v) = 1$ if $(u, v) \in E$
  - $dist(u, v) = 2$ if $(u, v) \notin E$
- Note that $G'$ satisfies the triangle inequality.
- $G$ has dominating set of size $k$ iff there exists $k$ centers $C^*$ with $r(C^*) = 1$.
- Thus, if $G$ has a dominating set of size $k$, a $(2 - \varepsilon)$-approximation algorithm for CENTER SELECTION would find a solution $C^*$ with $r(C^*) = 1$ since it cannot use any edge of distance 2. □
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Weighted vertex cover

Definition. Given a graph $G = (V, E)$, a vertex cover is a set $S \subseteq V$ such that each edge in $E$ has at least one end in $S$.

Weighted vertex cover. Given a graph $G$ with vertex weights, find a vertex cover of minimum weight.

![Graph with weights](image)

weight $= 2 + 2 + 4$  
weight $= 11$
**Pricing method.** Each edge must be covered by some vertex. Edge \( e = (i, j) \) pays price \( p_e \geq 0 \) to use both vertex \( i \) and \( j \).

**Fairness.** Edges incident to vertex \( i \) should pay \( \leq w_i \) in total.

\[
\text{for each vertex } i : \sum_{e=(i,j)} p_e \leq w_i
\]

**Fairness lemma.** For any vertex cover \( S \) and any fair prices \( p_e \): \( \sum_e p_e \leq w(S) \).

**Pf.**

\[
\sum_{e \in E} p_e \leq \sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in S} w_i = w(S). \]

each edge \( e \) covered by at least one node in \( S \)

sum fairness inequalities for each node in \( S \)
Pricing method

Set prices and find vertex cover simultaneously.

**WEIGHTED-VERTEX-COVER** \((G, w)\)

\[
S \leftarrow \emptyset.
\]

**FOREACH** \(e \in E\)

\[
p_e \leftarrow 0.
\]

**WHILE** (there exists an edge \((i, j)\) such that neither \(i\) nor \(j\) is tight)

Select such an edge \(e = (i, j)\).

Increase \(p_e\) as much as possible until \(i\) or \(j\) tight.

\[
S \leftarrow \text{set of all tight nodes}.
\]

**RETURN** \(S\).
Pricing method example

(a) Pricing method example

(b) Pricing method example

(c) Pricing method example

(d) Pricing method example
Pricing method: analysis

**Theorem.** Pricing method is a 2-approximation for \textsc{Weighted-Vertex-Cover}.

**Pf.**

- Algorithm terminates since at least one new node becomes tight after each iteration of while loop.

- Let $S = \text{set of all tight nodes upon termination of algorithm.}$
  
  $S$ is a vertex cover: if some edge $(i,j)$ is uncovered, then neither $i$ nor $j$ is tight. But then while loop would not terminate.

- Let $S^*$ be optimal vertex cover. We show $w(S) \leq 2w(S^*)$.

\[
w(S) = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e = (i,j)} p_e \leq \sum_{i \in V} \sum_{e = (i,j)} p_e = 2 \sum_{e \in E} p_e \leq 2w(S^*).
\]

\*all nodes in $S$ are tight\]
\*each edge counted twice\]
\*prices $\geq 0$\]
\*fairness lemma
11. **Approximation Algorithms**

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- **LP rounding**: weighted vertex cover
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Weighted vertex cover

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in $S$.

total weight $= 6 + 9 + 10 + 32 = 57$
Weighted vertex cover: ILP formulation

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in $S$.

Integer linear programming formulation.

- Model inclusion of each vertex $i$ using a 0/1 variable $x_i$.

$$x_i = \begin{cases} 
0 & \text{if vertex } i \text{ is not in vertex cover} \\
1 & \text{if vertex } i \text{ is in vertex cover}
\end{cases}$$

Vertex covers in 1–1 correspondence with 0/1 assignments: $S = \{i \in V : x_i = 1\}$.

- Objective function: minimize $\Sigma_i w_i x_i$.

- For every edge $(i, j)$, must take either vertex $i$ or $j$ (or both): $x_i + x_j \geq 1$. 
Weighted vertex cover: ILP formulation

Weighted vertex cover. Integer linear programming formulation.

\[(ILP) \quad \min \sum_{i \in V} w_i x_i \]
\[\text{s.t.} \quad x_i + x_j \geq 1 \quad (i, j) \in E \]
\[x_i \in \{0, 1\} \quad i \in V \]

Observation. If \(x^*\) is optimal solution to \(ILP\), then \(S = \{ i \in V : x_i^* = 1 \}\) is a min-weight vertex cover.
**Integer linear programming**

Given integers $a_{ij}$, $b_i$, and $c_j$, find integers $x_j$ that satisfy:

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad x \geq 0 \\
& \quad x \text{ integral}
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad 1 \leq i \leq m \\
& \quad x_j \geq 0 \quad 1 \leq j \leq n \\
& \quad x_j \text{ integral} \quad 1 \leq j \leq n
\end{align*}
\]

**Observation.** Vertex cover formulation proves that INTEGER-PROGRAMMING is an **NP**-hard search problem.
Linear programming

Given integers $a_{ij}$, $b_i$, and $c_j$, find real numbers $x_j$ that satisfy:

$$\begin{align*}
\min & \quad c^\top x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}$$

$$\begin{align*}
\min & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad 1 \leq i \leq m \\
& \quad x_j \geq 0 \quad 1 \leq j \leq n
\end{align*}$$

Linear. No $x^2$, $xy$, $arccos(x)$, $x(1-x)$, etc.


Interior point algorithms. [Karmarkar 1984, Renegar 1988, ... ]

Can solve LP both in poly-time and in practice.
LP feasible region

LP geometry in 2D.

The region satisfying the inequalities:
\[ x_1 \geq 0, \ x_2 \geq 0 \]
\[ x_1 + 2x_2 \geq 6 \]
\[ 2x_1 + x_2 \geq 6 \]
Weighted vertex cover: LP relaxation

Linear programming relaxation.

\[
(LP) \quad \min \sum_{i \in V} w_i x_i \\
\text{s.t.} \quad x_i + x_j \geq 1 \quad (i, j) \in E \\
\quad \quad \quad \quad x_i \geq 0 \quad i \in V
\]

Observation. Optimal value of \( LP \) is \( \leq \) optimal value of \( ILP \).

Pf. \( LP \) has fewer constraints.

Note. \( LP \) is not equivalent to weighted vertex cover.
(even if all weights are 1)

Q. How can solving \( LP \) help us find a low-weight vertex cover?
A. Solve \( LP \) and round fractional values.
Weighted vertex cover: LP rounding algorithm

**Lemma.** If \( x^* \) is optimal solution to \( LP \), then \( S = \{ i \in V : x^*_i \geq \frac{1}{2} \} \) is a vertex cover whose weight is at most twice the min possible weight.

**Pf.** [\( S \) is a vertex cover]
- Consider an edge \((i, j) \in E\).
- Since \( x^*_i + x^*_j \geq 1 \), either \( x^*_i \geq \frac{1}{2} \) or \( x^*_j \geq \frac{1}{2} \) (or both) \( \Rightarrow \) \((i, j)\) covered.

**Pf.** [\( S \) has desired cost]
- Let \( S^* \) be optimal vertex cover. Then

\[
\sum_{i \in S^*} w_i \geq \sum_{i \in S} w_i x^*_i \geq \frac{1}{2} \sum_{i \in S} w_i
\]

LP is a relaxation  \( x^*_i \geq \frac{1}{2} \)

**Theorem.** The rounding algorithm is a 2-approximation algorithm.

**Pf.** Lemma + fact that \( LP \) can be solved in poly-time.
Weighted vertex cover inapproximability

**Theorem.** [Dinur–Safra 2004] If $P \neq NP$, then no $\rho$-approximation for WEIGHTED-VERTEX-COVER for any $\rho < 1.3606$ (even if all weights are 1).

On the Hardness of Approximating Minimum Vertex Cover

Irit Dinur* Samuel Safra†

May 26, 2004

**Abstract**

We prove the Minimum Vertex Cover problem to be NP-hard to approximate to within a factor of 1.3606, extending on previous PCP and hardness of approximation technique. To that end, one needs to develop a new proof framework, and borrow and extend ideas from several fields.

Open research problem. Close the gap.
11. APPROXIMATION ALGORITHMS

- load balancing
- center selection
- pricing method: weighted vertex cover
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- generalized load balancing
- knapsack problem
Generalized load balancing

**Input.** Set of $m$ machines $M$; set of $n$ jobs $J$.
- Job $j \in J$ must run contiguously on an **authorized machine** in $M_j \subseteq M$.
- Job $j \in J$ has processing time $t_j$.
- Each machine can process at most one job at a time.

**Def.** Let $J_i$ be the subset of jobs assigned to machine $i$.
The load of machine $i$ is $L_i = \sum_{j \in J_i} t_j$.

**Def.** The makespan is the maximum load on any machine $= \max_i L_i$.

**Generalized load balancing.** Assign each job to an authorized machine to minimize makespan.
Generalized load balancing: integer linear program and relaxation

**ILP formulation.** \( x_{ij} \) = time machine \( i \) spends processing job \( j \).

\[
(IP) \quad \min \quad L \\
\text{s. t.} \quad \sum_i x_{ij} = t_j \quad \text{for all } j \in J \\
\quad \sum_j x_{ij} \leq L \quad \text{for all } i \in M \\
x_{ij} \in \{0, t_j\} \quad \text{for all } j \in J \text{ and } i \in M_j \\
x_{ij} = 0 \quad \text{for all } j \in J \text{ and } i \notin M_j
\]

**LP relaxation.**

\[
(LP) \quad \min \quad L \\
\text{s. t.} \quad \sum_i x_{ij} = t_j \quad \text{for all } j \in J \\
\quad \sum_j x_{ij} \leq L \quad \text{for all } i \in M \\
x_{ij} \geq 0 \quad \text{for all } j \in J \text{ and } i \in M_j \\
x_{ij} = 0 \quad \text{for all } j \in J \text{ and } i \notin M_j
\]
Generalized load balancing: lower bounds

Lemma 1. The optimal makespan $L^* \geq \max_j t_j$.
Proof. Some machine must process the most time-consuming job. □

Lemma 2. Let $L$ be optimal value to the LP. Then, optimal makespan $L^* \geq L$.
Proof. LP has fewer constraints than ILP formulation. □
Generalized load balancing: structure of LP solution

Lemma 3. Let $x$ be solution to LP. Let $G(x)$ be the graph with an edge between machine $i$ and job $j$ if $x_{ij} > 0$. Then $G(x)$ is acyclic.

Pf. (deferred)
Generalized load balancing: rounding

Rounded solution. Find $LP$ solution $x$ where $G(x)$ is a forest. Root forest $G(x)$ at some arbitrary machine node $r$.

- If job $j$ is a leaf node, assign $j$ to its parent machine $i$.
- If job $j$ is not a leaf node, assign $j$ to any one of its children.

Lemma 4. Rounded solution only assigns jobs to authorized machines.

Pf. If job $j$ is assigned to machine $i$, then $x_{ij} > 0$. $LP$ solution can only assign positive value to authorized machines. □

![Diagram of a tree representing the assignment of jobs to machines. Each internal job node is assigned to an arbitrary child, and each leaf is assigned to its parent.]
Generalized load balancing: analysis

Lemma 5. If job $j$ is a leaf node and machine $i = \text{parent}(j)$, then $x_{ij} = t_j$.

Pf.
- Since $i$ is a leaf, $x_{ij} = 0$ for all $j \neq \text{parent}(i)$.
- LP constraint guarantees $\sum_i x_{ij} = t_j$.

Lemma 6. At most one non-leaf job is assigned to a machine.

Pf. The only possible non-leaf job assigned to machine $i$ is $\text{parent}(i)$.
Generalized load balancing: analysis

**Theorem.** Rounded solution is a 2-approximation.

**Pf.**

- Let \( J(i) \) be the jobs assigned to machine \( i \).
- By **Lemma 6**, the load \( L_i \) on machine \( i \) has two components:

  - **leaf nodes:**
    \[
    \sum_{\substack{j \in J(i) \ \text{is a leaf}}} t_j = \sum_{\substack{j \in J(i) \ \text{is a leaf}}} x_{ij} \leq \sum_{j \in J} x_{ij} \leq L \leq L^* 
    \]
    
    \( \sum_{\substack{j \in J(i) \ \text{is a leaf}}} t_j \) \( \sum_{\substack{j \in J(i) \ \text{is a leaf}}} x_{ij} \) \( \sum_{j \in J} x_{ij} \) \( L \) \( L^* \)

  - **parent:** \( t_{\text{parent}(i)} \leq L^* \)

- Thus, the overall load \( L_i \leq 2L^* \). ■
Generalized load balancing: flow formulation

Flow formulation of $LP$.

$$\sum_{i} x_{ij} = t_j \quad \text{for all } j \in J$$

$$\sum_{j} x_{ij} \leq L \quad \text{for all } i \in M$$

$$x_{ij} \geq 0 \quad \text{for all } j \in J \text{ and } i \in M_j$$

$$x_{ij} = 0 \quad \text{for all } j \in J \text{ and } i \notin M_j$$

**Observation.** Solution to feasible flow problem with value $L$ are in 1-to-1 correspondence with $LP$ solutions of value $L$. 
Generalized load balancing: structure of solution

Lemma 3. Let \((x, L)\) be solution to \(LP\). Let \(G(x)\) be the graph with an edge from machine \(i\) to job \(j\) if \(x_{ij} > 0\). We can find another solution \((x', L)\) such that \(G(x')\) is acyclic.

\textbf{Pf.} Let \(C\) be a cycle in \(G(x)\).

\begin{itemize}
  \item Augment flow along the cycle \(C\). \hspace{1cm} \text{flow conservation maintained}
  \item At least one edge from \(C\) is removed (and none are added).
  \item Repeat until \(G(x')\) is acyclic.
\end{itemize}
Conclusions

Running time. The bottleneck operation in our 2-approximation is solving one LP with \( mn + 1 \) variables.

Remark. Can solve LP using flow techniques on a graph with \( m + n + 1 \) nodes: given \( L \), find feasible flow if it exists. Binary search to find \( L^* \).


- Job \( j \) takes \( t_{ij} \) time if processed on machine \( i \).
- 2-approximation algorithm via LP rounding.
- If \( P \neq NP \), then no \( \rho \)-approximation exists for any \( \rho < 3/2 \).
11. Approximation Algorithms

- load balancing
- center selection
- pricing method: weighted vertex cover
- LP rounding: weighted vertex cover
- generalized load balancing
- knapsack problem
Polynomial-time approximation scheme

**PTAS.** \((1 + \varepsilon)\)-approximation algorithm for any constant \(\varepsilon > 0\).

- Load balancing. [Hochbaum–Shmoys 1987]
- Euclidean TSP. [Arora, Mitchell 1996]

**Consequence.** PTAS produces arbitrarily high quality solution, but trades off accuracy for time.

**This section.** PTAS for knapsack problem via rounding and scaling.
Knapsack problem

Knapsack problem.

• Given \( n \) objects and a knapsack.
• Item \( i \) has value \( v_i > 0 \) and weighs \( w_i > 0 \).  \[\text{we assume } w_i \leq W \text{ for each } i\]
• Knapsack has weight limit \( W \).
• Goal: fill knapsack so as to maximize total value.

Ex: \{ 3, 4 \} has value 40.

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original instance (\( W = 11 \))
Knapsack is NP-complete

**Knapsack.** Given a set $X$, weights $w_i \geq 0$, values $v_i \geq 0$, a weight limit $W$, and a target value $V$, is there a subset $S \subseteq X$ such that:

$$\sum_{i \in S} w_i \leq W$$

$$\sum_{i \in S} v_i \geq V$$

**Subset-Sum.** Given a set $X$, values $u_i \geq 0$, and an integer $U$, is there a subset $S \subseteq X$ whose elements sum to exactly $U$?

**Theorem.** Subset-Sum $\leq_P$ Knapsack.

**Pf.** Given instance $(u_1, \ldots, u_n, U)$ of Subset-Sum, create Knapsack instance:

$$v_i = w_i = u_i$$

$$\sum_{i \in S} u_i \leq U$$

$$V = W = U$$

$$\sum_{i \in S} u_i \geq U$$
Knapsack problem: dynamic programming I

**Def.** \( OPT(i, w) = \text{max value subset of items } 1, \ldots, i \text{ with weight limit } w. \)

**Case 1.** \( OPT \) does not select item \( i \).

- \( OPT \) selects best of \( 1, \ldots, i - 1 \) using up to weight limit \( w \).

**Case 2.** \( OPT \) selects item \( i \).

- New weight limit = \( w - w_i \).
- \( OPT \) selects best of \( 1, \ldots, i - 1 \) using up to weight limit \( w - w_i \).

\[
OPT(i, w) = \begin{cases} 
0 & \text{if } i = 0 \\
OPT(i - 1, w) & \text{if } w_i > w \\
\max\{ OPT(i - 1, w), \ v_i + OPT(i - 1, w - w_i) \} & \text{otherwise}
\end{cases}
\]

**Theorem.** Computes the optimal value in \( O(n W) \) time.

- Not polynomial in input size.
- Polynomial in input size if weights are small integers.
Knapsack problem: dynamic programming II

**Def.** $OPT(i, v) = \min \text{ weight of a knapsack for which we can obtain a solution of value } \geq v \text{ using a subset of items } 1, \ldots, i.$

**Note.** Optimal value is the largest value $v$ such that $OPT(n, v) \leq W.$

**Case 1.** $OPT$ does not select item $i$.
- $OPT$ selects best of $1, \ldots, i-1$ that achieves value $\geq v$.

**Case 2.** $OPT$ selects item $i$.
- Consumes weight $w_i$, need to achieve value $\geq v - v_i$.
- $OPT$ selects best of $1, \ldots, i-1$ that achieves value $\geq v - v_i$.

\[ OPT(i, v) = \begin{cases} 
0 & \text{if } v \leq 0 \\
\infty & \text{if } i = 0 \text{ and } v > 0 \\
\min \{OPT(i - 1, v), w_i + OPT(i - 1, v - v_i)\} & \text{otherwise}
\]
Theorem. Dynamic programming algorithm II computes the optimal value in $O(n^2 v_{\text{max}})$ time, where $v_{\text{max}}$ is the maximum of any value.

Pf.

• The optimal value $V^* \leq n v_{\text{max}}$.
• There is one subproblem for each item and for each value $v \leq V^*$.
• It takes $O(1)$ time per subproblem. 

Remark 1. Not polynomial in input size!
Remark 2. Polynomial time if values are small integers.
Intuition for approximation algorithm.

- Round all values up to lie in smaller range.
- Run dynamic programming algorithm II on rounded/scaled instance.
- Return optimal items in rounded instance.

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original instance (W = 11)  
rounded instance (W = 11)
Knapsack problem: polynomial-time approximation scheme

Round up all values:

- $0 < \varepsilon \leq 1$ = precision parameter.
- $v_{\text{max}}$ = largest value in original instance.
- $\theta = $ scaling factor $= \varepsilon \frac{v_{\text{max}}}{2n}$.

$$\bar{v}_i = \left\lfloor \frac{v_i}{\theta} \right\rfloor \theta, \quad \hat{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil$$

Observation. Optimal solutions to problem with $\bar{v}$ are equivalent to optimal solutions to problem with $\hat{v}$.

Intuition. $\bar{v}$ close to $v$ so optimal solution using $\bar{v}$ is nearly optimal; $\hat{v}$ small and integral so dynamic programming algorithm II is fast.
Knapsack problem: polynomial-time approximation scheme

**Theorem.** If $S$ is solution found by rounding algorithm and $S^*$ is any other feasible solution, then  
\[
(1 + \epsilon) \sum_{i \in S} v_i \geq \sum_{i \in S^*} v_i
\]

**Pf.** Let $S^*$ be any feasible solution satisfying weight constraint.

\[
\begin{align*}
\sum_{i \in S^*} v_i & \leq \sum_{i \in S^*} \bar{v}_i \quad \text{always round up} \\
& \leq \sum_{i \in S} \bar{v}_i \quad \text{solve rounded instance optimally} \\
& \leq \sum_{i \in S} (v_i + \theta) \quad \text{never round up by more than } \theta \\
& \leq \sum_{i \in S} v_i + n\theta \quad |S| \leq n \\
& = \sum_{i \in S} v_i + \frac{1}{2} \epsilon v_{\text{max}} \quad \theta = \epsilon v_{\text{max}} / 2n \\
& \leq (1 + \epsilon) \sum_{i \in S} v_i \\
\end{align*}
\]

choosing $S^* = \{ \max \}$

\[
\begin{align*}
v_{\text{max}} & \leq \sum_{i \in S} v_i + \frac{1}{2} \epsilon v_{\text{max}} \leq \sum_{i \in S} v_i + \frac{1}{2} v_{\text{max}} \\
& \leq 2 \sum_{i \in S} v_i \\
\end{align*}
\]

subset containing only the item of largest value

$v_{\text{max}} \leq 2 \sum_{i \in S} v_i$
Knapsack problem: polynomial-time approximation scheme

**Theorem.** For any $\varepsilon > 0$, the rounding algorithm computes a feasible solution whose value is within a $(1 + \varepsilon)$ factor of the optimum in $O(n^3 / \varepsilon)$ time.

**Pf.**

- We have already proved the accuracy bound.
- Dynamic program II running time is $O(n^2 \hat{v}_{\text{max}})$, where

\[
\hat{v}_{\text{max}} = \left\lceil \frac{v_{\text{max}}}{\theta} \right\rceil = \left\lceil \frac{2n}{\varepsilon} \right\rceil
\]