NP-Completeness

Princeton University • COS 423 • Theory of Algorithms • Spring 2001 • Kevin Wayne

Properties of Algorithms

A given problem can be solved by many different algorithms. Which ALGORITHMS will be useful in practice?

A working definition: (Jack Edmonds, 1962)

- Efficient: polynomial time for ALL inputs.
- Inefficient: "exponential time" for SOME inputs.

Robust definition has led to explosion of useful algorithms for wide spectrum of problems.

- Notable exception: simplex algorithm.

Exponential Growth

Exponential growth dwarfs technological change.

- Suppose each electron in the universe had power of today's supercomputers.
- And each works for the life of the universe in an effort to solve TSP problem using N! algorithm.

Some Numbers	
Quantity	Number
Home PC instructions / second	10^9
Supercomputer instructions / second	10^{12}
Seconds per year	10^9
Age of universe †	10^{13}
Electrons in universe †	10^{79}

† Estimated

- Will not succeed for 1,000 city TSP!

$1000! \gg 10^{1000} \gg 10^{79} \times 10^{13} \times 10^{9} \times 10^{12}$

Properties of Problems

Which PROBLEMS will we be able to solve in practice?

- Those with efficient algorithms.
- How can I tell if I am trying to solve such a problem?
  - Theory of NP-completeness helps.

Yes	Probably No
Shortest path	Longest path
Euler cycle	Hamiltonian cycle
Min cut	Max cut
2-SAT	3-SAT
PLANAR-2-COLOR	PLANAR-3-COLOR
PLANAR-4-COLOR	PLANAR-3-COLOR
Matching	3D-Matching
Baseball elimination	Soccer elimination
Bipartite vertex cover	Vertex cover

Unknown
Primality
Factoring
Graph isomorphism

P

Decision problem X.
- X is a (possibly infinite) set of binary strings.
- Instance: finite binary string s, of length |s|.
- Algorithm A solves X if A(s) = YES $\Leftrightarrow$ s $\in$ X.

Polynomial time.
- Algorithm A runs in polynomial-time if for every instance s, A terminates in at most p(s) "steps", where p is some polynomial.

Definition of P.
- Set of all decision problems solvable in polynomial time on a deterministic Turing machine.

Examples:
- MULTIPLE: Is the integer y a multiple of x?
- RELPRIME: Are the integers x and y relatively prime?
- PERFECT-MATCHING: Given graph G, is there a perfect matching?

Strong Church-Turing Thesis

Definition of P fundamental because of SCT.

Strong Church-Turing thesis:
- P is the set of decision problems solvable in polynomial time on REAL computers.

Evidence supporting thesis:
- True for all physical computers.
- Can create deterministic TM that efficiently simulates any real general-purpose machine (and vice versa).

Possible exception?
- Quantum computers: no conventional gates.

Efficient Certification

Certification algorithm.
- Design an algorithm that checks whether proposed solution is a YES instance.

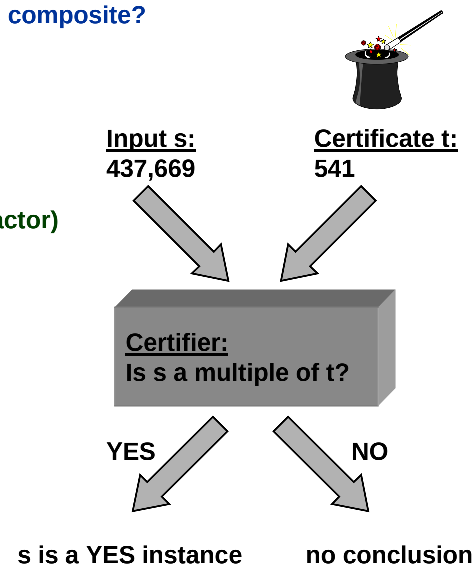
Algorithm C is an efficient certifier for X if:
- C is a polynomial-time algorithm that takes two inputs s and t.
- There exists a polynomial p( ) so that for every string s, s $\in$ X $\Leftrightarrow$ there exists a string t such that $|t| \le p(|s|)$ and C(s, t) = YES.

Intuition.
- Efficient certifier views things from "managerial" viewpoint.
- It doesn't determine whether s $\in$ X on its own.
- Rather, it evaluates a proposed proof t that s $\in$ X.
- Accepts if and only if given a "short" proof of this fact.

Certifiers and Certificates

COMPOSITE: Given integer s, is s composite?

Observation. s is composite $\Leftrightarrow$ there exists an integer $1 < t < s$ such that s is a multiple of t.
- YES instance: s = 437,669.
  - certificate t = 541 or 809 (a factor)



Certifiers and Certificates

COMPOSITE: Given integer s, is s composite?

Observation. s is composite ⇔ there exists an integer 1 < t < s such that s is a multiple of t.

- YES instance: s = 437,669.
  - certificate t = 541 or 809 (a factor)
- NO instance: s = 437,677.
  - no witness can fool verifier into saying YES
- Conclusion: COMPOSITE ∈ NP.

Input s:
437,677

Certificate t:
???

Certifier:
Is s a multiple of t?

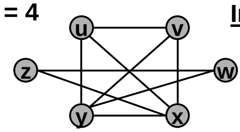
YES → s is a YES instance

NO → no conclusion

Certifiers and Certificates

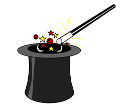
CLIQUE: Given an undirected graph, is there a subset S of k nodes such that there is an arc connecting every pair of nodes in S?

k = 4



Input s:

Certificate t:
{u, v, w, x}



Certifier:
1. For all pairs of nodes v and w in t, check that (v, w) is an edge in the graph.
2. Check that number of nodes in t ≤ k.

YES → s is a YES instance

NO → no conclusion

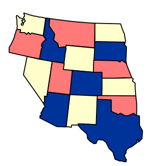
CLIQUE ∈ NP.

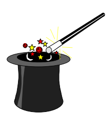
Certifiers and Certificates

3-COLOR: Given planar map, can it be colored with 3 colors?

Input s:

Certificate t:





Certifier:
1. Check that s and t describe same map.
2. Count number of distinct colors in t.
3. Check all pairs of adjacent states.

YES → s is a YES instance

NO → no conclusion

3-COLOR ∈ NP.

NP

Definition of NP:
- Does NOT mean "not polynomial."

Definition of NP:
- Set of all decision problems for which there exists an efficient certifier.
- Definition important because it links many fundamental problems.

Claim: P ⊆ NP.
Proof: Consider problem X ∈ P.
- Then, there exists efficient algorithm A(s) that solves X.
- Efficient certifier B(s, t): return A(s).

NP

Definition of EXP:

- Set of all decision problems solvable in **exponential** time on a deterministic Turing machine.

Claim: NP $\subseteq$ EXP.

Proof: Consider problem X $\in$ NP.

- Then, there exists efficient certifier C(s, t) for X.
- To solve input s, run C(s, t) on all strings t with $|t| \leq p(|s|)$.
- Return YES, if C(s, t) returns YES for any of these.

Useful alternate definition of NP:

- Set of all decision problems solvable in polynomial time on a **NONDETERMINISTIC** Turing machine.
- Intuition: act of searching for t is viewed as a non-deterministic search over the space of possible proofs. Nondeterministic TM can try all possible solutions in parallel.

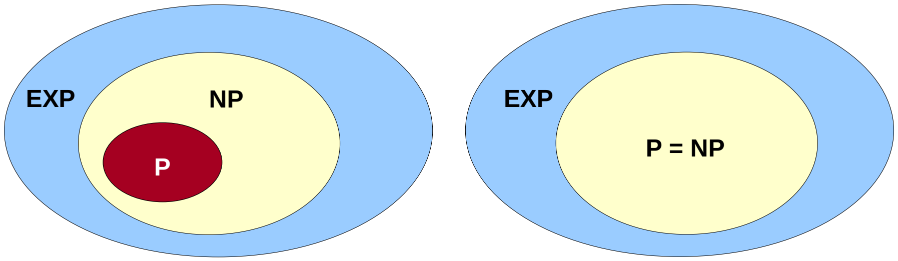
The Main Question

Does P = NP? (Edmonds, 1962)

- Is the original **DECISION** problem as easy as **CERTIFICATION**?
- Does nondeterminism help you solve problems faster?

Most important open problem in computer science.

- If yes, staggering practical significance.
- Clay Foundation Millennium $1 million prize.



If $P \neq NP$ If $P = NP$

The Main Question

Generator (P)

- Factor integer s.
- Color a map with minimum # colors.
- Design airfoil of minimum drag.
- Prove a beautiful theorem.
- Write a beautiful sonnet.
- Devise a good joke.
- Vinify fine wine.
- Orate a good lecture.
- Ace an exam.

Certifier (NP)

- Is s a multiple of t?
- Check if all adjacent regions have different colors.
- Compute drag of airfoil.
- Understand its proof.
- Appreciate it.
- Laugh at it.
- Be a wine snob.
- Know when you've heard one.
- Verify TA's solutions.

Imagine the wealth of a society that produces optimal planes, bridges, rockets, theorems, art, music, wine, jokes.

The Main Question

Does P = NP?

- Is the original **DECISION** problem as easy as **CERTIFICATION**?

If yes, then:

- Efficient algorithms for 3-COLOR, TSP, FACTOR, . . .
- Cryptography is impossible (except for one-time pads) on conventional machines.
- Modern banking system will collapse.
- Harmonial bliss.

If no, then:

- Can't hope to write efficient algorithm for TSP.
- But maybe efficient algorithm still exists for FACTOR . . . .

The Main Question

Does P = NP?

- Is the original DECISION problem as easy as CERTIFICATION?

Probably no, since:

- Thousands of researchers have spent four frustrating decades in search of polynomial algorithms for many fundamental NP problems without success.
- Consensus opinion: $P \neq NP$.

But maybe yes, since:

- No success in proving $P \neq NP$ either.

Polynomial Transformation

Problem X polynomial reduces (Cook-Turing) to problem Y ($X \leq_P Y$) if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Problem X polynomial transforms (Karp) to problem Y if given any input x to X, we can construct an input y such that x is a YES instance of X if and only if y is a YES instance of Y.

- We require |y| to be of size polynomial in |x|.

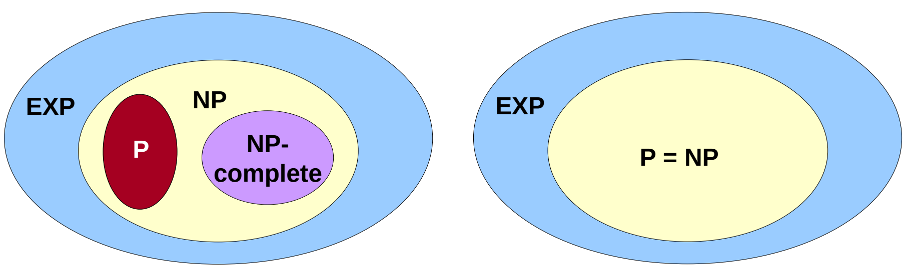
Polynomial transformation is polynomial reduction with just one call to oracle for Y, exactly at the end of the algorithm for X.

Note: all previous reductions were of this form!

NP-Complete

Definition of NP-complete:

- A problem Y in NP with the property that for every problem X in NP, X polynomial transforms to Y.
- "Hardest computational problems" in NP.



If $P \neq NP$ If $P = NP$

NP-Complete

Definition of NP-complete:

- A problem Y in NP with the property that for every problem X in NP, X polynomial transforms to Y.

Significance.

- Efficient algorithm for any NP-complete problem $\Rightarrow$ efficient algorithm for every other problem in NP.
- Links together a huge and diverse number of fundamental problems:
  - TSP, 3-COLOR, CNF-SAT, CLIQUE, ...
- Can implement any computer program in 3-COLOR.

Notorious complexity class.

- Only exponential algorithms known for these problems.
- Called "intractable" - unlikely that they can be solved given limited computing resources.

Some NP-Complete Problems

Most natural problems in NP are either in P or NP-complete.

Six basic genres and paradigmatic examples of NP-complete problems.

- Packing problems: SET-PACKING, INDEPENDENT-SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, CLIQUE.
- Constraint satisfaction problems: SAT, 3-SAT.
- Numerical problems: SUBSET-SUM, PARTITION, KNAPSACK.

Caveat: PRIME, FACTOR not known to be NP-complete.

The "World's First" NP-Complete Problem

CNF-SAT is NP-complete. (Cook-Levin, 1960's)

Stephen Cook

Idea of proof:

- Given problem X in NP, by definition, there exists nondeterministic TM M that solves X in polynomial time.
- Possible execution of M on input string s forms a branching tree of configurations, where each configuration gives snapshot of M (tape contents, head position, state of finite control) at some time step t.
- M is polynomial time $\Rightarrow$ polynomial tree depth $\Rightarrow$ polynomial number of tape cells in play.
- Use polynomial number of Boolean variables to model which symbol occupies cell i at time t, location of read head at time t, state of finite control, etc.
- Use polynomial number of clauses to ensure machine makes legal moves, starts and ends in appropriate configurations, etc.

Establishing NP-Completeness

Definition of NP-complete:

- A problem Y $\in$ NP with the property that for every problem X in NP, X polynomial transforms to Y.

Cook's theorem. CNF-SAT is NP-complete.

Recipe to establish NP-completeness of problem Y.

- Step 1. Show that Y $\in$ NP.
- Step 2. Show that CNF-SAT (or any other NP-complete problem) transforms to Y.

Example: CLIQUE is NP-complete.

✔ Step 1. CLIQUE $\in$ NP.

✔ Step 2. CNF-SAT polynomial transforms to CLIQUE.

Minesweeper Consistency Problem

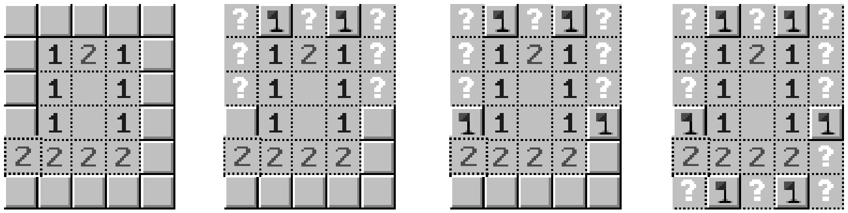
Minesweeper.

- Start: blank grid of squares.
- Some squares conceal mines; the rest are safe.
- Find location of mines without detonating any.
- Choose a square.
  - if mine underneath, it detonates and you lose
  - If no mine, computer tells you how many total mines in 8 neighboring squares

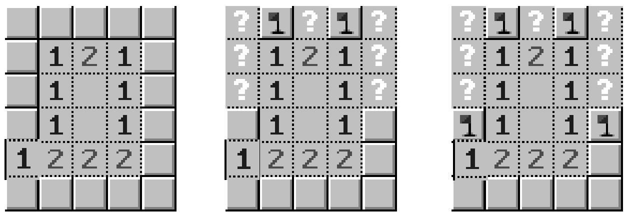
Minesweeper Consistency Problem

Minesweeper consistency problem.

- Given a state of what purports to be a Minesweeper games, is it logically consistent.



YES



NO

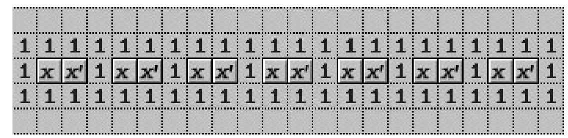
Minesweeper Consistency Problem

Minesweeper consistency problem.

- Given a state of what purports to be a Minesweeper games, is it logically consistent.

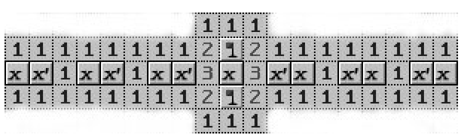
Claim. Minesweeper consistency is NP-complete.

- Proof idea: reduce from circuit satisfiability.
- Build circuit by laying out appropriate minesweeper configurations.

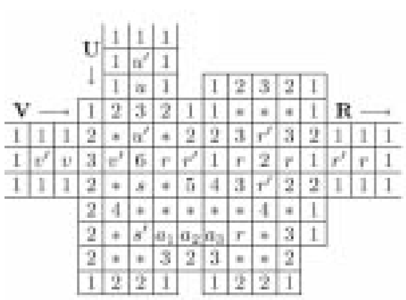


A Minesweeper Wire

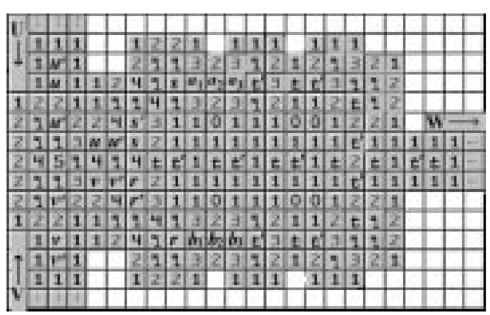
Minesweeper Consistency Problem



A NOT Gate

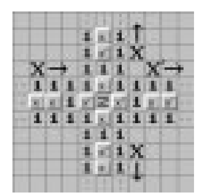


A OR Gate



An AND Gate

Minesweeper Consistency Problem



A 3-way Splitter

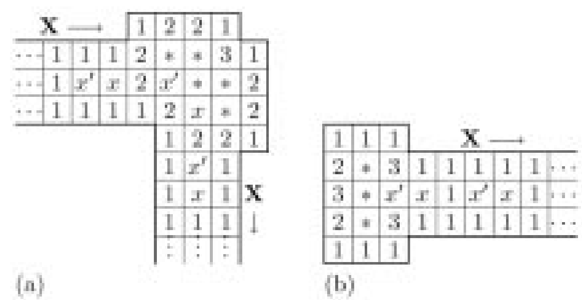
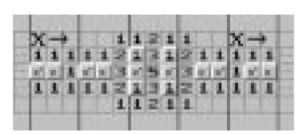
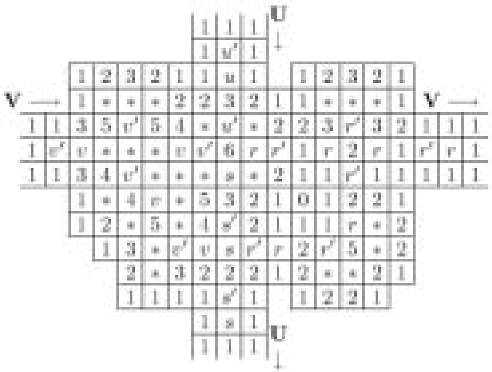


Figure 4: (a) A bent wire. (b) A terminated wire.



A Phase Changer

Minesweeper Consistency Problem

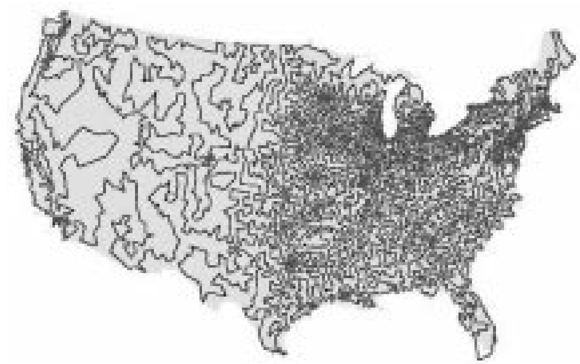


A Wire Crossing

Coping With NP-Completeness

Hope that worst case doesn't occur.

- Complexity theory deals with worst case behavior. The instance(s) you want to solve may be "easy."
  - TSP where all points are on a line or circle
  - 13,509 US city TSP problem solved



(Cook et. al., 1998)

Coping With NP-Completeness

Hope that worst case doesn't occur.

Change the problem.

- Develop a heuristic, and hope it produces a good solution.
- Design an approximation algorithm: algorithm that is guaranteed to find a high-quality solution in polynomial time.
  - active area of research, but not always possible unless P = NP!
  - Euclidean TSP tour within 1% of optimal
  - stay tuned

Sanjeev Arora (1997)

Coping With NP-Completeness

Hope that worst case doesn't occur.

Change the problem.

Exploit intractability.

- Cryptography.

Coping With NP-Completeness

Hope that worst case doesn't occur.

Change the problem.

Exploit intractability.

Keep trying to prove P = NP.

Coping With NP-Completeness

A person can be at most two of the following three things:

- Honest.
- Intelligent.
- A politician.

If a problem is NP-complete, you design an algorithm to do at most two of the following three things:

- Solve the problem exactly.
- Guarantee to solve the problem in polynomial time.
- Solve arbitrary instances of the problem.

Asymmetry of NP

Definition of NP: $X \in$ NP if there exists a certifier C(s, t) such that:

- Input string s is a YES instance if and only if there exists a short certificate t such that C(s, t) = YES.
- Alternatively, input string s is a NO instance if and only if for all short t, C(s, t) = NO.

Ex. HAM-CYCLE vs. NO-HAM-CYCLE.

- Given G and a proposed Hamiltonian cycle, easy to if check if it really is a Hamiltonian cycle.
- Given G, hard to assert that it is not Hamiltonian.

Ex. PRIME vs. COMPOSITE.

- Given integer s and proposed factor t, it is easy to check if s is a multiple of t.
- Appears harder to assert that an integer is not composite.

Co-NP

NP: Decision problem $X \in$ NP if there exists a certifier C(s, t) s.t.

- Input string s is a YES instance if and only if there exists a short certificate t such that C(s, t) = YES.
- Alternatively, input string s is a NO instance if and only if for all short t, C(s, t) = NO.

co-NP: $X \in$ co-NP if there exists a certifier C(s, t) s.t.

- Input string s is a NO instance if and only if there exists a short certificate t such that C(s, t) = YES.
- Alternatively, input string s is a YES instance if and only if for all short t, C(s, t) = NO.

Co-NP Certifiers and Certificates

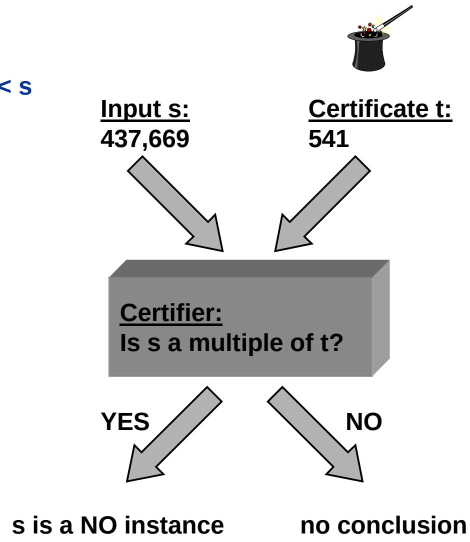
PRIME: Given integer s, is s prime?

Observation. s is composite if and only if there exists an integer 1 < t < s such that s is a multiple of t.

- NO instance: s = 437,669.

Conclusions.

- PRIME ∈ co-NP.
- COMPOSITE ∈ P.



Co-NP Verifiers and Certificates

COMPOSITE: Given integer s, is s composite?

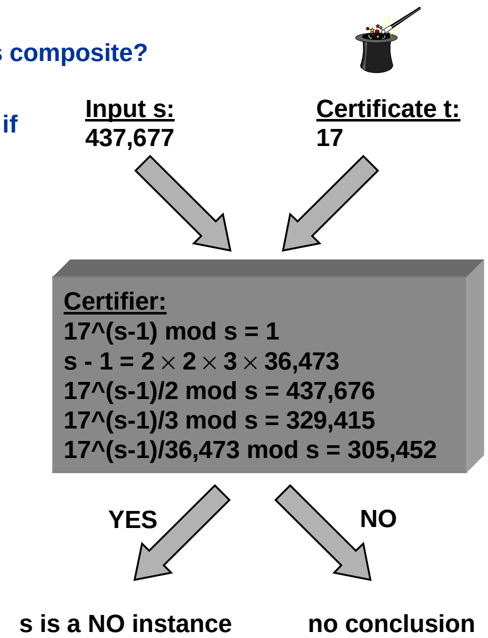
Fact (Pratt). s is prime if and only if s > 2 is odd and there exists an integer 1 < t < s s.t.

$$t^{s-1} \equiv 1 \pmod{s}$$
$$t^{(s-1)/p} \not\equiv 1 \pmod{s}$$
$$\text{for all prime divisors } p \text{ of } s\text{-}1$$

- NO instance: 437,677.

Conclusions.

- COMPOSITE ∈ co-NP.
- PRIME ∈ NP.



NP = co-NP ?

Fundamental question: does NP = co-NP?

- Do YES-instances have short certificates if and only if NO-instances have short certificates.
- Consensus opinion: no.

Theorem. If NP ≠ co-NP, then P ≠ NP.

Proof. We prove if P = NP, then NP = co-NP.

- Key idea: P is closed under complementation, so if P = NP, then NP is closed under complementation as well.
- More formally, using the assumption P = NP:

$$X \in NP \;\Rightarrow\; X \in P \;\Rightarrow \overline{X} \in P \;\Rightarrow \overline{X} \in NP \;\Rightarrow\; X \in \text{co-}NP$$

$$X \in \text{co-}NP \;\Rightarrow\; \overline{X} \in NP \;\Rightarrow \overline{X} \in P \;\Rightarrow X \in P \;\Rightarrow\; X \in NP$$

- Thus, NP ⊆ co-NP and co-NP ⊆ NP, so co-NP = NP.

Good Characterizations

Good characterization: NP ∩ co-NP.

- If problem X is in NP and co-NP, then:
  - for YES instance, there is a short certificate
  - for NO instance, there is a short certificate
- Provides conceptual leverage for reasoning about a problem.

Examples.

- MAX-FLOW: given a network, is there an s-t flow of value ≥ W.
  - if yes, can exhibit s-t flow that achieves this value
  - if no, can exhibit s-t cut whose capacity is less than W
- PERFECT-MATCHING: given a bipartite graph, is there a perfect matching.
  - if yes, can exhibit a perfect matching
  - if no, can exhibit a set of vertices S ⊆ L such that the total number of neighbors of S is strictly less than |S|

Good Characterizations

Observation. $P \subseteq NP \cap$ co-NP.

- Proof of max-flow min-cut theorem and Hall's theorem led to stronger results that max-flow and bipartite matching are in P.
- Sometimes finding a good characterization seems easier than finding an efficient algorithm: linear programming.

Fundamental question: does $P = NP \cap$ co-NP?

- Mixed opinions.
- Many examples where problem found to have a non-trivial good characterization, but only years later discovered to be in P.
- Note: PRIME $\in$ NP $\cap$ co-NP, but not known to be in P.

A Note on Terminology

Knuth. (SIGACT News 6, January 1974, p. 12 – 18)

Find an adjective x that sounds good in sentences like.

- FIND-TSP-TOUR is x.
- It is x to decide whether a given graph has a Hamiltonian cycle.
- It is unknown whether FACTOR is an x problem.

Note: x does not necessarily imply that a problem is in NP or even a decision problem.

Knuth's original suggestions.

- Hard.
- Tough.
- Herculean.
- Formidable.
- Arduous.

A Note on Terminology

Some English word write-ins.

- Impractical.
- Bad.
- Heavy.
- Tricky.
- Intricate.
- Prodigious.
- Difficult.
- Intractable.
- Costly.
- Obdurate.
- Obstinate.
- Exorbitant.
- Interminable.

A Note on Terminology

Hard-boiled. (Ken Steiglitz)

- In honor of Cook.

Hard-ass. (Al Meyer)

- Hard as satisfiability.

Sisyphean. (Bob Floyd)

- Problem of Sisyphus was time-consuming.
- Hercules needed great strength.
- Problem: Sisyphus never finished his task $\Rightarrow$ unsolvable.

Ulyssean. (Don Knuth)

- Ulysses was note for his persistence and also finished.

A Note on Terminology: Made-Up Words

Exparent. (Mike Paterson)
- exponential + apparent

Perarduous. (Mike Paterson)
- through, in space or time + completely

Supersat. (Al Meyer)
- greater than or equal to satisfiability

Polychronious. (Ed Reingold)
- enduringly long; chronic
- Appears in Webster's 2nd unabridged, but apparently in no other dictionary.

A Note on Terminology: Acronyms

PET. (Shen Lin)
- Probably exponential time.
- Provably exponential time, previously exponential time.

GNP. (Al Meyer)
- Greater than or equal to NP in difficulty.
- Costing more than GNP to resolve.

A Note on Terminology: Consensus

NP-complete.
- A problem in NP such that every other problem in NP transforms to it.

NP-hard. (Bell Labs, Steve Cook, Ron Rivest, Sartaj Sahni)
- A problem such that every problem in NP transforms to it.

Knuth's conclusion.
- "create research workers are as full of ideas for new terminology as they are empty of enthusiasm for adopting it."

Summary

Many fundamental problems are NP-complete.
- TSP, 3-CNF-SAT, 3-COLOR, CLIQUE, ...

Theory says we probably won't be able to design efficient algorithms for NP-complete problems.
- You will likely run into these problems in your scientific life.
- If you know about NP-completeness, you can identify them and avoid wasting time.