6. Dynamic Programming II

- sequence alignment
- Hirschberg’s algorithm
- Bellman–Ford algorithm
- distance vector protocols
- negative cycles in a digraph
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String similarity

Q. How similar are two strings?

Ex. occurrence and occurrence.

6 mismatches, 1 gap

1 mismatch, 1 gap

0 mismatches, 3 gaps
Edit distance

**Edit distance.** [Levenshtein 1966, Needleman–Wunsch 1970]

- Gap penalty $\delta$; mismatch penalty $\alpha_{pq}$.
- Cost = sum of gap and mismatch penalties.

\[
\begin{align*}
\text{cost} &= \delta + \alpha_{CG} + \alpha_{TA} \\
\end{align*}
\]

**Applications.** Unix diff, speech recognition, computational biology, ...
Sequence alignment

**Goal.** Given two strings $x_1 x_2 \ldots x_m$ and $y_1 y_2 \ldots y_n$ find a min-cost alignment.

**Def.** An alignment $M$ is a set of ordered pairs $x_i - y_j$ such that each item occurs in at most one pair and no crossings.

**Def.** The cost of an alignment $M$ is:

$$\text{cost}(M) = \sum_{(x_i, y_j) \in M} \alpha_{x_i, y_j} + \sum_{i \text{ : } x_i \text{ unmatched}} \delta + \sum_{j \text{ : } y_j \text{ unmatched}} \delta$$

$x_i - y_j$ and $x_i' - y_j'$ cross if $i < i'$, but $j > j'$

**Diagram:**

```
   x1  x2  x3  x4  x5  x6
  C   T   A   C   C   -   G
-   -   T   A   C   A   T   G
  y1  y2  y3  y4  y5  y6
```

an alignment of CTACCG and TACATG:

$M = \{ x_2 - y_1, x_3 - y_2, x_4 - y_3, x_5 - y_4, x_6 - y_6 \}$
Sequence alignment: problem structure

**Def.** \( OPT(i, j) = \) min cost of aligning prefix strings \( x_1 x_2 ... x_i \) and \( y_1 y_2 ... y_j \).

**Goal.** \( OPT(m, n) \).

**Case 1.** \( OPT(i, j) \) matches \( x_i - y_j \).
Pay mismatch for \( x_i - y_j \) + min cost of aligning \( x_1 x_2 ... x_{i-1} \) and \( y_1 y_2 ... y_{j-1} \).

**Case 2a.** \( OPT(i, j) \) leaves \( x_i \) unmatched.
Pay gap for \( x_i \) + min cost of aligning \( x_1 x_2 ... x_{i-1} \) and \( y_1 y_2 ... y_{j-1} \).

**Case 2b.** \( OPT(i, j) \) leaves \( y_j \) unmatched.
Pay gap for \( y_j \) + min cost of aligning \( x_1 x_2 ... x_i \) and \( y_1 y_2 ... y_{j-1} \).

\[
OPT(i, j) = \begin{cases} 
  j\delta & \text{if } i = 0 \\
  \min \begin{cases} 
    \alpha_{x_i y_j} + OPT(i-1, j-1) \\
    \delta + OPT(i-1, j) \\
    \delta + OPT(i, j-1) \\
  \end{cases} & \text{otherwise} \\
  i\delta & \text{if } j = 0 
\end{cases}
\]

(Proof via exchange argument)
SEQUENCE-ALIGNMENT \((m, n, x_1, \ldots, x_m, y_1, \ldots, y_n, \delta, \alpha)\)

FOR \(i = 0\) TO \(m\)
\[
M[i, 0] \leftarrow i \delta.
\]

FOR \(j = 0\) TO \(n\)
\[
M[0, j] \leftarrow j \delta.
\]

FOR \(i = 1\) TO \(m\)

FOR \(j = 1\) TO \(n\)
\[
M[i, j] \leftarrow \min \{ \alpha[x_i, y_j] + M[i-1, j-1], \\
\delta + M[i-1, j], \\
\delta + M[i, j-1] \}.
\]

RETURN \(M[m, n]\).
Sequence alignment: analysis

**Theorem.** The dynamic programming algorithm computes the edit distance (and optimal alignment) of two strings of lengths $m$ and $n$ in $\Theta(mn)$ time and $\Theta(mn)$ space.

**Pf.**

- Algorithm computes edit distance.
- Can trace back to extract optimal alignment itself. □

**Q.** Can we avoid using quadratic space?

**A.** Easy to compute optimal value in $O(mn)$ time and $O(m + n)$ space.

- Compute $\text{OPT}(i, \bullet)$ from $\text{OPT}(i - 1, \bullet)$.
- **But**, no longer easy to recover optimal alignment itself.
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Theorem. There exists an algorithm to find an optimal alignment in \( O(mn) \) time and \( O(m + n) \) space.

- Clever combination of divide-and-conquer and dynamic programming.
- Inspired by idea of Savitch from complexity theory.
Hirschberg’s algorithm

Edit distance graph.

- Let $f(i, j)$ be shortest path from $(0,0)$ to $(i, j)$.
- Lemma: $f(i, j) = OPT(i, j)$ for all $i$ and $j$. 
Hirschberg’s algorithm

Edit distance graph.

- Let \( f(i, j) \) be shortest path from \((0,0)\) to \((i, j)\).
- Lemma: \( f(i, j) = OPT(i, j) \) for all \( i \) and \( j \).

Pf of Lemma. [ by strong induction on \( i + j \) ]

- Base case: \( f(0, 0) = OPT(0, 0) = 0 \).
- Inductive hypothesis: assume true for all \((i', j')\) with \( i' + j' < i + j \).
- Last edge on shortest path to \((i, j)\) is from \((i - 1, j - 1)\), \((i - 1, j)\), or \((i, j - 1)\).
- Thus,

\[
    f(i, j) = \min \{ \alpha_{x_i y_j} + f(i - 1, j - 1), \ \delta + f(i - 1, j), \ \delta + f(i, j - 1) \} \\
    = \min \{ \alpha_{x_i y_j} + OPT(i - 1, j - 1), \ \delta + OPT(i - 1, j), \ \delta + OPT(i, j - 1) \} \\
    = OPT(i, j) \quad \blacksquare
\]
Hirschberg’s algorithm

Edit distance graph.

- Let \( f(i, j) \) be shortest path from \((0,0)\) to \((i, j)\).
- Lemma: \( f(i, j) = OPT(i, j) \) for all \( i \) and \( j \).
- Can compute \( f(\bullet, j) \) for any \( j \) in \( O(mn) \) time and \( O(m + n) \) space.
Hirschberg’s algorithm

Edit distance graph.

- Let $g(i, j)$ be shortest path from $(i, j)$ to $(m, n)$.
- Can compute by reversing the edge orientations and inverting the roles of $(0, 0)$ and $(m, n)$. 

\[
\begin{align*}
\varepsilon & \rightarrow y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6 \\
x_1 & \rightarrow y_1 \\
x_2 & \rightarrow y_2 \\
x_3 & \rightarrow y_3 \\
0, 0 & \rightarrow y_1 \\
m, n & \rightarrow y_6
\end{align*}
\]
Hirschberg’s algorithm

Edit distance graph.
- Let $g(i, j)$ be shortest path from $(i, j)$ to $(m, n)$.
- Can compute $g(\bullet, j)$ for any $j$ in $O(mn)$ time and $O(m + n)$ space.
Hirschberg’s algorithm

**Observation 1.** The cost of a shortest path that uses \((i, j)\) is \(f(i, j) + g(i, j)\).
Hirschberg’s algorithm

**Observation 2.** Let $q$ be an index that minimizes $f(q, n/2) + g(q, n/2)$. Then, there exists a shortest path from $(0, 0)$ to $(m, n)$ that uses $(q, n/2)$. 
Hirschberg’s algorithm

**Divide.** Find index $q$ that minimizes $f(q, n/2) + g(q, n/2)$; align $x_q$ and $y_{n/2}$.

**Conquer.** Recursively compute optimal alignment in each piece.
Hirschberg’s algorithm: running time analysis warmup

**Theorem.** Let $T(m, n) = \max$ running time of Hirschberg’s algorithm on strings of lengths at most $m$ and $n$. Then, $T(m, n) = O(m \cdot n \log n)$.

**Pf.**
\[
T(m, n) \leq 2T(m, n/2) + O(m \cdot n)
\]
\[
\Rightarrow T(m, n) = O(m \cdot n \log n).
\]

**Remark.** Analysis is not tight because two subproblems are of size $(q, n/2)$ and $(m − q, n/2)$. In next slide, we save $\log n$ factor.
Hirschberg’s algorithm: running time analysis

**Theorem.** Let $T(m, n) = \max$ running time of Hirschberg’s algorithm on strings of lengths at most $m$ and $n$. Then, $T(m, n) = O(mn)$.

**Pf.** [ by induction on $n$ ]

- $O(mn)$ time to compute $f(\bullet, n/2)$ and $g(\bullet, n/2)$ and find index $q$.
- $T(q, n/2) + T(m - q, n/2)$ time for two recursive calls.
- Choose constant $c$ so that: $T(m, 2) \leq cm$
  
  $T(2, n) \leq cn$

  $T(m, n) \leq c mn + T(q, n/2) + T(m - q, n/2)$

  
  Claim. $T(m, n) \leq 2 cm n$.

  Base cases: $m = 2$ or $n = 2$.

  Inductive hypothesis: $T(m, n) \leq 2 cm n$ for all $(m', n')$ with $m' + n' < m + n$.

  $T(m, n) \leq T(q, n/2) + T(m - q, n/2) + cm n$

  \leq 2 c q n/2 + 2 c (m - q) n/2 + cm n$

  $= c q n + cm n - c q n + cm n$

  $= 2 cm n$ \hfill $\blacksquare$
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Shortest paths

Shortest-path problem. Given a digraph \( G = (V, E) \), with arbitrary edge weights or costs \( c_{vw} \), find cheapest path from node \( s \) to node \( t \).

cost of path = 9 − 3 + 1 + 11 = 18
Shortest paths: failed attempts

**Dijkstra.** May not produce shortest paths when edge weights are negative.

**Reweighting.** Adding a constant to every edge weight does not necessarily make Dijkstra’s algorithm produce shortest paths.
**Def.** A negative cycle is a directed cycle such that the sum of its edge weights is negative.

A negative cycle $W$:

$$c(W) = \sum_{e \in W} c_e < 0$$
Shortest paths and negative cycles

Lemma 1. If some path from $v$ to $t$ contains a negative cycle, then there does not exist a cheapest path from $v$ to $t$.

\textbf{Pf.} If there exists such a cycle $W$, then can build a $v\rightarrow t$ path of arbitrarily negative weight by detouring around cycle as many times as desired. $\blacksquare$
Shortest paths and negative cycles

Lemma 2. If $G$ has no negative cycles, then there exists a cheapest path from $v$ to $t$ that is simple (and has $\leq n-1$ edges).

Pf.

- Consider a cheapest $v \rightarrow t$ path $P$ that uses the fewest edges.
- If $P$ contains a cycle $W$, can remove portion of $P$ corresponding to $W$ without increasing the cost. $\blacksquare$

\[
c(W) \geq 0
\]
Shortest-paths and negative-cycle problems

**Single-destination shortest-paths problem.** Given a digraph $G = (V, E)$ with edge weights $c_{vw}$ and no negative cycles and a distinguished node $t$, find cheapest $v \rightarrow t$ path for each node $v$.

**Negative-cycle problem.** Given a digraph $G = (V, E)$ with edge weights $c_{vw}$, find a negative cycle (if one exists).
Shortest paths: dynamic programming

**Def.** \(OPT(i, v) = \text{cost of shortest } v \rightarrow t \text{ path that uses } \leq i \text{ edges.}\)

- Case 1: Cheapest \(v \rightarrow t\) path uses \(\leq i - 1\) edges.
  - \(OPT(i, v) = OPT(i - 1, v)\)

- Case 2: Cheapest \(v \rightarrow t\) path uses exactly \(i\) edges.
  - if \((v, w)\) is first edge, then \(OPT\) uses \((v, w)\), and then selects best \(w \rightarrow t\) path using \(\leq i - 1\) edges

\[
OPT(i, v) = \begin{cases} 
\infty & \text{if } i = 0 \\
\min \left\{ OPT(i - 1, v), \min_{(v, w) \in E} \left\{ OPT(i - 1, w) + c_{vw} \right\} \right\} & \text{otherwise}
\end{cases}
\]

**Observation.** If no negative cycles, \(OPT(n - 1, v) = \text{cost of cheapest } v \rightarrow t \text{ path.}\)

**Pf.** By Lemma 2, cheapest \(v \rightarrow t\) path is simple. ■
**Shortest-paths** \((V, E, c, t)\)

**Foreach** node \(v \in V\)

\[
M[0, v] \leftarrow \infty.
\]

\[
M[0, t] \leftarrow 0.
\]

**For** \(i = 1\) **to** \(n - 1\)

**Foreach** node \(v \in V\)

\[
M[i, v] \leftarrow M[i - 1, v].
\]

**Foreach** edge \((v, w) \in E\)

\[
M[i, v] \leftarrow \min \{ M[i, v], M[i - 1, w] + c_{vw} \}.
\]
Shortest paths: implementation

**Theorem 1.** Given a digraph $G = (V, E)$ with no negative cycles, the dynamic programming algorithm computes the cost of a cheapest $v \rightarrow t$ path for each node $v$ in $\Theta(mn)$ time and $\Theta(n^2)$ space.

**Pf.**

- Table requires $\Theta(n^2)$ space.
- Each iteration $i$ takes $\Theta(m)$ time since we examine each edge once. □

**Finding the shortest paths.**

- Approach 1: Maintain a $successor(i, v)$ that points to next node on cheapest $v \rightarrow t$ path using at most $i$ edges.
- Approach 2: Compute optimal costs $M[i, v]$ and consider only edges with $M[i, v] = M[i - 1, w] + c_{vw}$. 
Shortest paths: practical improvements

Space optimization. Maintain two 1d arrays (instead of 2d array).

\[ d(v) = \text{cost of a cheapest } v \rightarrow t \text{ path that we have found so far.} \]

\[ \text{successor}(v) = \text{next node on a } v \rightarrow t \text{ path.} \]

Performance optimization. If \( d(w) \) was not updated in iteration \( i - 1 \), then no reason to consider edges entering \( w \) in iteration \( i \).
Bellman–Ford: efficient implementation

**Bellman–Ford** \((V, E, c, t)\)

**FOREACH** node \(v \in V\)

\(d(v) \leftarrow \infty.\)

\(successor(v) \leftarrow null.\)

\(d(t) \leftarrow 0.\)

**FOR** \(i = 1\) **TO** \(n - 1\)

**FOREACH** node \(w \in V\)

**IF** \((d(w)\) was updated in previous iteration) **AND** \((d(v) > d(w) + c_{vw})\)

\(d(v) \leftarrow d(w) + c_{vw}.\)

\(successor(v) \leftarrow w.\)

**IF no** \(d(w)\) **value changed in iteration** \(i\), **STOP**.
Bellman–Ford: analysis

**Claim.** After the \(i^{th}\) pass of Bellman Ford, \(d(v)\) equals the cost of a cheapest \(v \rightarrow t\) path using at most \(i\) edges.

**Counterexample.** Claim is false!

![Diagram](image)

If node \(w\) considered before node \(v\), then \(d(v) = 3\) after 1 pass.
Bellman–Ford: analysis

**Lemma 3.** Throughout Bellman–Ford algorithm, \( d(v) \) is the cost of some \( v \rightarrow t \) path; after the \( i^{th} \) pass, \( d(v) \) is no larger than the cost of a cheapest \( v \rightarrow t \) path using \( \leq i \) edges.

**Pf.** [by induction on \( i \)]

- Assume true after \( i^{th} \) pass.
- Let \( P \) be any \( v \rightarrow t \) path with \( i + 1 \) edges.
- Let \( (v, w) \) be first edge on path and let \( P' \) be subpath from \( w \) to \( t \).
- By inductive hypothesis, \( d(w) \leq c(P') \) since \( P' \) is a \( w \rightarrow t \) path with \( i \) edges.
- After considering \( v \) in pass \( i + 1 \): 
  \[
  d(v) \leq c_{vw} + d(w) \\
  \leq c_{vw} + c(P') \\
  = c(P)
  \]

**Theorem 2.** Given a digraph with no negative cycles, Bellman–Ford computes the costs of the cheapest \( v \rightarrow t \) paths in \( O(mn) \) time and \( \Theta(n) \) extra space.

**Pf.** Lemmas 2 + 3. □
Bellman–Ford: analysis

Claim. Throughout the Bellman–Ford algorithm, following $\text{successor}(v)$ pointers gives a directed path from $v$ to $t$ of cost $d(v)$.

Counterexample. Claim is false!

- Cost of successor $v \to t$ path may have strictly lower cost than $d(v)$.

Consider nodes in order: $t$, $1$, $2$, $3$
Bellman–Ford: analysis

Claim. Throughout the Bellman–Ford algorithm, following \( \text{successor}(v) \) pointers gives a directed path from \( v \) to \( t \) of cost \( d(v) \).

Counterexample. Claim is false!
- Cost of successor \( v \to t \) path may have strictly lower cost than \( d(v) \).

Consider nodes in order: \( t, 1, 2, 3 \)

\[
\begin{align*}
s(2) &= 1 \\
d(2) &= 20 \\
s(1) &= 3 \\
d(1) &= 2 \\
d(t) &= 0
\end{align*}
\]

\[
\begin{array}{c}
\begin{array}{c}
2 \\
1 \\
3 \\
t
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{cccc}
10 & 10 & 1 & 1
\end{array}
\end{array}
\]
Bellman–Ford: analysis

**Claim.** Throughout the Bellman–Ford algorithm, following $successor(v)$ pointers gives a directed path from $v$ to $t$ of cost $d(v)$.

**Counterexample.** Claim is false!

- Cost of successor $v \rightarrow t$ path may have strictly lower cost than $d(v)$.
- Successor graph may have cycles.

consider nodes in order: $t$, 1, 2, 3, 4

![Diagram](image_url)
Bellman–Ford: analysis

**Claim.** Throughout the Bellman–Ford algorithm, following $successor(v)$ pointers gives a directed path from $v$ to $t$ of cost $d(v)$.

**Counterexample.** Claim is false!

- Cost of successor $v \rightarrow t$ path may have strictly lower cost than $d(v)$.
- Successor graph may have cycles.

consider nodes in order: $t$, 1, 2, 3, 4

![Graph diagram]

- $d(3) = 10$
- $d(2) = 8$
- $d(t) = 0$
- $d(4) = 11$
- $d(1) = 3$
Lemma 4. If the successor graph contains a directed cycle $W$, then $W$ is a negative cycle.

Pf.

- If $\text{successor}(v) = w$, we must have $d(v) \geq d(w) + c_{vw}$.
  (LHS and RHS are equal when $\text{successor}(v)$ is set; $d(w)$ can only decrease; $d(v)$ decreases only when $\text{successor}(v)$ is reset)
- Let $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ be the nodes along the cycle $W$.
- Assume that $(v_k, v_1)$ is the last edge added to the successor graph.
- Just prior to that:
  \[
  \begin{align*}
  d(v_1) & \geq d(v_2) + c(v_1, v_2) \\
  d(v_2) & \geq d(v_3) + c(v_2, v_3) \\
  \vdots & \vdots \\
  d(v_{k-1}) & \geq d(v_k) + c(v_{k-1}, v_k) \\
  d(v_k) & > d(v_1) + c(v_k, v_1)
  \end{align*}
  \]

  holds with strict inequality since we are updating $d(v_k)$

- Adding inequalities yields $c(v_1, v_2) + c(v_2, v_3) + \ldots + c(v_{k-1}, v_k) + c(v_k, v_1) < 0$. ▪
Bellman–Ford: finding the shortest paths

Theorem 3. Given a digraph with no negative cycles and a node \( t \), Bellman–Ford finds cheapest \( v \rightarrow t \) paths for each node \( v \) in \( O(mn) \) time and \( \Theta(n) \) extra space.

Pf.

- The successor graph cannot have a negative cycle. [Lemma 4]
- Thus, following the successor pointers from \( v \) yields a directed path to \( t \).
- Let \( v = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = t \) be the nodes along this path \( P \).
- Upon termination, if \( \text{successor}(v) = w \), we must have \( d(v) = d(w) + c_{vw} \).

(LHS and RHS are equal when \( \text{successor}(v) \) is set; \( d(\cdot) \) did not change)

Thus,

\[
\begin{align*}
    d(v_1) &= d(v_2) + c(v_1, v_2) \\
    d(v_2) &= d(v_3) + c(v_2, v_3) \\
    &\vdots \\
    d(v_{k-1}) &= d(v_k) + c(v_{k-1}, v_k)
\end{align*}
\]

Adding equations yields \( d(s) = d(t) + c(v_1, v_2) + c(v_2, v_3) + \ldots + c(v_{k-1}, v_k) \). □
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Distance vector protocols

Communication network.
- Node $\approx$ router.
- Edge $\approx$ direct communication link.
- Cost of edge $\approx$ delay on link.  
  naturally nonnegative, but Bellman–Ford used anyway!

Dijkstra’s algorithm. Requires global information of network.

Bellman–Ford. Uses only local knowledge of neighboring nodes.

Synchronization. We don’t expect routers to run in lockstep. The order in which each foreach loop executes is not important. Moreover, algorithm still converges even if updates are asynchronous.
Distance vector protocols

Distance vector protocols. [“routing by rumor”]

- Each router maintains a vector of shortest-path lengths to every other node (distances) and the first hop on each path (directions).
- Algorithm: each router performs \( n \) separate computations, one for each potential destination node.

**Ex.** RIP, Xerox XNS RIP, Novell’s IPX RIP, Cisco’s IGRP, DEC’s DNA Phase IV, AppleTalk’s RTMP.

**Caveat.** Edge costs may change during algorithm (or fail completely).

```
d(s) = 2  d(v) = 1  d(t) = 0
```

“counting to infinity”
Path vector protocols

Link state routing.
- Each router also stores the entire path.
- Based on Dijkstra’s algorithm.
- Avoids “counting-to-infinity” problem and related difficulties.
- Requires significantly more storage.

Ex. Border Gateway Protocol (BGP), Open Shortest Path First (OSPF).
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Detecting negative cycles

Negative cycle detection problem. Given a digraph $G = (V, E)$, with edge weights $c_{vw}$, find a negative cycle (if one exists).
Detecting negative cycles: application

**Currency conversion.** Given $n$ currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

**Remark.** Fastest algorithm very valuable!

\[
0.741 \times 1.366 \times 0.995 = 1.00714497
\]
Detecting negative cycles

Lemma 5. If $OPT(n, v) = OPT(n - 1, v)$ for all $v$, then no negative cycle can reach $t$.


Lemma 6. If $OPT(n, v) < OPT(n - 1, v)$ for some node $v$, then (any) cheapest path from $v$ to $t$ contains a cycle $W$. Moreover $W$ is a negative cycle.

Pf. [by contradiction]
• Since $OPT(n, v) < OPT(n - 1, v)$, we know that shortest $v \rightarrow t$ path $P$ has exactly $n$ edges.
• By pigeonhole principle, $P$ must contain a directed cycle $W$.
• Deleting $W$ yields a $v \rightarrow t$ path with $< n$ edges $\Rightarrow W$ has negative cost. □
Detecting negative cycles

**Theorem 4.** Can find a negative cycle in $\Theta(mn)$ time and $\Theta(n^2)$ space.

**Pf.**

- Add new node $t$ and connect all nodes to $t$ with 0-cost edge.
- $G$ has a negative cycle iff $G'$ has a negative cycle than can reach $t$.
- If $OPT(n, v) = OPT(n - 1, v)$ for all nodes $v$, then no negative cycles.
- If not, then extract directed cycle from path from $v$ to $t$.
  (cycle cannot contain $t$ since no edges leave $t$) □
Detecting negative cycles

**Theorem 5.** Can find a negative cycle in $O(mn)$ time and $O(n)$ extra space.

**Pf.**

- Run Bellman–Ford for $n$ passes (instead of $n - 1$) on modified digraph.
- If no $d(v)$ values updated in pass $n$, then no negative cycles.
- Otherwise, suppose $d(s)$ updated in pass $n$.
- Define $\text{pass}(v) =$ last pass in which $d(v)$ was updated.
- Observe $\text{pass}(s) = n$ and $\text{pass}(\text{successor}(v)) \geq \text{pass}(v) - 1$ for each $v$.
- Following successor pointers, we must eventually repeat a node.
- Lemma 4 $\Rightarrow$ this cycle is a negative cycle.

**Remark.** See p. 304 for improved version and early termination rule.
(Tarjan’s subtree disassembly trick)