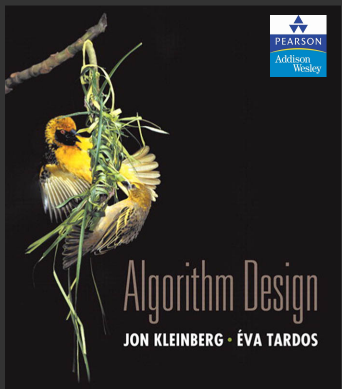
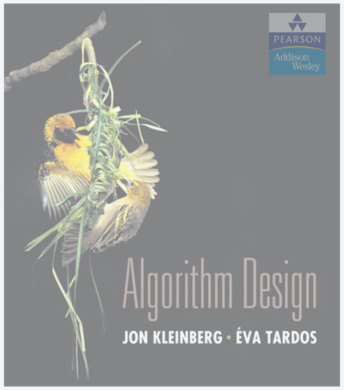



# 3. Graphs

- basic definitions and applications
- graph connectivity and graph traversal
- testing bipartiteness
- connectivity in directed graphs
- DAGs and topological ordering

Lecture slides by Kevin Wayne
Copyright © 2005 Pearson-Addison Wesley
http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 4/1/19 10:09 PM




# 3. Graphs

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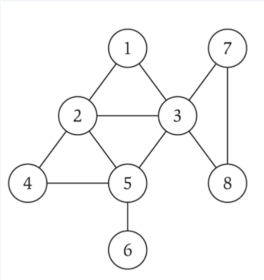
## Undirected graphs

Notation. $G = (V, E)$

- $V$ = nodes (or vertices).
- $E$ = edges (or arcs) between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: $n = |V|, m = |E|$.



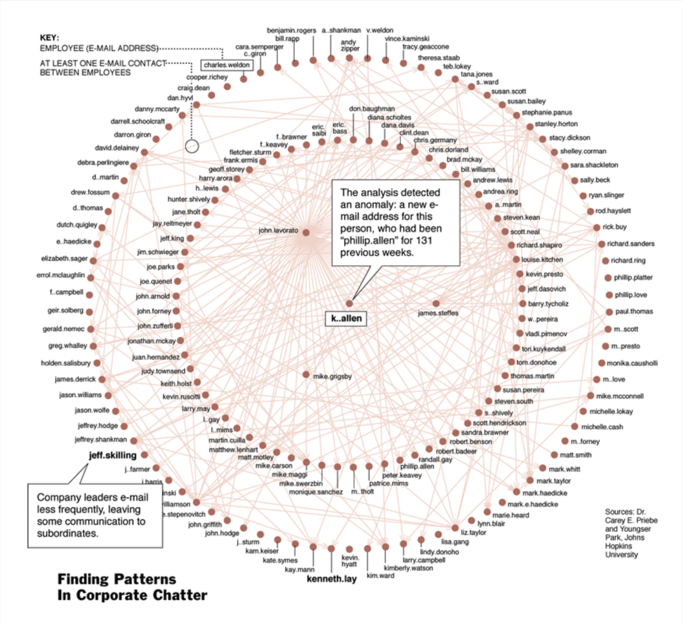

$V = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$

$E = \{ 1–2, 1–3, 2–3, 2–4, 2–5, 3–5, 3–7, 3–8, 4–5, 5–6, 7–8 \}$

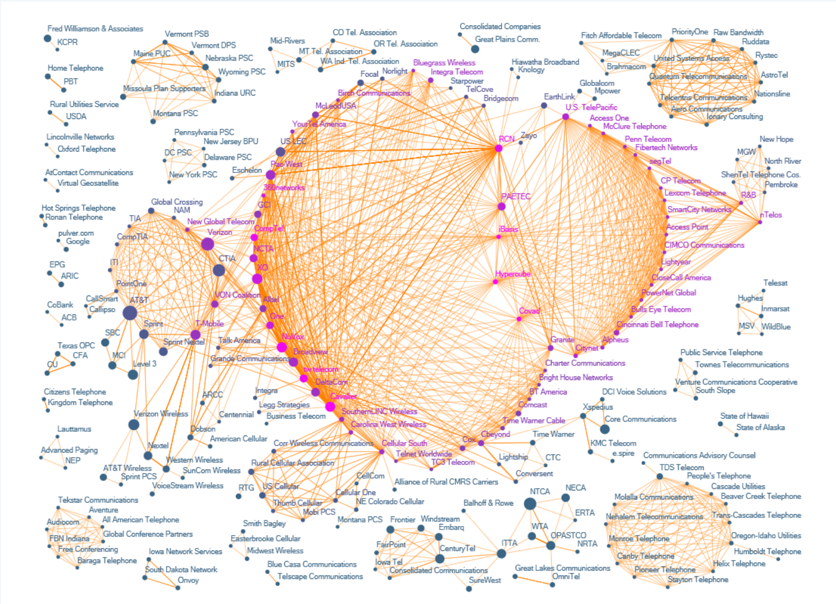
$m = 11, n = 8$



## One week of Enron emails



## The evolution of FCC lobbying coalitions



"The Evolution of FCC Lobbying Coalitions" by Pierre de Vries in JoSS Visualization Symposium 2010



## Framingham heart study

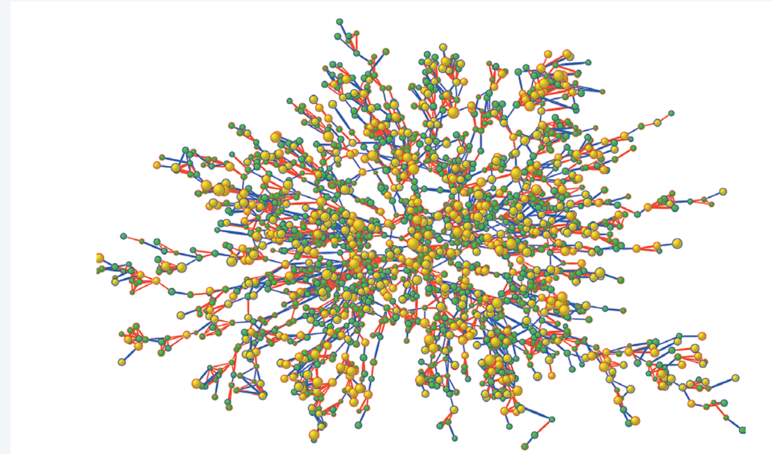


**Figure 1. Largest Connected Subcomponent of the Social Network in the Framingham Heart Study in the Year 2000.**
Each circle (node) represents one person in the data set. There are 2200 persons in this subcomponent of the social network. Circles with red borders denote women, and circles with blue borders denote men. The size of each circle is proportional to the person's body-mass index. The interior color of the circles indicates the person's obesity status: yellow denotes an obese person (body-mass index, ≥30) and green denotes a nonobese person. The colors of the ties between the nodes indicate the relationship between them: purple denotes a friendship or marital tie and orange denotes a familial tie.

"The Spread of Obesity in a Large Social Network over 32 Years" by Christakis and Fowler in New England Journal of Medicine, 2007



## Some graph applications

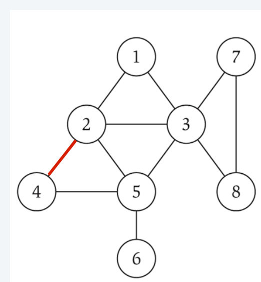
| graph | node | edge |
|---|---|---|
| **communication** | telephone, computer | fiber optic cable |
| **circuit** | gate, register, processor | wire |
| **mechanical** | joint | rod, beam, spring |
| **financial** | stock, currency | transactions |
| **transportation** | street intersection, airport | highway, airway route |
| **internet** | class C network | connection |
| **game** | board position | legal move |
| **social relationship** | person, actor | friendship, movie cast |
| **neural network** | neuron | synapse |
| **protein network** | protein | protein-protein interaction |
| **molecule** | atom | bond |



## Graph representation: adjacency matrix

Adjacency matrix. $n$-by-$n$ matrix with $A_{uv} = 1$ if $(u, v)$ is an edge.

- Two representations of each edge.
- Space proportional to $n^2$.
- Checking if $(u, v)$ is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



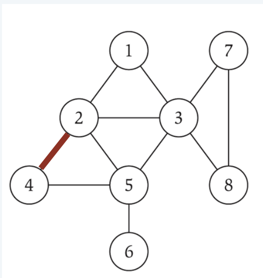

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 3 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 5 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 7 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 8 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |

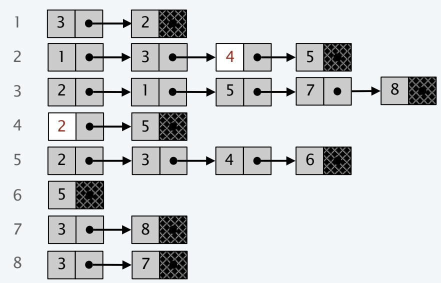
## Graph representation: adjacency lists

Adjacency lists. Node-indexed array of lists.

- Two representations of each edge.
- Space is $\Theta(m + n)$.
- Checking if $(u, v)$ is an edge takes $O(degree(u))$ time.
- Identifying all edges takes $\Theta(m + n)$ time.

degree = number of neighbors of $u$



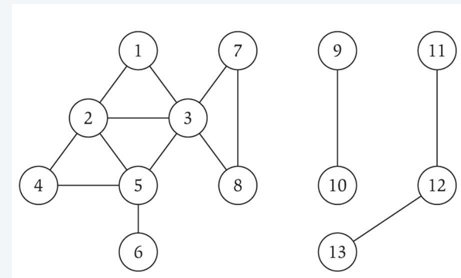



## Paths and connectivity

Def. A path in an undirected graph $G = (V, E)$ is a sequence of nodes $v_1, v_2, \ldots, v_k$ with the property that each consecutive pair $v_{i-1}, v_i$ is joined by a different edge in $E$.

Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes $u$ and $v$, there is a path between $u$ and $v$.

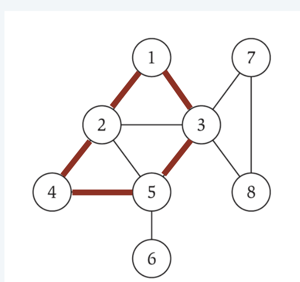





## Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_k$ in which $v_1 = v_k$ and $k \geq 2$.

Def. A cycle is simple if all nodes are distinct (except for $v_1$ and $v_k$).




**cycle C = 1-2-4-5-3-1**

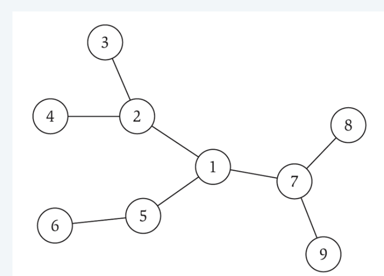


## Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Theorem. Let $G$ be an undirected graph on $n$ nodes. Any two of the following statements imply the third:

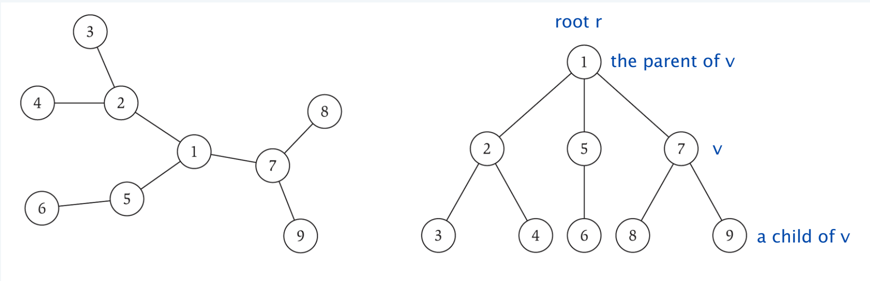
- $G$ is connected.
- $G$ does not contain a cycle.
- $G$ has $n - 1$ edges.



## Rooted trees

Given a tree $T$, choose a root node $r$ and orient each edge away from $r$.

Importance. Models hierarchical structure.



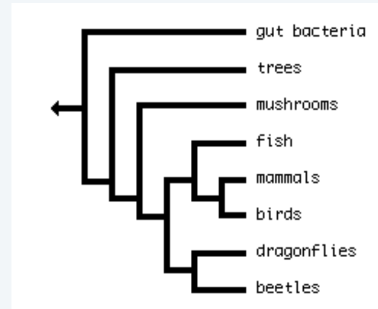

a tree

the same tree, rooted at 1



## Phylogeny trees

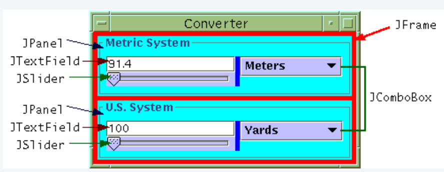
Describe evolutionary history of species.

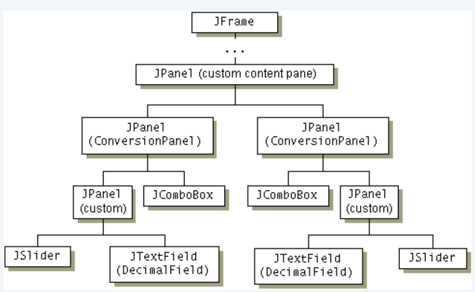





## GUI containment hierarchy

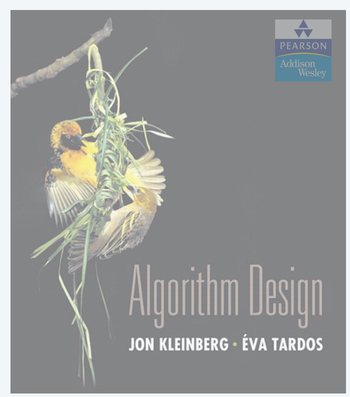
Describe organization of GUI widgets.




http://java.sun.com/docs/books/tutorial/uiswing/overview/anatomy.html



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## Connectivity

s-t connectivity problem. Given two nodes $s$ and $t$, is there a path between $s$ and $t$?

s-t shortest path problem. Given two nodes $s$ and $t$, what is the length of a shortest path between $s$ and $t$?

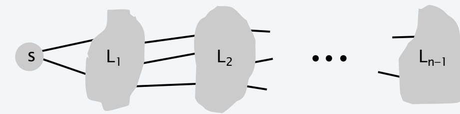
Applications.

- Friendster.
- Maze traversal.
- Kevin Bacon number.
- Fewest hops in a communication network.



## Breadth-first search

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.




BFS algorithm.

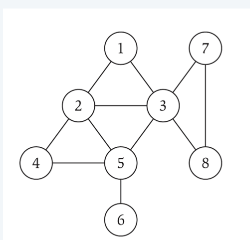
- $L_0 = \{ s \}$.
- $L_1$ = all neighbors of $L_0$.
- $L_2$ = all nodes that do not belong to $L_0$ or $L_1$, and that have an edge to a node in $L_1$.
- $L_{i+1}$ = all nodes that do not belong to an earlier layer, and that have an edge to a node in $L_i$.

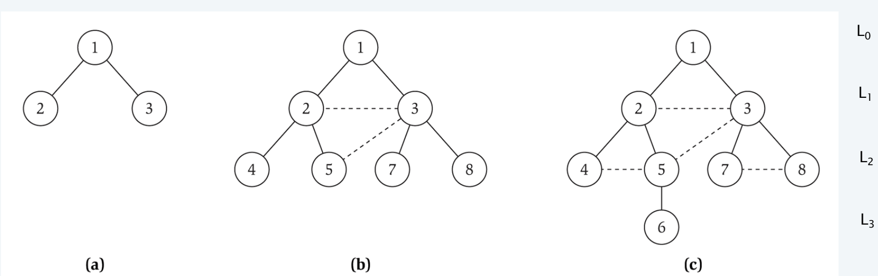
Theorem. For each $i$, $L_i$ consists of all nodes at distance exactly $i$ from $s$. There is a path from $s$ to $t$ iff $t$ appears in some layer.



## Breadth-first search

Property. Let $T$ be a BFS tree of $G = (V, E)$, and let $(x, y)$ be an edge of $G$. Then, the levels of $x$ and $y$ differ by at most 1.








## Breadth-first search: analysis

Theorem. The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency representation.

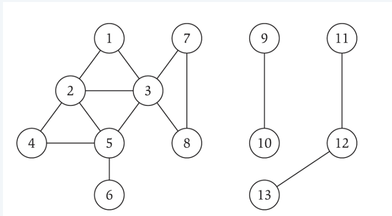
Pf.

- Easy to prove $O(n^2)$ running time:
  - at most $n$ lists $L[i]$
  - each node occurs on at most one list; for loop runs $\le n$ times
  - when we consider node $u$, there are $\le n$ incident edges $(u, v)$, and we spend $O(1)$ processing each edge
- Actually runs in $O(m + n)$ time:
  - when we consider node $u$, there are $degree(u)$ incident edges $(u, v)$
  - total time processing edges is $\Sigma_{u \in V}\ degree(u) \ = \ 2m$. ▪

each edge (u, v) is counted exactly twice
in sum: once in degree(u) and once in degree(v)

## Connected component

Connected component. Find all nodes reachable from $s$.




Connected component containing node $1 = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$.

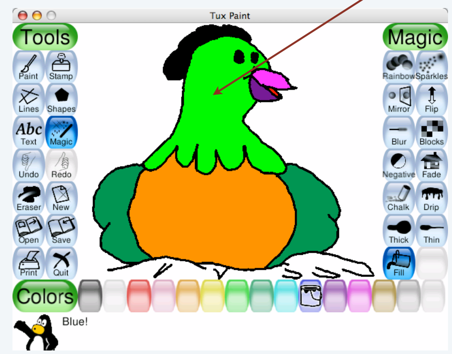


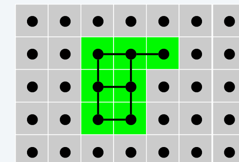
## Flood fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

recolor lime green blob to blue





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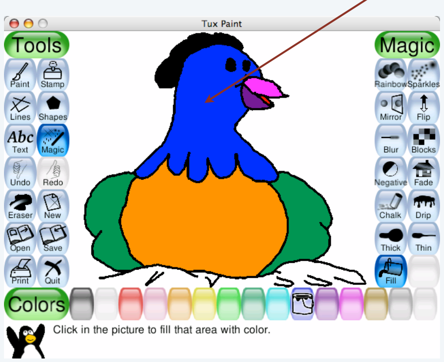
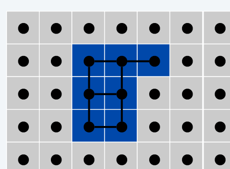
- Blob: connected component of lime pixels.

recolor lime green blob to blue



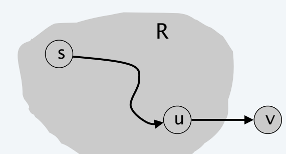





## Connected component

Connected component. Find all nodes reachable from $s$.

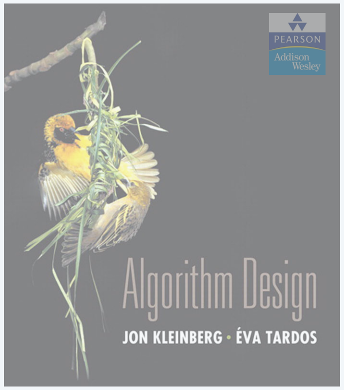
```
R will consist of nodes to which s has a path
Initially R = {s}
While there is an edge (u, v) where u ∈ R and v ∉ R
  Add v to R
Endwhile
```




**it's safe to add v**

Theorem. Upon termination, $R$ is the connected component containing $s$.

- BFS = explore in order of distance from $s$.
- DFS = explore in a different way.



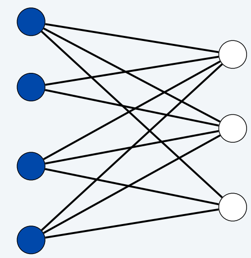

## 3. Graphs

- *basic definitions and applications*
- *graph connectivity and graph traversal*
- *testing bipartiteness*
- *connectivity in directed graphs*
- *DAGs and topological ordering*

## Bipartite graphs

Def. An undirected graph $G = (V, E)$ is bipartite if the nodes can be colored blue or white such that every edge has one white and one blue end.

Applications.
- Stable matching: med-school residents = blue, hospitals = white.
- Scheduling: machines = blue, jobs = white.



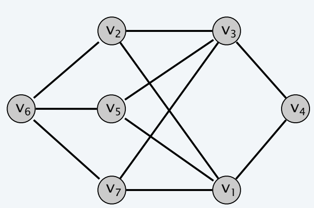
a bipartite graph



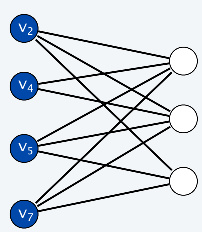
## Testing bipartiteness

Many graph problems become:
- Easier if the underlying graph is bipartite (matching).
- Tractable if the underlying graph is bipartite (independent set).

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.




a bipartite graph G



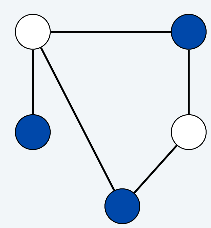

another drawing of G



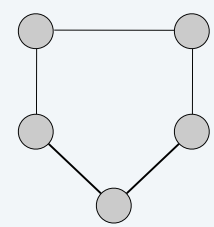
## An obstruction to bipartiteness

Lemma. If a graph $G$ is bipartite, it cannot contain an odd-length cycle.

Pf. Not possible to 2-color the odd-length cycle, let alone $G$.



bipartite
(2-colorable)



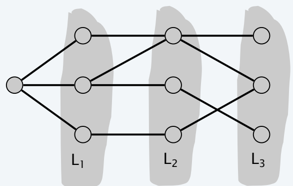
not bipartite
(not 2-colorable)

## Bipartite graphs

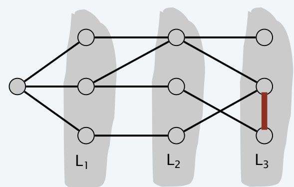
Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).




**Case (i)**




**Case (ii)**



## Bipartite graphs

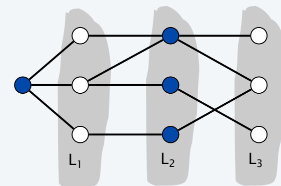
Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

Pf. (i)

- Suppose no edge joins two nodes in same layer.
- By BFS property, each edge joins two nodes in adjacent levels.
- Bipartition: white = nodes on odd levels, blue = nodes on even levels.




**Case (i)**



## Bipartite graphs

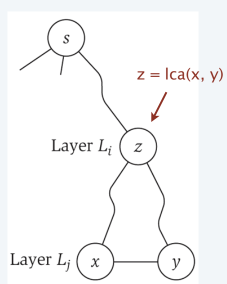
Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

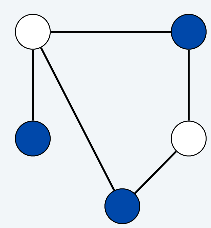
- Suppose $(x, y)$ is an edge with $x$, $y$ in same level $L_j$.
- Let $z = lca(x, y)$ = lowest common ancestor.
- Let $L_i$ be level containing $z$.
- Consider cycle that takes edge from $x$ to $y$, then path from $y$ to $z$, then path from $z$ to $x$.
- Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd. ▪



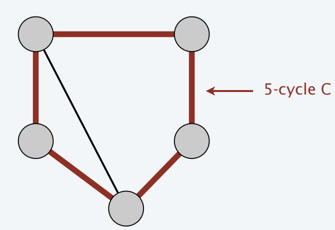



## The only obstruction to bipartiteness

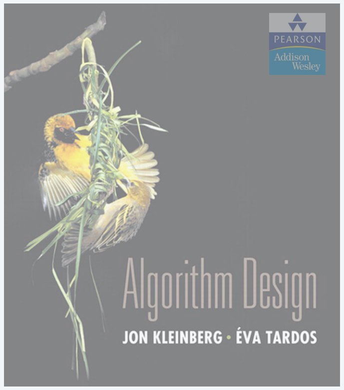
Corollary. A graph $G$ is bipartite iff it contains no odd-length cycle.



**bipartite**
**(2-colorable)**




**not bipartite**
**(not 2-colorable)**



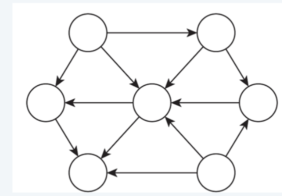

# 3. Graphs



## Directed graphs

Notation. $G = (V, E)$.

- Edge $(u, v)$ leaves node $u$ and enters node $v$.



Ex. Web graph: hyperlink points from one web page to another.

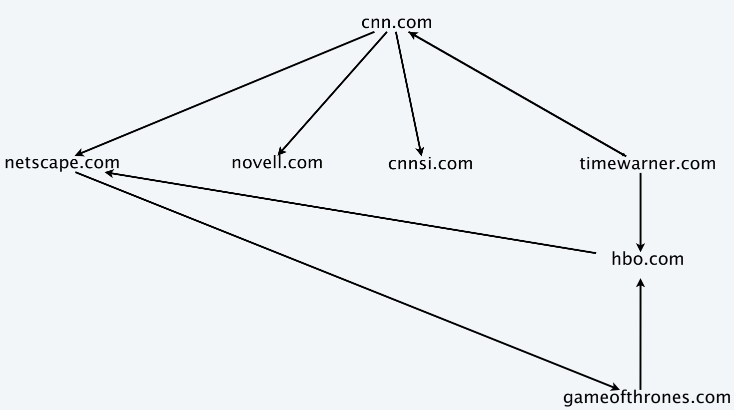
- Orientation of edges is crucial.
- Modern web search engines exploit hyperlink structure to rank web pages by importance.



## World wide web

Web graph.

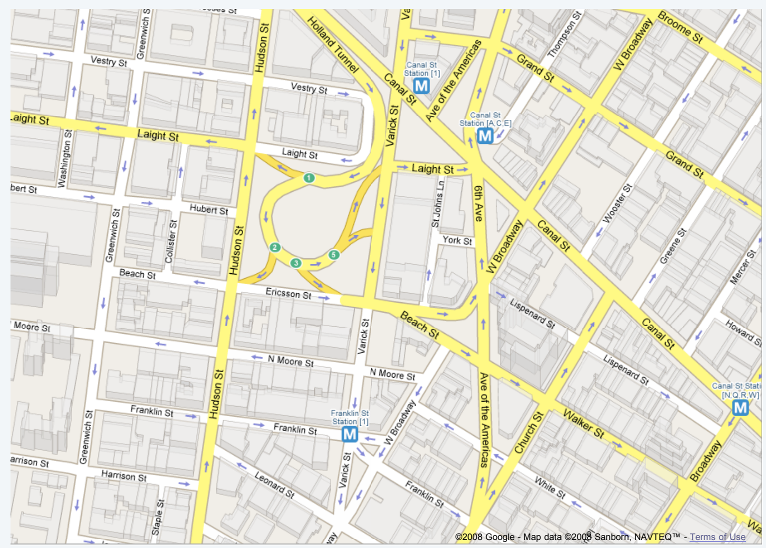
- Node: web page.
- Edge: hyperlink from one page to another (orientation is crucial).
- Modern search engines exploit hyperlink structure to rank web pages by importance.





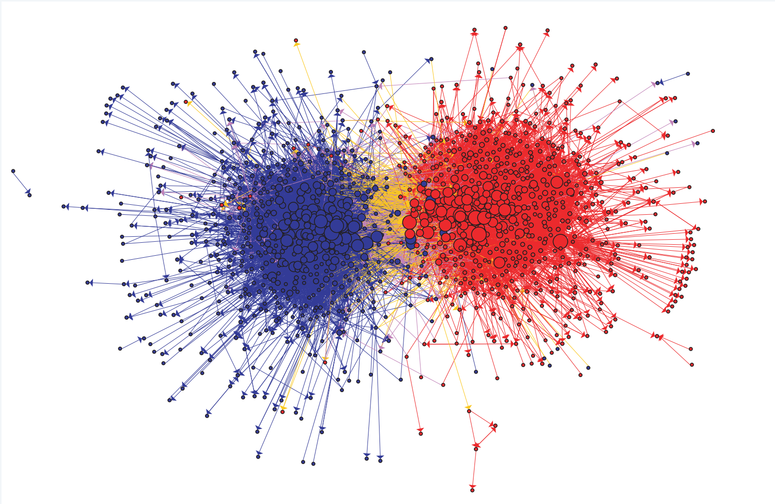

## Road network

Node = intersection; edge = one-way street.



## Political blogosphere graph

Node = political blog; edge = link.



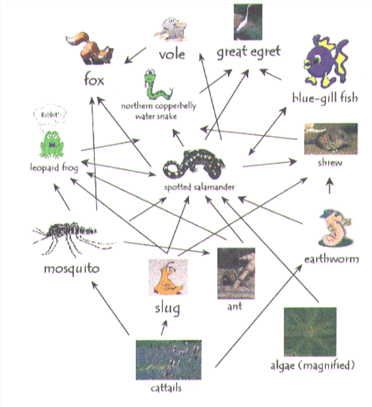
**The Political Blogosphere and the 2004 U.S. Election: Divided They Blog, Adamic and Glance, 2005**



## Ecological food web

Food web graph.

- Node = species.
- Edge = from prey to predator.




Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff



## Some directed graph applications

| directed graph | node | directed edge |
|---|---|---|
| **transportation** | street intersection | one-way street |
| **web** | web page | hyperlink |
| **food web** | species | predator-prey relationship |
| **WordNet** | synset | hypernym |
| **scheduling** | task | precedence constraint |
| **financial** | bank | transaction |
| **cell phone** | person | placed call |
| **infectious disease** | person | infection |
| **game** | board position | legal move |
| **citation** | journal article | citation |
| **object graph** | object | pointer |
| **inheritance hierarchy** | class | inherits from |
| **control flow** | code block | jump |



## Graph search

Directed reachability. Given a node $s$, find all nodes reachable from $s$.

Directed s→t shortest path problem. Given two nodes $s$ and $t$, what is the length of a shortest path from $s$ to $t$?

Graph search. BFS extends naturally to directed graphs.

Web crawler. Start from web page $s$. Find all web pages linked from $s$, either directly or indirectly.

## Strong connectivity

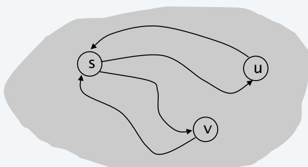
Def. Nodes $u$ and $v$ are mutually reachable if there is both a path from $u$ to $v$ and also a path from $v$ to $u$.

Def. A graph is strongly connected if every pair of nodes is mutually reachable.

Lemma. Let $s$ be any node. $G$ is strongly connected iff every node is reachable from $s$, and $s$ is reachable from every node.

Pf. $\Rightarrow$ Follows from definition.
Pf. $\Leftarrow$ Path from $u$ to $v$: concatenate $u \rightsquigarrow s$ path with $s \rightsquigarrow v$ path.
Path from $v$ to $u$: concatenate $v \rightsquigarrow s$ path with $s \rightsquigarrow u$ path. ▪

ok if paths overlap

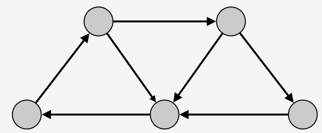




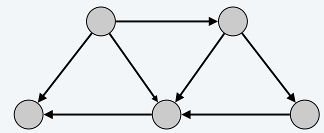

## Strong connectivity: algorithm

Theorem. Can determine if $G$ is strongly connected in $O(m + n)$ time.
Pf.

- Pick any node $s$.
- Run BFS from $s$ in $G$.
- Run BFS from $s$ in $G^{reverse}$. (reverse orientation of every edge in G)
- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from previous lemma. ▪



strongly connected

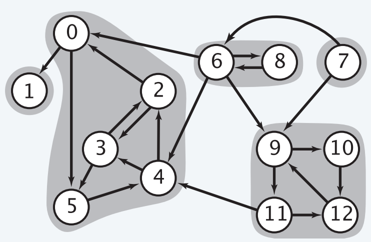


not strongly connected

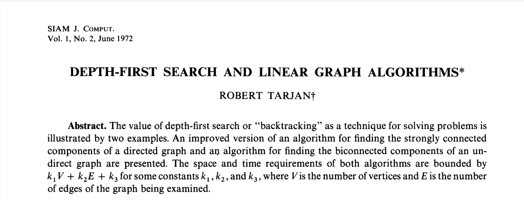


## Strong components

Def. A strong component is a maximal subset of mutually reachable nodes.




Theorem. [Tarjan 1972] Can find all strong components in $O(m + n)$ time.



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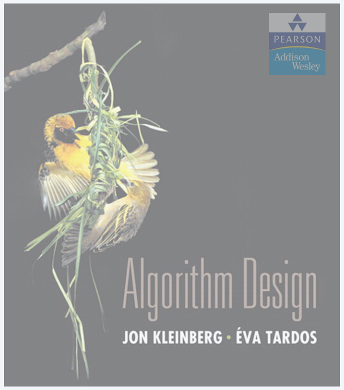
DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

**Abstract.** The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an un-direct graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants $k_1$, $k_2$, and $k_3$, where $V$ is the number of vertices and $E$ is the number of edges of the graph being examined.





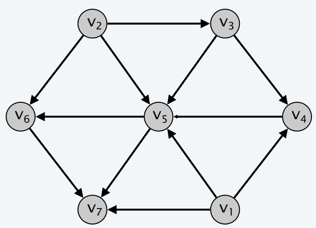

## 3. Graphs

- basic definitions and applications
- graph connectivity and graph traversal
- testing bipartiteness
- connectivity in directed graphs
- ***DAGs and topological ordering***

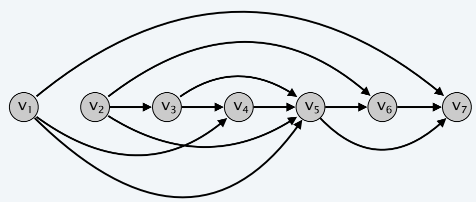
## Directed acyclic graphs

Def. A DAG is a directed graph that contains no directed cycles.

Def. A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge $(v_i, v_j)$ we have $i < j$.




**a DAG**




**a topological ordering**



## Precedence constraints

Precedence constraints. Edge $(v_i, v_j)$ means task $v_i$ must occur before $v_j$.

Applications.

- Course prerequisite graph: course $v_i$ must be taken before $v_j$.
- Compilation: module $v_i$ must be compiled before $v_j$.
- Pipeline of computing jobs: output of job $v_i$ needed to determine input of job $v_j$.

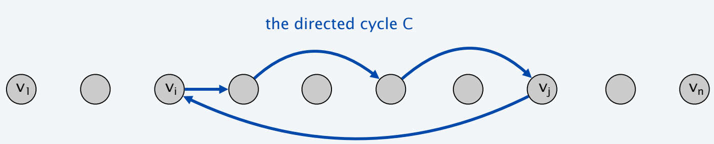


## Directed acyclic graphs

Lemma. If $G$ has a topological order, then $G$ is a DAG.

Pf. [by contradiction]

- Suppose that $G$ has a topological order $v_1, v_2, \ldots, v_n$ and that $G$ also has a directed cycle $C$. Let's see what happens.
- Let $v_i$ be the lowest-indexed node in $C$, and let $v_j$ be the node just before $v_i$; thus $(v_j, v_i)$ is an edge.
- By our choice of $i$, we have $i < j$.
- On the other hand, since $(v_j, v_i)$ is an edge and $v_1, v_2, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction. ▪




**the supposed topological order: $v_1, \ldots, v_n$**



## Directed acyclic graphs

Lemma. If $G$ has a topological order, then $G$ is a DAG.

Q. Does every DAG have a topological ordering?

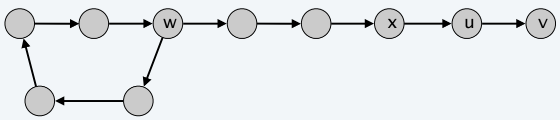
Q. If so, how do we compute one?

## Directed acyclic graphs

Lemma. If $G$ is a DAG, then $G$ has a node with no entering edges.

Pf. [by contradiction]

- Suppose that $G$ is a DAG and every node has at least one entering edge. Let's see what happens.
- Pick any node $v$, and begin following edges backward from $v$. Since $v$ has at least one entering edge $(u, v)$ we can walk backward to $u$.
- Then, since $u$ has at least one entering edge $(x, u)$, we can walk backward to $x$.
- Repeat until we visit a node, say $w$, twice.
- Let $C$ denote the sequence of nodes encountered between successive visits to $w$. $C$ is a cycle. ▪





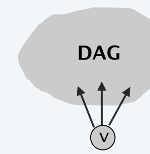

## Directed acyclic graphs

Lemma. If $G$ is a DAG, then $G$ has a topological ordering.

Pf. [by induction on $n$]

- Base case: true if $n = 1$.
- Given DAG on $n > 1$ nodes, find a node $v$ with no entering edges.
- $G - \{ v \}$ is a DAG, since deleting $v$ cannot create cycles.
- By inductive hypothesis, $G - \{ v \}$ has a topological ordering.
- Place $v$ first in topological ordering; then append nodes of $G - \{ v \}$
- in topological order. This is valid since $v$ has no entering edges. ▪

```
To compute a topological ordering of G:
Find a node v with no incoming edges and order it first
Delete v from G
Recursively compute a topological ordering of G−{v}
  and append this order after v
```






## Topological sorting algorithm: running time

Theorem. Algorithm finds a topological order in $O(m + n)$ time.

Pf.

- Maintain the following information:
  - $count(w)$ = remaining number of incoming edges
  - $S$ = set of remaining nodes with no incoming edges
- Initialization: $O(m + n)$ via single scan through graph.
- Update: to delete $v$
  - remove $v$ from $S$
  - decrement $count(w)$ for all edges from $v$ to $w$; and add $w$ to $S$ if $count(w)$ hits $0$
  - this is $O(1)$ per edge ▪