Polynomial-Time Reductions

Princeton University • COS 423 • Theory of Algorithms • Spring 2001 • Kevin Wayne

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Contents.

Polynomial-Time Reduction

Intuitively, problem X reduces to problem Y if:

- Any instance of X can be "rephrased" as an instance of Y.

Formally, problem X polynomial reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.
  - computational model supplemented by special piece of hardware that solves instances of Y in a single step

Remarks.

- We pay for time to write down instances sent to black box $\Rightarrow$ instances of Y are of polynomial size.
- Note: Cook-Turing reducibility (not Karp or many-to-one).
- Notation: $X \le_P Y$ (or more precisely $X \le_P^T Y$).

Polynomial-Time Reduction

Purpose. Classify problems according to relative difficulty.

Design algorithms. If $X \le_P Y$ and Y can be solved in polynomial-time, then X can be solved in polynomial time.

Establish intractability. If $X \le_P Y$ and X cannot be solved in polynomial-time, then X cannot be solved in polynomial time.

Anti-symmetry. If $X \le_P Y$ and $Y \le_P X$, we use notation $X \equiv_P Y$.

Transitivity. If $X \le_P Y$ and $Y \le_P Z$, then $X \le_P Z$.

- Proof idea: compose the two algorithms.
- Given an oracle for Z, can solve instance of X:
  - run the algorithm for X using a oracle for Y
  - each time oracle for Y is called, simulate it in a polynomial number of steps by using algorithm for Y, plus oracle calls to Z

Polynomial-Time Reduction

Basic strategies.

- **Reduction by simple equivalence.**
- Reduction from special case to general case.
- Reduction from general case to special case.
- Reduction by encoding with gadgets.

Compositeness and Primality

COMPOSITE: Given the decimal representation of an integer x, does x have a nontrivial factor?

PRIME: Given the decimal representation of an integer x, is x prime?

Claim. COMPOSITE $\equiv_P$ PRIME.

- COMPOSITE $\leq_P$ PRIME.
- PRIME $\leq_P$ COMPOSITE.

COMPOSITE (x)

```
IF (PRIME(x) = TRUE)
   RETURN FALSE
ELSE
   RETURN TRUE
```

PRIME (x)

```
IF (COMPOSITE(x) = TRUE)
   RETURN FALSE
ELSE
   RETURN TRUE
```

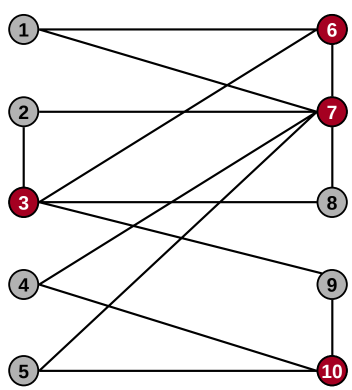

Vertex Cover

VERTEX COVER: Given an undirected graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and if $(v, w) \in E$ then either $v \in S$, $w \in S$ or both.

Ex.

- Is there a vertex cover of size 4?

  YES.



Vertex Cover

VERTEX COVER: Given an undirected graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and if $(v, w) \in E$ then either $v \in S$, $w \in S$ or both.

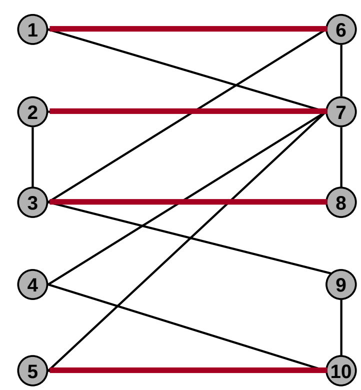
Ex.

- Is there a vertex cover of size 4?

  YES.

- Is there a vertex cover of size 3?

  NO.



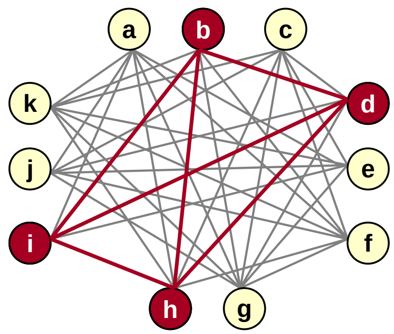
Clique

CLIQUE: Given N people and their pairwise relationships. Is there a group of S people such that every pair in the group knows each other.

Ex.

- **People:** a, b, c, d, e, . . . , k.
- **Friendships:** (a, e), (a, f), (a, g), . . ., (h, k).
- **Clique size:** S = 4.

YES Instance

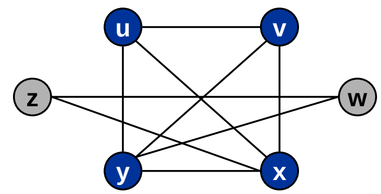


Friendship Graph

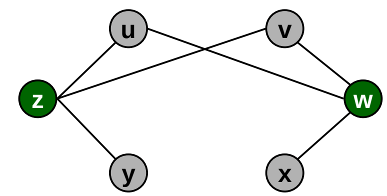
Vertex Cover and Clique

Claim. VERTEX COVER $\equiv_P$ CLIQUE.

- Given an undirected graph G = (V, E), its complement is G' = (V, E'), where E' = { (v, w) : (v, w) $\notin$ E}.
- G has a clique of size k if and only if G' has a vertex cover of size |V| - k.



G
Clique = {u, v, x, y}



G'
Vertex cover = {w, z}

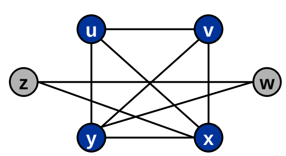
Vertex Cover and Clique

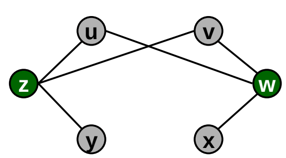
Claim. VERTEX COVER $\equiv_P$ CLIQUE.

- Given an undirected graph G = (V, E), its complement is G' = (V, E'), where E' = { (v, w) : (v, w) $\notin$ E}.
- G has a clique of size k if and only if G' has a vertex cover of size |V| - k.

Proof. $\Rightarrow$

- Suppose G has a clique S with |S| = k.
- Consider S' = V – S.
- |S'| = |V| - k.
- To show S' is a cover, consider any edge (v, w) $\in$ E'.
  - then (v, w) $\notin$ E
  - at least one of v or w is not in S (since S forms a clique)
  - at least one of v or w is in S'
  - hence (v, w) is covered by S'





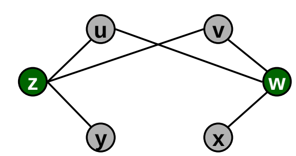
Vertex Cover and Clique

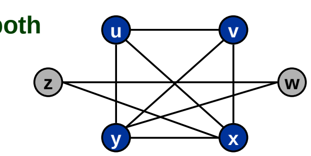
Claim. VERTEX COVER $\equiv_P$ CLIQUE.

- Given an undirected graph G = (V, E), its complement is G' = (V, E'), where E' = { (v, w) : (v, w) $\notin$ E}.
- G has a clique of size k if and only if G' has a vertex cover of size |V| - k.

Proof. $\Leftarrow$

- Suppose G' has a cover S' with |S'| = |V| - k.
- Consider S = V – S'.
- Clearly |S| = k.
- To show S is a clique, consider some edge (v, w) $\in$ E'.
  - if (v, w) $\in$ E', then either v $\in$ S', w $\in$ S', or both
  - by contrapositive, if v $\notin$ S' and w $\notin$ S', then (v, w) $\in$ E
  - thus S is a clique in G





Polynomial-Time Reduction

Basic strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction from general case to special case.
- Reduction by encoding with gadgets.

Compositeness Reduces to Factoring

COMPOSITE: Given an integer x, does x have a nontrivial factor?

FACTOR: Given two integers x and y, does x have a nontrivial factor less than y?

Claim. COMPOSITE $\leq_P$ FACTOR.

Proof. Given an oracle for FACTOR, we solve COMPOSITE.

- Is 350 composite?
- Does 350 have a nontrivial factor less than 350?

COMPOSITE (x)

```
IF (FACTOR(x, x) = TRUE)
   RETURN TRUE
ELSE
   RETURN FALSE
```


Primality Testing and Factoring

We established:

- PRIME $\leq_P$ COMPOSITE $\leq_P$ FACTOR.

Natural question:

- Does FACTOR $\leq_P$ PRIME ?
- Consensus opinion = NO.

State-of-the-art.

- PRIME in randomized P and conjectured to be in P.
- FACTOR not believed to be in P.

RSA cryptosystem.

- Based on dichotomy between two problems.
- To use, must generate large primes efficiently.
- Can break with efficient factoring algorithm.

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of at most k of these sets whose union is equal of U?

Sample application.

- n available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The ith piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using small number of pieces of software.

Ex. U = {1, 2, 3, . . . , 12}, k = 3.

- S_1 = {1, 2, 3, 4, 5, 6} S_2 = {5, 6, 8, 9}
- S_3 = {1, 4, 7, 10} S_4 = {2, 5, 7, 8, 11}
- S_5 = {3, 6, 9, 12} S_6 = {10, 11}

YES: S_3, S_4, S_5.

Vertex Cover Reduces to Set Cover

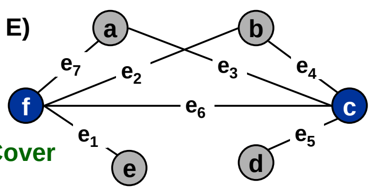
SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_n$ of subsets of U, and an integer k, does there exist a collection of at most k of these sets whose union is equal to U?

VERTEX COVER: Given an undirected graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and if $(v, w) \in E$ then either $v \in S$, $w \in S$ or both.

Claim. VERTEX-COVER $\leq_P$ SET-COVER.

Proof. Given black box that solves instances of SET-COVER.

G = (V, E)
k = 2



Vertex Cover

U = {1, 2, 3, 4, 5, 6, 7}
k = 2

S_a = {3, 7} S_b = {2, 4}
S_c = {3, 4, 5, 6} S_d = {5}
S_e = {1} S_f = {1, 2, 6, 7}

Set Cover

Vertex Cover Reduces to Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_n$ of subsets of U, and an integer k, does there exist a collection of at most k of these sets whose union is equal to U?

VERTEX COVER: Given an undirected graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and if $(v, w) \in E$ then either $v \in S$, $w \in S$ or both.

Claim. VERTEX-COVER $\leq_P$ SET-COVER.

Proof. Given black box that solves instances of SET-COVER.

- Let G = (V, E), k be an instance of VERTEX-COVER.
- Create SET-COVER instance:
  - k = k, U = E, $S_v = \{e \in E : e \text{ incident to } v\}$
- Set-cover of size at most k if and only if vertex cover of size at most k.

Polynomial-Time Reduction

Basic strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction from general case to special case.
- Reduction by encoding with gadgets.

Factoring and Finding Factors

FACTOR: Given two integers x and y, does x have a nontrivial factor less than y?

FACTORIZE: Given an integer x, find its prime factorization.

Claim. FACTORIZE $\equiv_P$ FACTOR.

Proof: FACTOR $\leq_P$ FACTORIZE.

- Reduction from special case to general case.

S = prime factorization is of polynomial size →

FACTOR (x)

```
S ← FACTORIZE(x)
d ← smallest factor in S
IF (d < y)
   RETURN TRUE
ELSE
   RETURN FALSE
```

Factoring and Finding Factors

FACTOR: Given two integers x and y, does x have a nontrivial factor less than y?

FACTORIZE: Given an integer x, find its prime factorization.

Claim. FACTORIZE $\equiv_P$ FACTOR.

Proof: FACTORIZE $\leq_P$ FACTOR.

- Reduction from general case to special case.

find smallest factor via binary search

S = global variable containing set of factors

FACTORIZE(x)

```
IF (FACTOR(x, x) = NO)
   S ← S ∪ {x}
   RETURN

left = 1, right = x
WHILE (right > left + 1)
   mid = (left + right) / 2
   IF (FACTOR(x, mid) = TRUE)
      right = mid
   ELSE
      left = mid
S ← S ∪ {left}
FACTORIZE(x / left)
```


Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A Boolean formula that is the conjunction of clauses. $B = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

CNF-SAT: Given propositional formula in conjunctive normal form, does it have a satisfying truth assignment?

$$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$$

YES instance
x_1 = true
x_1 = true
x_1 = false

SAT Reduces to 3-SAT

3-CNF-SAT: CNF-SAT, where each clause has 3 distinct literals.

Claim. CNF-SAT $\leq_P$ 3-CNF-SAT.

- Case 3: clause C_j contains exactly 3 terms.
- Case 2: clause C_j contains exactly 2 terms.
  - add 1 new term, and replace C_j with 2 clauses

$$C_j = \overline{x_3} \vee x_7 \quad \Rightarrow \quad \begin{aligned} C'_{j1} &= \overline{x_3} \vee x_7 \vee y \\ C'_{j2} &= \overline{x_3} \vee x_7 \vee \overline{y} \end{aligned}$$

- Case 1: clause C_j contains exactly 1 term.
  - add 4 new terms, and replace C_j with 4 clauses

$$C_j = \overline{x_3} \quad \Rightarrow \quad \begin{aligned} C'_{j1} &= \overline{x_3} \vee y_1 \vee y_2 \\ C'_{j2} &= \overline{x_3} \vee y_1 \vee \overline{y_2} \\ C'_{j3} &= \overline{x_3} \vee \overline{y_1} \vee y_2 \\ C'_{j4} &= \overline{x_3} \vee \overline{y_1} \vee \overline{y_2} \end{aligned}$$

SAT Reduces to 3-SAT

3-CNF-SAT: CNF-SAT, where each clause has 3 distinct literals.

Claim. CNF-SAT $\leq_P$ 3-CNF-SAT.

- Case 4: clause C_j contains $\ell \geq 4$ terms.
  - introduce ℓ - 1 extra Boolean variables
  - replace C_j with ℓ clauses

$$C_j = x_1 \vee \overline{x_3} \vee \overline{x_4} \vee x_5 \vee x_6 \vee \overline{x_9} \quad \Rightarrow \quad \begin{aligned} C'_{j1} &= x_1 \vee x_1 \vee y_1 \\ C'_{j2} &= \overline{y_1} \vee \overline{x_3} \vee y_2 \\ C'_{j3} &= \overline{y_2} \vee \overline{x_4} \vee y_3 \\ C'_{j4} &= \overline{y_3} \vee x_5 \vee y_4 \\ C'_{j5} &= \overline{y_4} \vee x_6 \vee y_5 \\ C'_{j6} &= \overline{y_5} \vee \overline{x_9} \vee \overline{x_9} \end{aligned}$$

SAT Reduces to 3-SAT

- Case 4: clause C_j contains $\ell \ge 4$ terms.

$$C_j = t_{j1} \vee t_{j2} \vee t_{j3} \vee \cdots \vee t_{j\ell} \quad \Rightarrow \quad \begin{array}{lclclcl} C'_{j1} & = & t_{j1} & \vee & t_{j1} & \vee & y_1 \\ C'_{j2} & = & \overline{y_1} & \vee & t_{j2} & \vee & y_2 \\ C'_{j3} & = & \overline{y_2} & \vee & t_{j3} & \vee & y_3 \\ C'_{j4} & = & \overline{y_3} & \vee & t_{j4} & \vee & y_4 \\ C'_{j5} & = & \overline{y_4} & \vee & t_{j5} & \vee & y_5 \\ \vdots & & \vdots & & \vdots & & \vdots \\ C'_{j\ell} & = & \overline{y_{\ell-1}} & \vee & t_{j\ell} & \vee & t_{j\ell} \end{array}$$

k = 4

set TRUE

Claim. CNF-SAT instance is satisfiable if and only if 3-CNF-SAT instance is.

Proof. $\Rightarrow$ Suppose SAT instance is satisfiable.

- If SAT assignment sets $t_{jk} = 1$, 3-SAT assignment sets:
  - $t_{jk} = 1$
  - $y_m = 1$ for all $m < k$; $y_m = 0$ for all $m \ge k$

SAT Reduces to 3-SAT

- Case 2: clause C_j contains $\ell \ge 4$ terms.

$$C_j = t_{j1} \vee t_{j2} \vee t_{j3} \vee \cdots \vee t_{j\ell} \quad \Rightarrow \quad \begin{array}{lclclcl} C'_{j1} & = & t_{j1} & \vee & t_{j1} & \vee & y_1 \\ C'_{j2} & = & \overline{y_1} & \vee & t_{j2} & \vee & y_2 \\ C'_{j3} & = & \overline{y_2} & \vee & t_{j3} & \vee & y_3 \\ C'_{j4} & = & \overline{y_3} & \vee & t_{j4} & \vee & y_4 \\ C'_{j5} & = & \overline{y_4} & \vee & t_{j5} & \vee & y_5 \\ \vdots & & \vdots & & \vdots & & \vdots \\ C'_{j\ell} & = & \overline{y_{\ell-1}} & \vee & t_{j\ell} & \vee & t_{j\ell} \end{array}$$

Claim. CNF-SAT instance is satisfiable if and only if 3-CNF-SAT instance is.

Proof. $\Leftarrow$ Suppose 3-SAT instance is satisfiable.

- If 3-SAT assignment sets $t_{jk} = 1$, SAT assignment sets $t_{jk} = 1$.
- Consider clause C_j. We claim $t_{jk} = 1$ for some k.
  - each of ℓ - 1 new Boolean variables y_j can only make one of ℓ new clauses true
  - the remaining clause must be satisfied by an original term t_{jk}

Polynomial-Time Reduction

Basic strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction from general case to special case.
- Reduction by encoding with gadgets.

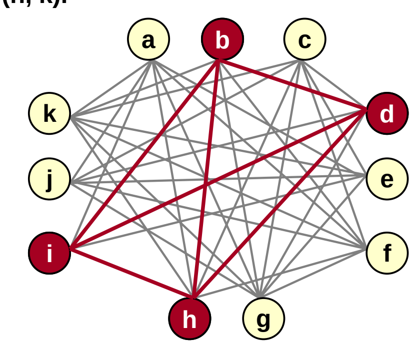
Clique

CLIQUE: Given N people and their pairwise relationships. Is there a group of C people such that every pair in the group knows each other.

Ex.

- People: a, b, c, d, e, . . . , k.
- Friendships: (a, e), (a, f), (a, g), . . ., (h, k).
- Clique size: C = 4.

YES Instance



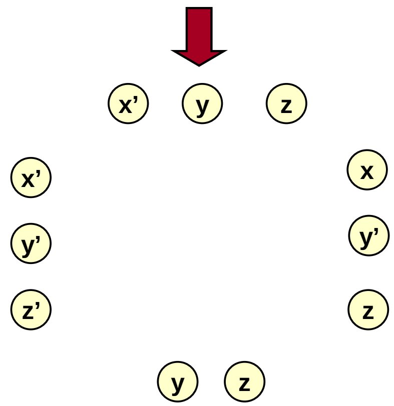
Friendship Graph

Satisfiability Reduces to Clique

Claim. CNF-SAT $\leq_P$ CLIQUE.

- Given instance of CNF-SAT, create a person for each literal in each clause.

first clause

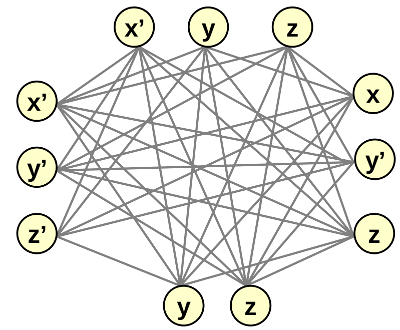


(x' + y + z) (x + y' + z) (y + z) (x' + y' + z')
C = 4 clauses

Satisfiability Reduces to Clique

Claim. CNF-SAT $\leq_P$ CLIQUE.

- Given instance of CNF-SAT, create a person for each literal in each clause.
- Two people know each other except if:
  - they come from the same clause
  - they represent a literal and its negation

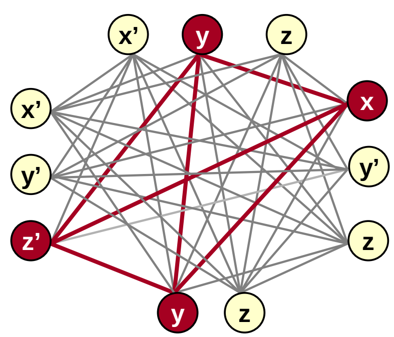


(x' + y + z) (x + y' + z) (y + z) (x' + y' + z')
C = 4 clauses

Satisfiability Reduces to Clique

Claim. CNF-SAT $\leq_P$ CLIQUE.

- Given instance of CNF-SAT, create a person for each literal in each clause.
- Two people know each other except if:
  - they come from the same clause
  - they represent a literal and its negation
- Clique of size C $\Rightarrow$ satisfiable assignment.
  - set variable in clique to true
  - (x, y, z) = (true, true, false)

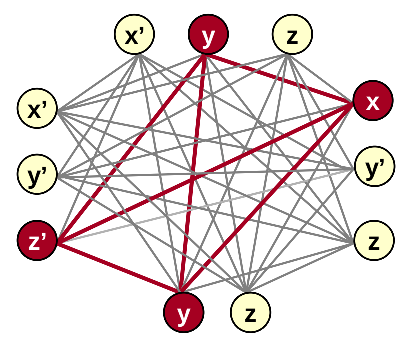


(x' + y + z) (x + y' + z) (y + z) (x' + y' + z')
C = 4 clauses

Satisfiability Reduces to Clique

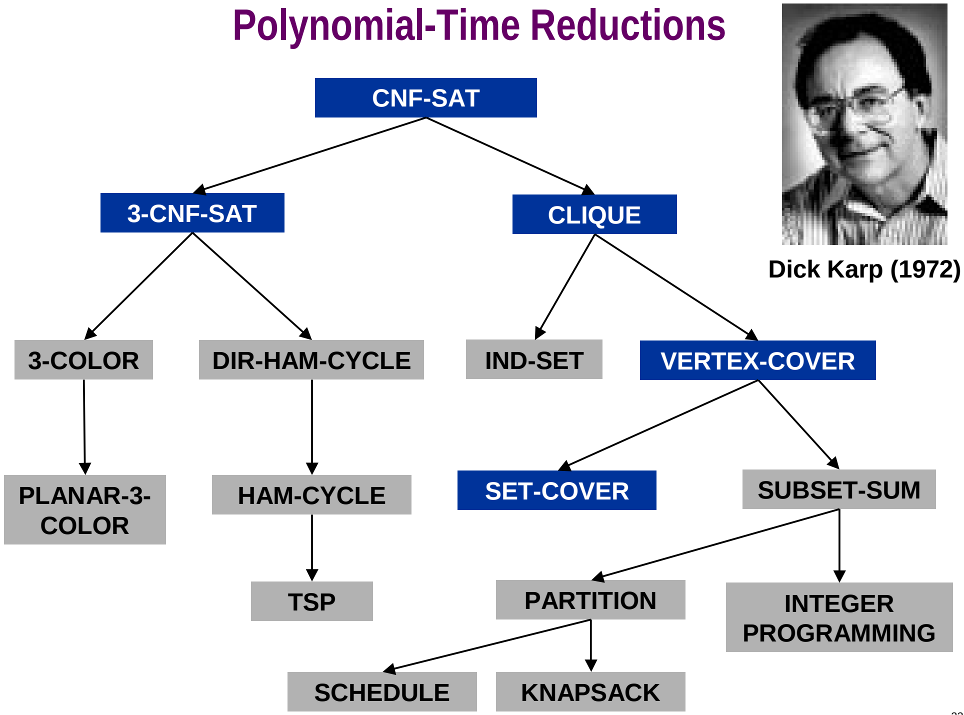
Claim. CNF-SAT $\leq_P$ CLIQUE.

- Given instance of CNF-SAT, create a person for each literal in each clause.
- Two people know each other except if:
  - they come from the same clause
  - they represent a literal and its negation
- Clique of size C $\Rightarrow$ satisfiable assignment.
- Satisfiable assignment $\Rightarrow$ clique of size C.
  - (x, y, z) = (true, true, false)
  - choose one true literal from each clause



(x' + y + z) (x + y' + z) (y + z) (x' + y' + z')
C = 4 clauses

Polynomial-Time Reductions



Dick Karp (1972)

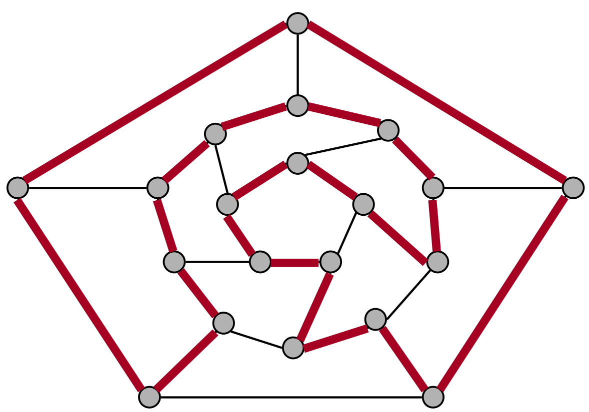
Problem Genres

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- **Sequencing problems: HAMILTONIAN-CYCLE, TSP.**
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK, FACTOR.

Hamiltonian Cycle

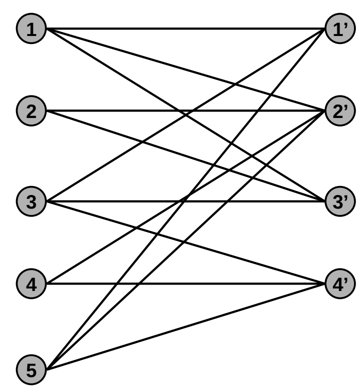
HAMILTONIAN-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle C that contains every vertex in V.



YES: vertices and faces of a dodecahedron.

Hamiltonian Cycle

HAMILTONIAN-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle C that contains every vertex in V.



NO: bipartite graph with odd number of nodes.

Finding a Hamiltonian Cycle

HAM-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle C that contains every vertex in V.

FIND-HAM-CYCLE: given an undirected graph G = (V, E), output a Hamiltonian cycle if one exists, otherwise output any cycle.

Claim. HAM-CYLCE $\equiv_P$ FIND-HAM-CYCLE.

Proof. $\le_P$

HAM-CYCLE (G)

```
C ← FIND-HAM-CYCLE(G)
IF  (C is Hamiltonian)
   RETURN TRUE
ELSE
   RETURN FALSE
```


Finding a Hamiltonian Cycle

HAM-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle C that contains every vertex in V.

FIND-HAM-CYCLE: given an undirected graph G = (V, E), output a Hamiltonian cycle if one exists, otherwise output any cycle.

Claim. HAM-CYLCE $\equiv_P$ FIND-HAM-CYCLE.

Proof. $\ge_P$

FIND-HAM-CYCLE (G)

```
IF (HAM-CYCLE(G) = FALSE)
   RETURN FALSE

A ← E
FOR EACH e ∈ E
   IF (HAM-CYCLE(V, A - {e}) = TRUE)
      A ← A - {e}

RETURN unique cycle remaining in G
```

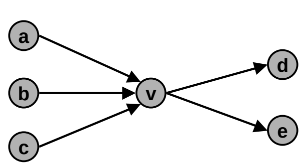

Directed Hamiltonian Cycle

DIR-HAM-CYCLE: given a directed graph G = (V, E), does there exists a simple directed cycle C that contains every vertex in V.

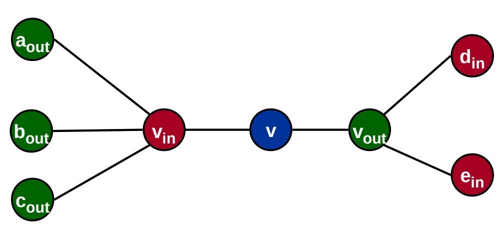
Claim. DIR-HAM-CYCLE $\le_P$ HAM-CYCLE.

Proof.

- Given a directed graph G = (V, E), construct an undirected graph G' with 3n vertices.



G



G'

Directed Hamiltonian Cycle

Claim. G has a Hamiltonian cycle if and only if G' does.

Proof. $\Rightarrow$

- Suppose G has a directed Hamiltonian cycle C.
- Then G' has an undirected Hamiltonian cycle.

Proof. $\Leftarrow$

- Suppose G' has an undirected Hamiltonian cycle C'.
- C' must visit nodes in G' using one of following two orders:
  - . . . , G, R, B, G, R, B, G, R, B, . . .
  - . . . , R, G, B, R, G, B, R, G, B, . . .
- Blue nodes in C' make up directed Hamiltonian cycle C in G, or reverse of one.

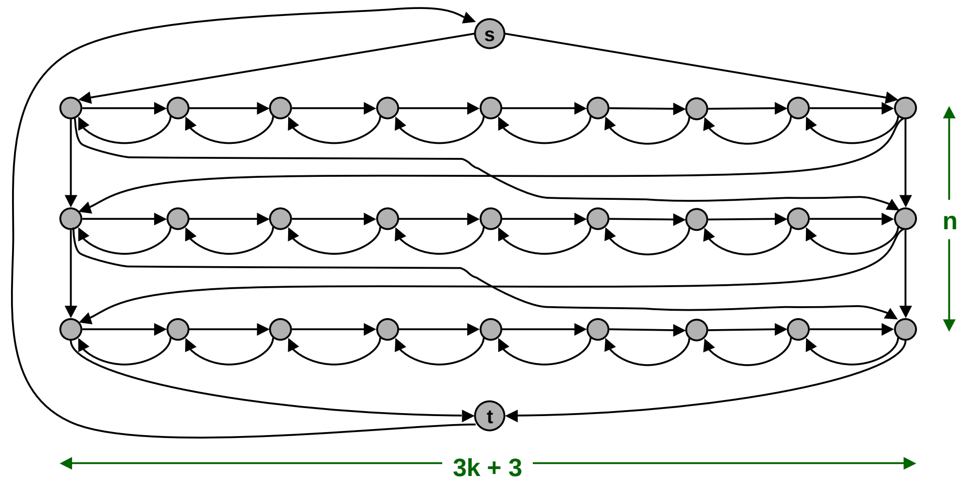
3-SAT Reduces to Directed Hamiltonian Cycle

Claim. 3-CNF-SAT $\leq_P$ DIR-HAM-CYCLE.

- Why not reduce from some other problem?
  - Need to find another problem that is sufficiently close. (could reduce from VERTEX-COVER)
  - If don't succeed, start from 3-CNF-SAT since its combinatorial structure is very basic.
  - Downside: reduction will require certain level of complexity.

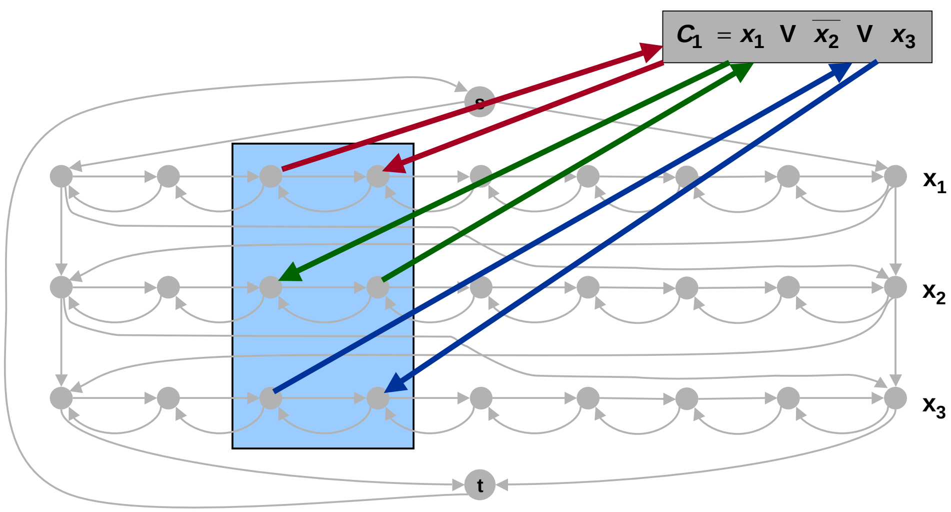
3-SAT Reduces to Directed Hamiltonian Cycle

Proof: Given 3-CNF-SAT instance with n variables x_i and k clauses C_j.
- Construct G to have 2^n Hamiltonian cycles.
- Intuition: traverse path i from left to right $\Leftrightarrow$ set variable $x_i = 1$.



3-SAT Reduces to Directed Hamiltonian Cycle

Proof: Given 3-CNF-SAT instance with n variables x_i and k clauses C_j.
- Add node and 6 edges for each clause.



3-SAT Reduces to Directed Hamiltonian Cycle

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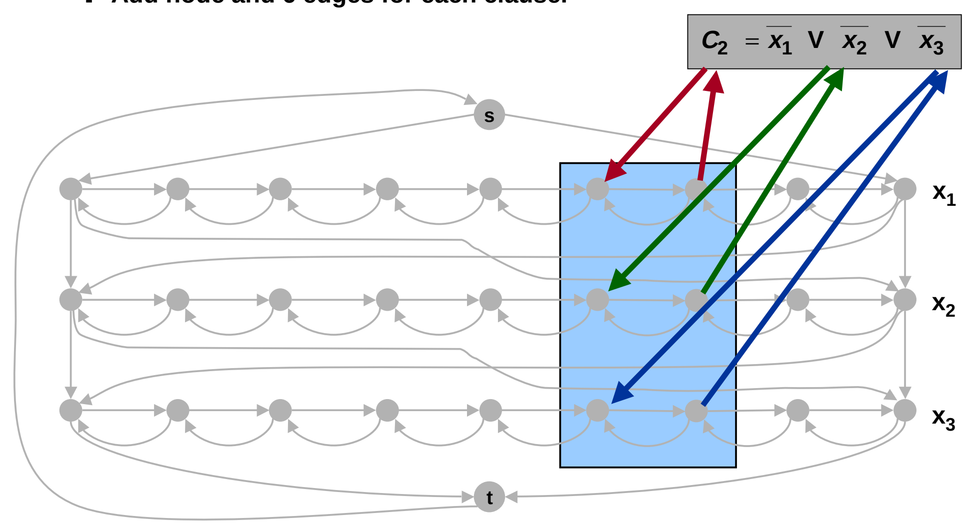


3-SAT Reduces to Directed Hamiltonian Cycle

Claim. 3-CNF-SAT instance is satisfiable if and only if corresponding graph G has a Hamiltonian cycle.

Proof. $\Rightarrow$

- Suppose 3-SAT instance has satisfying assignment x*.
- Then, define Hamiltonian cycle in G as follows:
  - if $x^*_i = 1$, traverse path P_i from left to right
  - if $x^*_i = 0$, traverse path P_i from right to left
  - for each clause C_j, there will be at least one path P_i in which we are going in "correct" direction to splice node C_j into tour

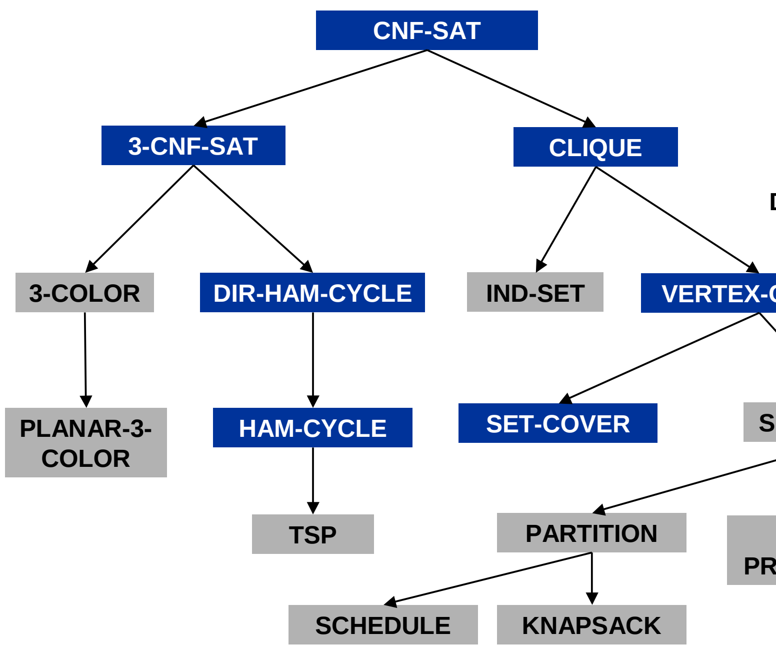
3-SAT Reduces to Directed Hamiltonian Cycle

Claim. 3-CNF-SAT instance is satisfiable if and only if corresponding graph G has a Hamiltonian cycle.

Proof. $\Rightarrow$

- Suppose G has a Hamiltonian cycle C.
- If C enters clause node C_j, it must depart on mate edge.
  - thus, nodes immediately before and after C_j are connected by an edge e in G
  - removing C_j from cycle, and replacing it with edge e yields Hamiltonian cycle on G - $\{C_j\}$
- Continuing in this way, we are left with Hamiltonian cycle C' in G - $\{C_1, C_2, \ldots, C_k\}$.
- Set $x^*_i = 1$ if path P_i if traversed from left to right, and 0 otherwise.
- Since C visits each clause node C_j, at least one of the paths is traversed in "correct" direction, and each clause is satisfied.

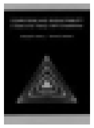
Polynomial-Time Reductions

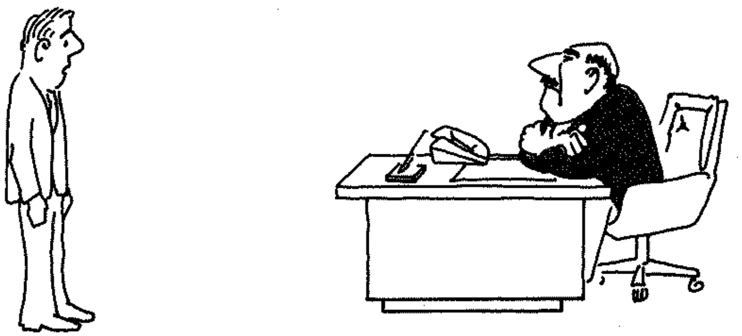


Dick Karp (1972)

Implications Reduction

Proof that a problem is as hard as CNF-SAT is usually taken as signal to abandon hope of finding an efficient algorithm.

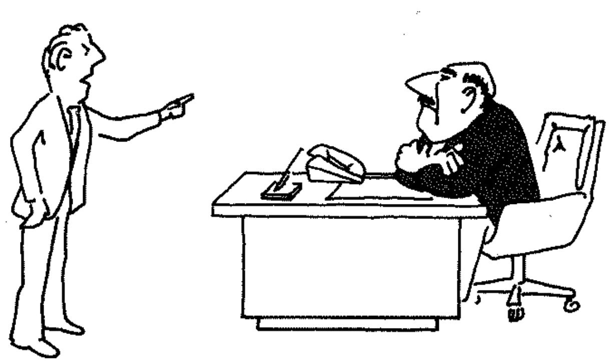


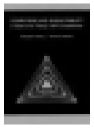


"I can't find an efficient algorithm, I guess I'm just too dumb."

Implications Reduction

Proof that a problem is as hard as CNF-SAT is usually taken as signal to abandon hope of finding an efficient algorithm.

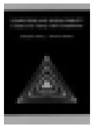




"I can't find an efficient algorithm, because no such algorithm is possible!"

Implications Reduction

Proof that a problem is as hard as CNF-SAT is usually taken as signal to abandon hope of finding an efficient algorithm.



"I can't find an efficient algorithm, but neither can all these famous people."

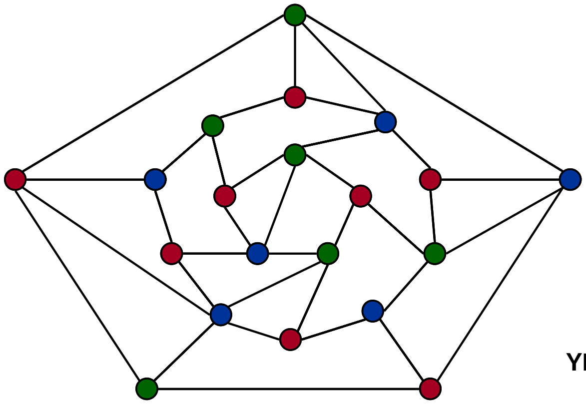
Problem Genres

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- **Partitioning problems: 3-COLOR.**
- Numerical problems: SUBSET-SUM, KNAPSACK, FACTOR.

3-Colorability

3-COLOR: Given an undirected graph does there exists a way to color the nodes R, G, and B such no adjacent nodes have the same color?



YES instance

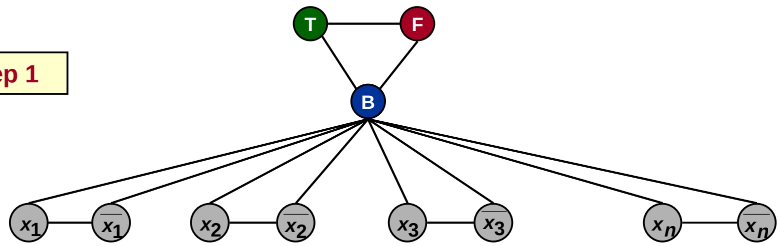
3-Colorability

Claim. 3-CNF-SAT $\leq_P$ 3-COLOR.

Proof: Given 3-SAT instance with n variables x_i and k clauses C_j.

- Create instance of 3-COLOR G = (V, E) as follows.
- Step 1:
  - create triangle R (false), G (true), or B
  - create nodes for each literal and connect to B
    - Each literal colored R or G.
  - create nodes for each literal, and connect literal to its negation
    - Each literal colored opposite of its negation.

Step 1

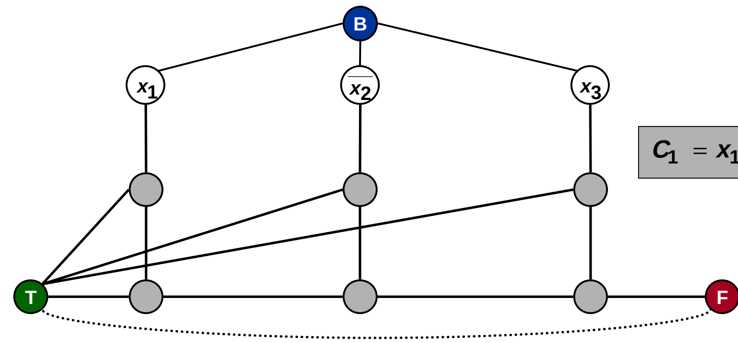


3-Colorability

Claim. 3-CNF-SAT $\leq_P$ 3-COLOR.

Proof: Given 3-SAT instance with n variables x_i and k clauses C_j.

- Step 2:
  - for each clause, add "gadget" of 6 new nodes and 13 new edges



$C_1 = x_1 \vee \overline{x_2} \vee x_3$

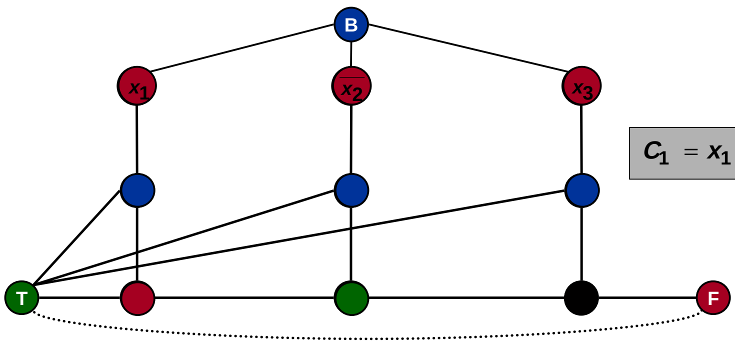
Step 2

3-Colorability

Claim. 3-CNF-SAT $\leq_P$ 3-COLOR.

Proof: Given 3-SAT instance with n variables x_i and k clauses C_j.

- Step 2:
  - for each clause, add "gadget" of 6 new nodes and 13 new edges
  - if 3-colorable, top row must have at least one green (true) node
    - Otherwise, middle row all blue.
    - Bottom row alternates between green and red $\Rightarrow$ contradiction.



$C_1 = x_1 \vee \overline{x_2} \vee x_3$

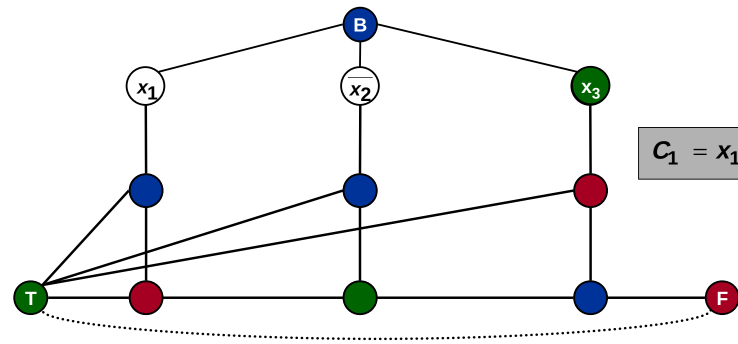
Step 2

3-Colorability

Claim. 3-CNF-SAT $\leq_P$ 3-COLOR.

Proof: Given 3-SAT instance with n variables x_i and k clauses C_j.

- Step 2:
  - for each clause, add "gadget" of 6 new nodes and 13 new edges
  - if top row has green (true) node, then 3-colorable
    - Color vertex below green node red, and one below that blue.
    - Color remaining middle row nodes blue.
    - Color remaining bottom nodes red or green, as forced.



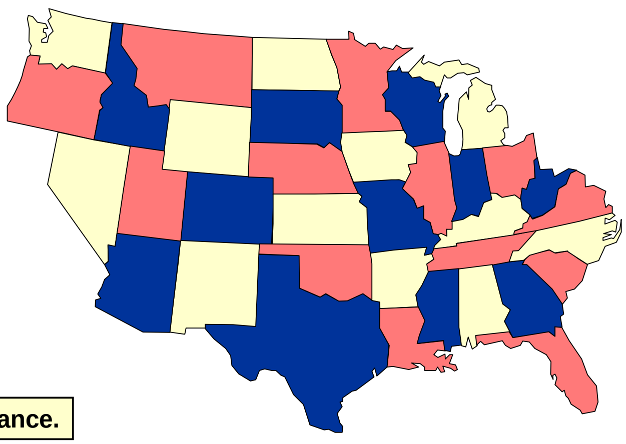
$C_1 = x_1 \vee \overline{x_2} \vee x_3$

Step 2

Planar 3-Colorability

PLANAR-3-COLOR.

- Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?

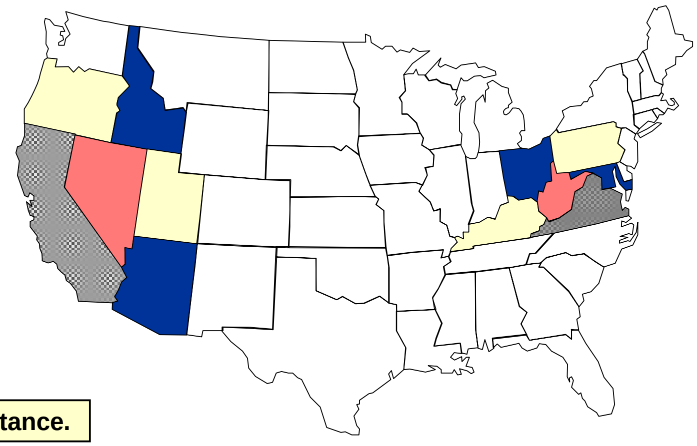


YES instance.

Planar 3-Colorability

PLANAR-3-COLOR.

- Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?

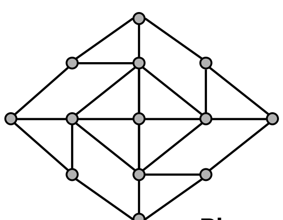


NO instance.

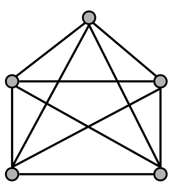
Planarity

Planarity. A graph is planar if it can be embedded on the plane (or sphere) in such a way that no two edges graph.

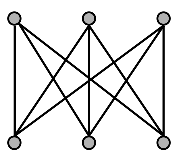
- Applications: VLSI circuit design, computer graphics.



Planar

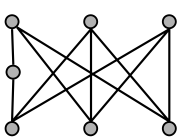


K_5: non-planar



$K_{3,3}$: non-planar

Kuratowski's Theorem. An undirected graph G is non-planar if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.



homeomorphic to $K_{3,3}$

Planarity Testing

Kuratowski's Theorem. An undirected graph G is non-planar if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.

Brute force: $O(n^6)$.

- Step 1. Contract all nodes of degree 2.
- Step 2. Check all subsets of 5 nodes to see if they form a K_5.
- Step 3. Check all subsets of 6 nodes to see if they form a $K_{3,3}$.

Cleverness: O(n).

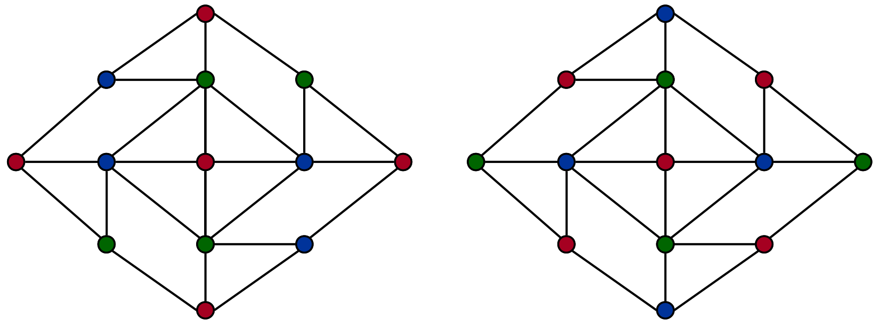
- Step 1. DFS.
- Step 2. Tarjan.

Planar 3-Colorability

Claim. 3-COLOR $\leq_P$ PLANAR-3-COLOR.

Proof sketch: Given instance of 3-COLOR, draw graph in plane, letting edges cross if necessary.

- Replace each edge crossing with the following planar gadget W.
  - in any 3-coloring of W, opposite corners have the same color
  - any assignment of colors to the corners in which opposite corners have the same color extends to a 3-coloring of W



Planar 4-Colorability

PLANAR-4-COLOR: Given a planar map, can it be colored using 4 colors so that no adjacent regions have the same color?

Intuition.

- If PLANAR-3-COLOR is hard, then so is PLANAR-4-COLOR and PLANAR-5-COLOR.
- Don't always believe your intuition!

Planar 4-Colorability

PLANAR-2-COLOR.

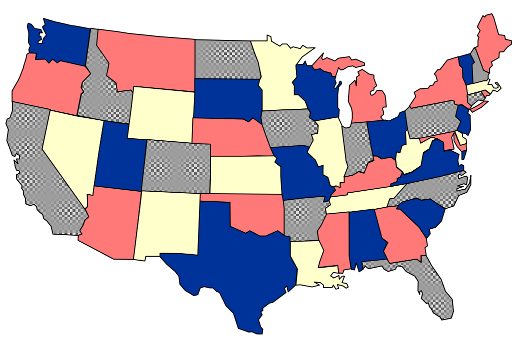
- Solvable in linear time.

PLANAR-3-COLOR.

- NP-complete.

PLANAR-4-COLOR.

- Solvable in O(1) time.



Theorem (Appel-Haken, 1976). Every planar map is 4-colorable.

- Resolved century-old open problem.
- Used 50 days of computer time to deal with many special cases.
- First major theorem to be proved using computer.

Problem Genres

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING.
- Numerical problems: SUBSET-SUM, KNAPSACK, FACTOR.

Subset Sum

SUBSET-SUM: Given a set X of integers and a target integer t, is there a subset $S \subseteq X$ whose elements sum to exactly t.

Example: X = {1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344}, t = 3754.

- YES: S = {1, 16, 64, 256, 1040, 1093, 1284}.

Remark.

- With arithmetic problems, input integers are encoded in binary.
- Polynomial reduction must be polynomial in binary encoding.

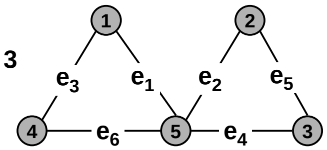
Subset Sum

Treat as base k+1 integer

Claim. VERTEX-COVER $\leq_P$ SUBSET-SUM.

Proof. Given instance G, k of VERTEX-COVER, create following instance of SUBSET-SUM.

k = 3



	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	0	1	0	0	0
v_2	0	1	0	0	1	0
v_3	0	0	0	1	1	0
v_4	0	0	1	0	0	1
v_5	1	1	0	1	0	1

Node-arc incidence matrix

		e_1	e_2	e_3	e_4	e_5	e_6	decimal
x_1	1	1	0	1	0	0	0	5,184
x_2	1	0	1	0	0	1	0	4,356
x_3	1	0	0	0	1	1	0	4,116
x_4	1	0	0	1	0	0	1	4,161
x_5	1	1	1	0	1	0	1	5,393
y_1	0	1	0	0	0	0	0	1,024
y_2	0	0	1	0	0	0	0	256
y_3	0	0	0	1	0	0	0	64
y_4	0	0	0	0	1	0	0	16
y_5	0	0	0	0	0	1	0	4
y_6	0	0	0	0	0	0	1	1
t	3	2	2	2	2	2	2	15,018
	k							

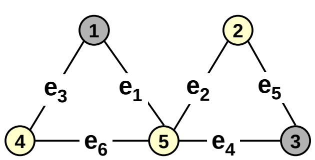
Subset Sum

Claim. G has vertex cover of size k if and only if there is a subset S that sums to exactly t.

Proof. $\Rightarrow$

- Suppose G has a vertex cover C of size k.
- Let $S = C \cup \{ y_j : |e_j \cap C| = 1 \}$
  - most significant bits add up to k
  - remaining bits add up to 2

		e_1	e_2	e_3	e_4	e_5	e_6	decimal
x_1	1	1	0	1	0	0	0	5,184
x_2	1	0	1	0	0	1	0	4,356
x_3	1	0	0	0	1	1	0	4,116
x_4	1	0	0	1	0	0	1	4,161
x_5	1	1	1	0	1	0	1	5,393
y_1	0	1	0	0	0	0	0	1,024
y_2	0	0	1	0	0	0	0	256
y_3	0	0	0	1	0	0	0	64
y_4	0	0	0	0	1	0	0	16
y_5	0	0	0	0	0	1	0	4
y_6	0	0	0	0	0	0	1	1
t	3	2	2	2	2	2	2	15,018
	k							



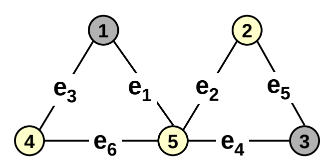
Subset Sum

Claim. G has vertex cover of size k if and only if there is a subset S that sums to exactly t.

Proof. $\Leftarrow$

- Suppose subset S sums to t.
- Let $C = S \cap \{x_1, \ldots, x_n\}$.
  - each edge has three 1's, so no carries possible
  - $|C| = k$
  - at least one x_i must contribute to sum for e_j

		e_1	e_2	e_3	e_4	e_5	e_6	decimal
x_1	1	1	0	1	0	0	0	5,184
x_2	1	0	1	0	0	1	0	4,356
x_3	1	0	0	0	1	1	0	4,116
x_4	1	0	0	1	0	0	1	4,161
x_5	1	1	1	0	1	0	1	5,393
y_1	0	1	0	0	0	0	0	1,024
y_2	0	0	1	0	0	0	0	256
y_3	0	0	0	1	0	0	0	64
y_4	0	0	0	0	1	0	0	16
y_5	0	0	0	0	0	1	0	4
y_6	0	0	0	0	0	0	1	1
t	3	2	2	2	2	2	2	15,018
	k							



Partition

SUBSET-SUM: Given a set X of integers and a target integer t, is there a subset $S \subseteq X$ whose elements sum to exactly t.

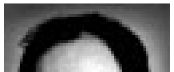
PARTITION: Given a set X of integers, is there a subset $S \subseteq X$ such that $\sum_{a \in S} a = \sum_{a \in X \setminus S} a$.

Claim. SUBSET-SUM $\le_P$ PARTITION.

Proof. Let (X, t) be an instance of SUBSET-SUM.

- Define W to be sum of integers in X: $W = \sum_{a \in X} a$.
- Create instance of PARTITION: $X' = X \cup \{2W - t\} \cup \{W + t\}$.
- SUBSET-SUM instance is yes if and only if PARTITION instance is.
  - in any partition of X'
    - Each half of partition sums to 2W.
    - Two new elements can't be in same partition.
    - Discard new elements ⇒ subset of X that sums to t.

Polynomial-Time Reductions



Dick Karp (1972)