7. Network Flows I

- Ford-Fulkerson pathological example
Ford-Fulkerson pathological example

**Intuition.** Let \( r \) satisfy \( r^2 = 1 - r \).
- Initial capacities are \( \{ 1, r \} \).
- After some augmentation, residual capacities are \( \{ 1, r, r^2 \} \).
- After some more, residual capacities are \( \{ 1, r, r^2, r^3 \} \).
- After some more, residual capacities are \( \{ 1, r, r^2, r^3, r^4 \} \).

\[
r = \frac{\sqrt{5} - 1}{2} \implies r^2 = 1 - r
\]
Ford-Fulkerson pathological example

network $G$

$u^2 = 1 - r$

sufficiently large that it won't ever be a bottleneck (we'll suppress)
Ford-Fulkerson pathological example

augmenting path 1: $s \rightarrow v \rightarrow w \rightarrow t$ (bottleneck capacity = 1)

$r^2 = 1 - r$
Ford-Fulkerson pathological example

augmenting path 2: $s \rightarrow x \rightarrow w \rightarrow v \rightarrow u \rightarrow t$ (bottleneck capacity = $r$)

$r^2 = 1 - r$
Ford-Fulkerson pathological example

augmenting path 3: $s \rightarrow v \rightarrow w \rightarrow x \rightarrow t$ (bottleneck capacity = $r$)

$$r^2 = 1 - r$$
augmenting path 4: \( s \rightarrow x \rightarrow w \rightarrow v \rightarrow u \rightarrow t \) (bottleneck capacity = \( r^2 \))

\[
r^2 = 1 - r
\]
Ford-Fulkerson pathological example

augmenting path 5: $s \rightarrow u \rightarrow v \rightarrow w \rightarrow t$ (bottleneck capacity = $r^2$)

$r^2 = 1 - r$
Ford-Fulkerson pathological example

augmenting path 6: $s \rightarrow x \rightarrow w \rightarrow v \rightarrow u \rightarrow t$ (bottleneck capacity = $r_3$)

\[ r^2 = 1 - r \]
Ford-Fulkerson pathological example

augmenting path 7: $s \rightarrow v \rightarrow w \rightarrow x \rightarrow t$ (bottleneck capacity = $r^3$)
Ford-Fulkerson pathological example

augmenting path 8: $s \rightarrow x \rightarrow w \rightarrow v \rightarrow u \rightarrow t$ (bottleneck capacity = $r^4$)

$r^2 = 1 - r$
Ford-Fulkerson pathological example

augmenting path 9: s→u→v→w→t (bottleneck capacity = r^4)

\[ r^2 = 1 - r \]
Ford-Fulkerson pathological example

after augmenting path 1: \( \{ 1 - r^0, 1, r - r^1 \} \) (flow = 1)

after augmenting path 5: \( \{ 1 - r^2, 1, r - r^3 \} \) (flow = \(1 + 2r + 2r^2\))

after augmenting path 9: \( \{ 1 - r^4, 1, r - r^5 \} \) (flow = \(1 + 2r + 2r^2 + 2r^3 + 2r^4\))

\[ r^2 = 1 - r \]
Ford-Fulkerson pathological example

**Theorem.** The Ford-Fulkerson algorithm may not terminate; moreover, it may converge a value not equal to the value of the maximum flow.

**Pf.**

- Using the given sequence of augmenting paths, after $(1 + 4k)^{th}$ such path, the value of the flow

\[
= 1 + 2 \sum_{i=1}^{2k} r^i
\]

\[
\leq 1 + 2 \sum_{i=1}^{\infty} r^i
\]

\[
= 1 + 2 \left( \frac{\sqrt{5} - 1}{2} \right)
\]

\[
r = \frac{\sqrt{5} - 1}{2}
\]

\[
= 3 + 2r
\]

\[
< 5
\]

- Value of maximum flow = $200 + 1$. ■