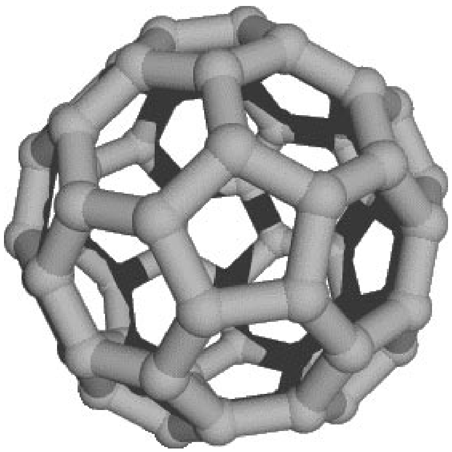
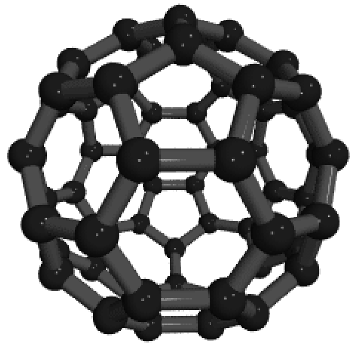
Linear Programming





Linear Programming

Significance.

- Quintessential tool for optimal allocation of scarce resources, among a number of competing activities.
- Powerful model generalizes many classic problems:
  - shortest path, max flow, multicommodity flow, MST, matching, 2-person zero sum games
- Ranked among most important scientific advances of 20th century.
  - accounts for a major proportion of all scientific computation
- Helps find "good" solutions to NP-hard optimization problems.
  - optimal solutions (branch-and-cut)
  - provably good solutions (randomized rounding)

Brewery Problem

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

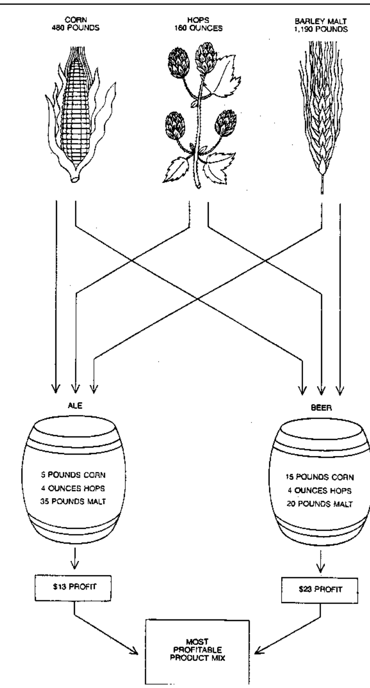
Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale	5	4	35	13
Beer	15	4	20	23
Quantity	480	160	1190	

How can brewer maximize profits?

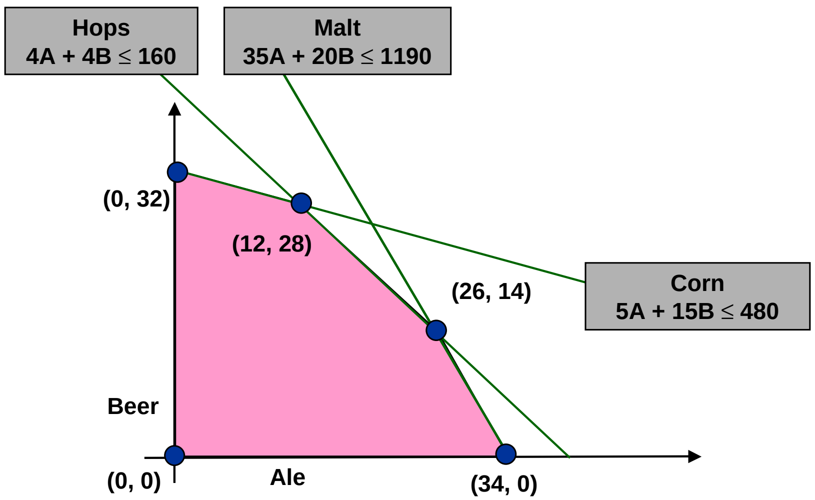
- Devote all resources to beer: 32 barrels of beer ⇒ $736.
- Devote all resources to ale: 34 barrels of ale ⇒ $442.
- 7½ barrels of ale, 29½ barrels of beer ⇒ $776.
- 12 barrels of ale, 28 barrels of beer ⇒ $800.

Brewery Problem

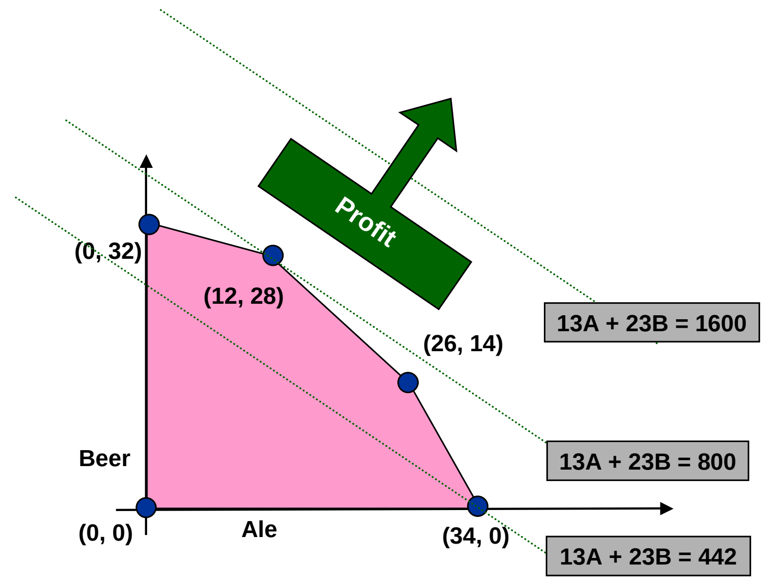
$$
\begin{array}{rrcrcr}
\text{max} & 13A & + & 23B & & \\
\text{s.t.} & 5A & + & 15B & \leq & 480 \\
 & 4A & + & 4B & \leq & 160 \\
 & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$



Brewery Problem: Feasible Region

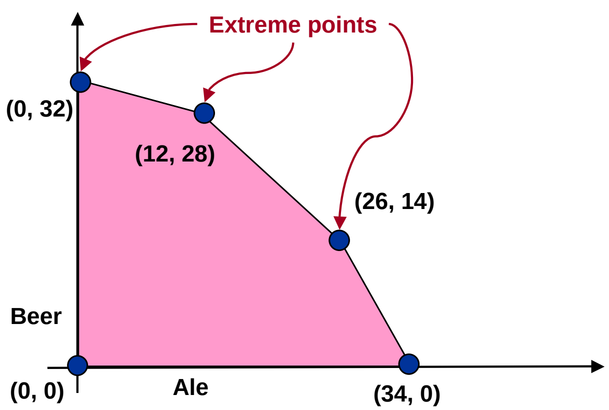


Brewery Problem: Objective Function



Brewery Problem: Geometry

Brewery problem observation. Regardless of coefficients of linear objective function, there exists an optimal solution that is an extreme point.



Linear Programming

LP "standard" form.

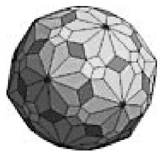
- Input data: rational numbers c_j, b_i, a_{ij} .
- Maximize linear objective function.
- Subject to linear inequalities.

$$
\begin{array}{llll}
(P) & \max & \sum_{j=1}^{n} c_j x_j & \\
 & \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \;\le\; b_i & 1 \le i \le m \\
 & & x_j \;\ge\; 0 & 1 \le j \le n
\end{array}
$$

$$
\begin{array}{llll}
(P) & \max & c \bullet x & \\
 & \text{s.t.} & Ax & \le \; b \\
 & & x & \ge \; 0
\end{array}
$$

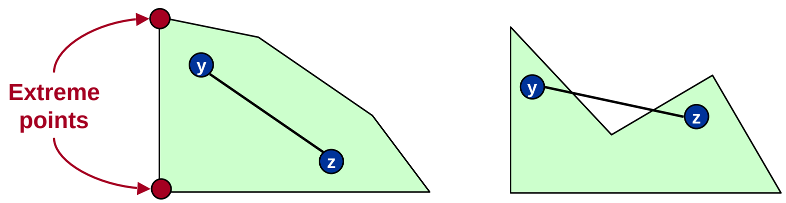
LP: Geometry

Geometry.
- Forms an n-dimensional polyhedron.



$$
\begin{aligned}
(P)\quad \max\ & \sum_{j=1}^{n} c_j x_j \\
\text{s. t.}\ & \sum_{j=1}^{n} a_{ij} x_j \ \leq\ b_i \quad 1 \leq i \leq m \\
& x_j \ \geq\ 0 \quad 1 \leq j \leq n
\end{aligned}
$$

- **Convex:** if y and z are feasible solutions, then so is ½y + ½z.
- **Extreme point:** feasible solution x that can't be written as ½y + ½z for any two distinct feasible solutions y and z.



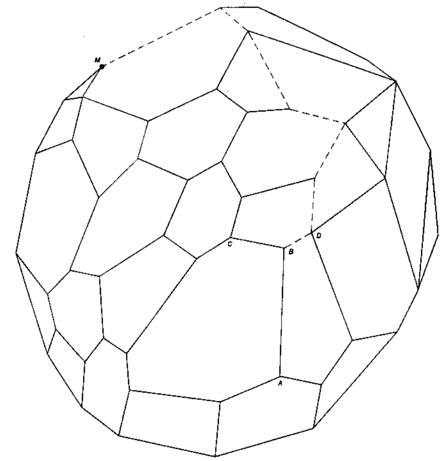
Convex

Not convex

LP: Geometry

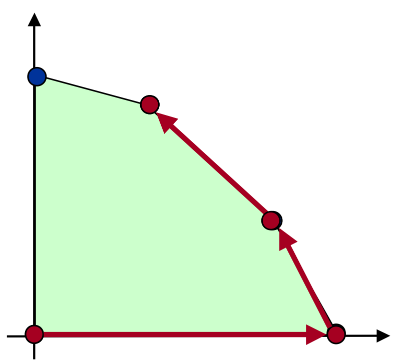
Extreme Point Theorem. If there exists an optimal solution to standard form LP (P), then there exists one that is an extreme point.
- Only need to consider finitely many possible solutions.

Greed. Local optima are global optima.



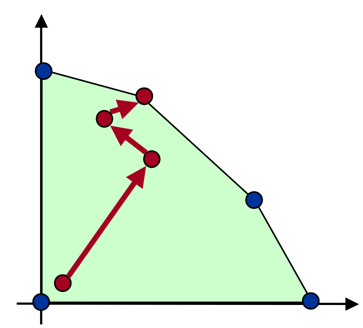
LP: Algorithms

Simplex. (Dantzig 1947)
- Developed shortly after WWII in response to logistical problems: used for 1948 Berlin airlift.
- Practical solution method that moves from one extreme point to a neighboring extreme point.
- Finite (exponential) complexity, but no polynomial implementation known.



LP: Polynomial Algorithms

Ellipsoid. (Khachian 1979, 1980)
- Solvable in polynomial time: $O(n^4 L)$ bit operations.
  - n = # variables
  - L = # bits in input
- Theoretical tour de force.
- Not remotely practical.



Karmarkar's algorithm. (Karmarkar 1984)
- $O(n^{3.5} L)$.
- Polynomial and reasonably efficient implementations possible.

Interior point algorithms.
- $O(n^3 L)$.
- Competitive with simplex!
  - will likely dominate on large problems soon
- Extends to even more general problems.

LP Duality

Primal problem.

max	$4x_1$	$+$	x_2	$+$	$5x_3$	$+$	$3x_4$		
s. t.	x_1	$-$	x_2	$-$	x_3	$+$	$3x_4$	$\leq$	1
	$5x_1$	$+$	x_2	$+$	$3x_3$	$+$	$8x_4$	$\leq$	55
	$-x_1$	$+$	$2x_2$	$+$	$3x_3$	$-$	$5x_4$	$\leq$	3
	x_1	,	x_2	,	x_3	,	x_4	$\geq$	0

Find a lower bound on optimal value.

- $(x_1, x_2, x_3, x_4) = (0, 0, 1, 0) \Rightarrow z^* \geq 5.$
- $(x_1, x_2, x_3, x_4) = (2, 1, 1, 1/3) \Rightarrow z^* \geq 15.$
- $(x_1, x_2, x_3, x_4) = (3, 0, 2, 0) \Rightarrow z^* \geq 22.$
- $(x_1, x_2, x_3, x_4) = (0, 14, 0, 5) \Rightarrow z^* \geq 29.$

LP Duality

Primal problem.

max	$4x_1$	$+$	x_2	$+$	$5x_3$	$+$	$3x_4$		
s. t.	x_1	$-$	x_2	$-$	x_3	$+$	$3x_4$	$\leq$	1
	$5x_1$	$+$	x_2	$+$	$3x_3$	$+$	$8x_4$	$\leq$	55
	$-x_1$	$+$	$2x_2$	$+$	$3x_3$	$-$	$5x_4$	$\leq$	3
	x_1	,	x_2	,	x_3	,	x_4	$\geq$	0

Find an upper bound on optimal value.

- Multiply 2nd inequality by 2: $10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110.$

  $\Rightarrow z^* = 4x_1 + x_2 + 5x_3 + 3x_4 \leq 10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110.$

- Adding 2nd and 3rd inequalities: $4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58.$

  $\Rightarrow z^* = 4x_1 + x_2 + 5x_3 + 3x_4 \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58.$

LP Duality

Primal problem.

max	$4x_1$	$+$	x_2	$+$	$5x_3$	$+$	$3x_4$		
s. t.	x_1	$-$	x_2	$-$	x_3	$+$	$3x_4$	$\leq$	1
	$5x_1$	$+$	x_2	$+$	$3x_3$	$+$	$8x_4$	$\leq$	55
	$-x_1$	$+$	$2x_2$	$+$	$3x_3$	$-$	$5x_4$	$\leq$	3
	x_1	,	x_2	,	x_3	,	x_4	$\geq$	0

Find an upper bound on optimal value.

- Adding 11 times 1st inequality to 6 times 3rd inequality:

  $\Rightarrow z^* = 4x_1 + x_2 + 5x_3 + 3x_4 \leq 5x_1 + x_2 + 7x_3 + 3x_4 \leq 29.$

Recall.

- $(x_1, x_2, x_3, x_4) = (0, 14, 0, 5) \Rightarrow z^* \geq 29.$

LP Duality

Primal problem.

max	$4x_1$	$+$	x_2	$+$	$5x_3$	$+$	$3x_4$		
s. t.	x_1	$-$	x_2	$-$	x_3	$+$	$3x_4$	$\leq$	1
	$5x_1$	$+$	x_2	$+$	$3x_3$	$+$	$8x_4$	$\leq$	55
	$-x_1$	$+$	$2x_2$	$+$	$3x_3$	$-$	$5x_4$	$\leq$	3
	x_1	,	x_2	,	x_3	,	x_4	$\geq$	0

General idea: add linear combination (y_1, y_2, y_3) of the constraints.

$$(y_1 + 5y_2 - y_3)\, x_1 + (-y_1 + y_2 + 2y_3)\, x_2 + (-y_1 + 3y_2 + 3y_3)\, x_3 + (3y_1 + 8y_2 - 5y_3)\, x_4 \leq y_1 + 55y_2 + 3y_3$$

Dual problem.

min	y_1	$+$	$55y_2$	$+$	$3y_3$		
s. t.	y_1	$+$	$5y_2$	$-$	y_3	$\geq$	4
	$-y_1$	$+$	y_2	$+$	$2y_3$	$\geq$	1
	$-y_1$	$+$	$3y_2$	$+$	$3y_3$	$\geq$	5
	$3y_1$	$+$	$8y_2$	$-$	$5y_3$	$\geq$	3
	y_1	,	y_2	,	y_3	$\geq$	0

LP Duality

Primal and dual linear programs: given rational numbers a_{ij}, b_i, c_j, find values x_i, y_j that optimize (P) and (D).

$$(P) \quad \max \sum_{j=1}^{n} c_j x_j \quad \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m, \qquad x_j \ge 0 \quad 1 \le j \le n$$

$$(D) \quad \min \sum_{i=1}^{m} b_i y_i \quad \text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad 1 \le j \le n, \qquad y_i \ge 0 \quad 1 \le i \le m$$

Duality Theorem (Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947).
If (P) and (D) are nonempty then max = min.

- Dual solution provides certificate of optimality $\Rightarrow$ decision version $\in$ NP $\cap$ co-NP.
- Special case: max-flow min-cut theorem.
- Sensitivity analysis.

LP Duality: Economic Interpretation

Brewer's problem: find optimal mix of beer and ale to maximize profits.

$$\begin{array}{llrcrcr} (P) & \max & 13A & + & 23B & & \\ & \text{s.t.} & 5A & + & 15B & \le & 480 \\ & & 4A & + & 4B & \le & 160 \\ & & 35A & + & 20B & \le & 1190 \\ & & A & , & B & \ge & 0 \end{array}$$

$A^* = 12$
$B^* = 28$
OPT = 800

Entrepreneur's problem: Buy individual resources from brewer at minimum cost.

- C, H, M = unit price for corn, hops, malt.
- Brewer won't agree to sell resources if 5C + 4H + 35M < 13.

$$\begin{array}{llrcrcrcr} (D) & \min & 480C & + & 160H & + & 1190M & & \\ & \text{s.t.} & 5C & + & 4H & + & 35M & \ge & 13 \\ & & 15C & + & 4H & + & 20M & \ge & 23 \\ & & C & , & H & , & M & \ge & 0 \end{array}$$

$C^* = 1$
$H^* = 2$
$M^* = 0$
OPT = 800

LP Duality: Economic Interpretation

Sensitivity analysis.

- How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?
  - ✐ corn \$1, hops \$2, malt \$0.
- Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?
  - ✐ At least 2 (\$1) + 5 (\$2) + 24 (0\$) = \$12 / barrel.

Standard Form

Standard form.

$$(P) \quad \max \sum_{j=1}^{n} c_j x_j \quad \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m, \qquad x_j \ge 0 \quad 1 \le j \le n$$

Easy to handle variants.

- $x + 2y - 3z \ge 17 \quad \Rightarrow \quad -x - 2y + 3z \le -17.$
- $x + 2y - 3z = 17 \quad \Rightarrow \quad x + 2y - 3z \le 17, \;\; -x - 2y + 3z \le -17.$
- $\min x + 2y - 3z \quad \Rightarrow \quad \max -x - 2y + 3z.$
- x unrestricted $\quad \Rightarrow \quad x = y - z, \;\; y \ge 0, z \ge 0.$

LP Application: Weighted Bipartite Matching

Assignment problem. Given a complete bipartite network $K_{n,n}$ and edge weights c_{ij}, find a perfect matching of minimum weight.

$$\begin{array}{llcll}
\min & \displaystyle\sum_{1\le i\le n}\ \sum_{1\le j\le n} c_{ij}\, x_{ij} & & & \\
\text{s. t.} & \displaystyle\sum_{1\le j\le n} x_{ij} & = & 1 & 1\le i\le n \\
& \displaystyle\sum_{1\le i\le n} x_{ij} & = & 1 & 1\le j\le n \\
& x_{ij} & \ge & 0 & 1\le i, j\le n
\end{array}$$

Birkhoff-von Neumann theorem (1946, 1953). All extreme points of the above polyhedron are {0-1}-valued.

Corollary. Can solve assignment problem using LP techniques since LP algorithms return optimal solution that is an extreme point.

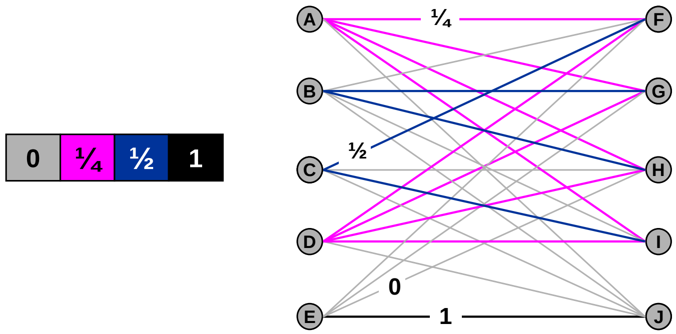
Remark. Polynomial combinatorial algorithms also exist.

LP Application: Weighted Bipartite Matching

Birkhoff-von Neumann theorem (1946, 1953). All extreme points of the above polytope are {0-1}-valued.

Proof (by contradiction). Suppose x is a fractional feasible solution.

- Consider $A = \{ (i, j) : 0 < x_{ij} \}$.

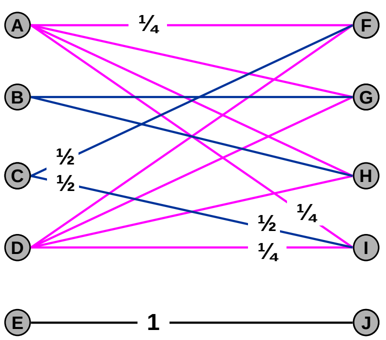


LP Application: Weighted Bipartite Matching

Birkhoff-von Neumann theorem (1946, 1953). All extreme points of the above polytope are {0-1}-valued.

Proof (by contradiction). Suppose x is a fractional feasible solution.

- Consider $A = \{ (i, j) : 0 < x_{ij} \}$.
- Claim: there exists a perfect matching in (V, A).
  - fractional flow gives fractional perfect matching

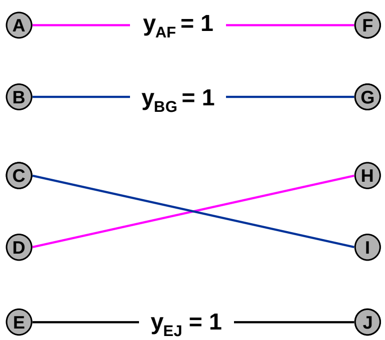


LP Application: Weighted Bipartite Matching

Birkhoff-von Neumann theorem (1946, 1953). All extreme points of the above polytope are {0-1}-valued.

Proof (by contradiction). Suppose x is a fractional feasible solution.

- Consider $A = \{ (i, j) : 0 < x_{ij} \}$.
- Claim: there exists a perfect matching in (V, A).
  - fractional flow gives fractional perfect matching
  - apply integrality theorem for max flow
- Define $\varepsilon = \min \{ x_{ij} : x_{ij} > 0\}$, $x^1 = (1 - \varepsilon)\, x + \varepsilon\, y$, $x^2 = (1+ \varepsilon)\, x - \varepsilon\, y$.
- $x = ½\, x^1 + ½\, x^2 \Rightarrow$ x not an extreme point.



LP Application: Multicommodity Flow

Multicommodity flow. Given a network G = (V, E) with edge capacities $u(e) \geq 0$, edge costs $c(e) \geq 0$, set of commodities K, and supply / demand $d^k(v)$ for commodity k at node v, find a minimum cost flow that satisfies all of the demand.

$$\begin{array}{llcll} \min & \sum_{k \in K} \sum_{e \in E} c^k(e)\, x^k(e) & & & \\ \text{s.t.} & \sum_{e \text{ in to } v} x^k(e) - \sum_{e \text{ out of } v} x^k(e) & = & d^k(v) & v \in V, k \in K \\ & \sum_{k \in K} \sum_{e \in E} x^k(e) & \leq & u(e) & e \in E \\ & x^k(e) & \geq & 0 & e \in E, k \in K \end{array}$$

Applications.

- Transportation networks.
- Communication networks (Akamai).
- Solving Ax =b with Gaussian elimination, preserving sparsity.
- VLSI design.

Weighted Vertex Cover

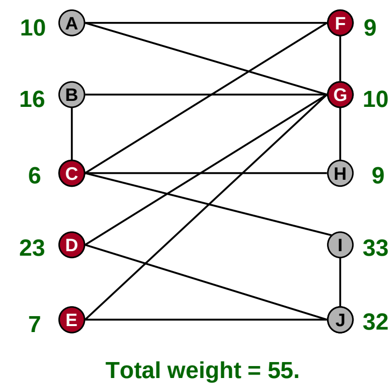
Weighted vertex cover. Given an undirected graph G = (V, E) with vertex weights $w_v \geq 0$, find a minimum weight subset of nodes S such that every edge is incident to at least one vertex in S.

- NP-hard even if all weights are 1.

Integer programming formulation.

$$\begin{array}{llcll} (ILP) \ \min & \sum_{v \in V} w_v\, x_v & & & \\ \text{s.t.} & x_v + x_w & \geq & 1 & (v,w) \in E \\ & x_v & \in & \{0,1\} & v \in V \end{array}$$

- If x* is optimal solution to (ILP), then $S = \{v \in V : x^*_v = 1\}$ is a min weight vertex cover.



Integer Programming

INTEGER-PROGRAMMING: Given rational numbers a_{ij}, b_i, c_j, find integers x_j that solve:

$$\begin{array}{llcll} \min & \sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & 1 \leq i \leq m \\ & x_j & \geq & 0 & 1 \leq j \leq n \\ & x_j & & \text{integral} & 1 \leq j \leq n \end{array}$$

Claim. INTEGER-PROGRAMMING is NP-hard.

Proof. VERTEX-COVER $\leq_P$ INTEGER-PROGRAMMING.

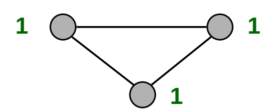
$$\begin{array}{llcll} \min & \sum_{v \in V} x_v & & & \\ \text{s.t.} & x_v + x_w & \geq & 1 & (v,w) \in E \\ & x_v & \geq & 0 & v \in V \\ & x_v & & \text{integral} & v \in V \end{array}$$

Weighted Vertex Cover

Linear programming relaxation.

$$\begin{array}{llcll} (LP) \ \min & \sum_{v \in V} w_v\, x_v & & & \\ \text{s.t.} & x_v + x_w & \geq & 1 & (v,w) \in E \\ & x_v & \geq & 0 & v \in V \end{array}$$

- Note: optimal value of (LP) is ≤ optimal value of (ILP), and may be strictly less.
  - clique on n nodes requires n-1 nodes in vertex cover
  - LP solution x* = ½ has value n / 2



- Provides lower bound for approximation algorithm.
- How can solving LP help us find good vertex cover?
  - Round fractional values.

Weighted Vertex Cover

Theorem. If x^* is optimal solution to (LP), then $S = \{v \in V : x^*_v \geq \frac{1}{2}\}$ is a vertex cover within a factor of 2 of the best possible.

- Provides 2-approximation algorithm.
- Solve LP, and then round.

S is a vertex cover.

- Consider an edge $(v,w) \in E$.
- Since $x^*_v + x^*_w \geq 1$, either x^*_v or $x^*_w \geq \frac{1}{2} \Rightarrow (v,w)$ covered.

S has small cost.

- Let S^* be optimal vertex cover.

$$\sum_{v \in S^*} w_v \geq \sum_{v \in V} w_v x^*_v \quad \Leftarrow \text{LP is relaxation}$$

$$\geq \sum_{v \in S} w_v x^*_v$$

$$\geq \frac{1}{2} \sum_{v \in S} w_v \quad \Leftarrow x^*_v \geq \frac{1}{2}$$

Weighted Vertex Cover

Good news.

- 2-approximation algorithm is basis for most practical heuristics.
  - can solve LP with min cut $\Rightarrow$ faster
  - primal-dual schema $\Rightarrow$ linear time 2-approximation
- PTAS for planar graphs.
- Solvable on bipartite graphs using network flow.

Bad news.

- NP-hard even on 3-regular planar graphs with unit weights.
- If $P \neq NP$, then no ρ-approximation for $\rho < 4/3$, even with unit weights. (Dinur-Safra, 2001)

Maximum Satisfiability

MAX-SAT: Given clauses $C_1, C_2, \ldots C_m$ in CNF over Boolean variables $x_1, x_2, \ldots x_n$, and integer weights $w_j \geq 0$ for each clause, find a truth assignment for the x_i that maximizes the total weight of clauses satisfied.

- NP-hard even if all weights are 1.
- Ex.

$$\begin{aligned}
C_1 &= x_2 \vee \overline{x_3} & w_1 &= 1 \\
C_2 &= x_2 \vee x_3 & w_2 &= 2 \\
C_3 &= \overline{x_1} \vee x_2 \vee \overline{x_3} & w_3 &= 3 \\
C_4 &= \overline{x_1} \vee \overline{x_2} & w_4 &= 4 \\
C_5 &= x_1 \vee \overline{x_2} & w_5 &= 5
\end{aligned}$$

$x_1 = 0$
$x_2 = 0$
$x_3 = 1$
weight = 14

Maximum Satisfiability: Johnson's Algorithm

Randomized polynomial time (RP). Polynomial time extended to allow calls to `random()` call in unit time.

- Polynomial algorithm A with one-sided error:
  - if x is YES instance: $\Pr[A(x) = \text{YES}] \geq \frac{1}{2}$
  - if x is NO instance: $\Pr[A(x) = \text{YES}] = 0$
- Fundamental open question: does P = RP?

Johnson's Algorithm: Flip a coin, and set each variable true with probability ½, independently for each variable.

Theorem: The "dumb" algorithm is a 2-approximation for MAX-SAT.

Maximum Satisfiability: Johnson's Algorithm

Proof: Consider random variable $Y_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

- Let $W = \sum_{j=1}^{m} w_j Y_j$.
- Let OPT = weight of the optimal assignment.
- Let ℓ_j be the number of distinct literals in clause C_j.

$$\begin{aligned} E[W] &= \sum_{j=1}^{m} w_j E[Y_j] && \leftarrow \text{linearity of expectation} \\ &= \sum_{j=1}^{m} w_j \Pr[\text{clause } C_j \text{ is satisfied}] \\ &= \sum_{j=1}^{m} w_j (1-(\tfrac{1}{2})^{\ell_j}) \\ &\geq \tfrac{1}{2} \sum_{j=1}^{m} w_j \\ &\geq \tfrac{1}{2} OPT. && \leftarrow \text{weights are} \geq 0 \end{aligned}$$

Maximum Satisfiability: Johnson's Algorithm

Corollary. If every clause has at least k literals, Johnson's algorithm is a $1 / (1 - (½)^k)$ approximation algorithm.

- 8/7 approximation algorithm for MAX E3SAT.

Theorem (Håstad, 1997). If MAX ESAT has an ρ-approximation for $\rho < 8/7$, then P = NP.

- Johnson's algorithm is best possible in some cases.

Maximum Satisfiability: Randomized Rounding

Idea 1. Used biased coin flips, not 50-50.

Idea 2. Solve linear program to determine coin biases.

$$z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases} \qquad y_i = \begin{cases} 1 & x_i \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

- P_j = indices of variables that occur un-negated in clause C_j.
  N_j = indices of variables that occur negated in clause C_j.

$$\begin{aligned} \text{(LP)} \quad \max \quad & \sum_j w_j z_j \\ \text{s. t.} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \\ & 0 \leq z_j \leq 1 \end{aligned}$$

Theorem (Goemans-Williamson, 1994). The algorithm is an $e / (e-1) \approx 1.582$-approximation algorithm.

Maximum Satisfiability: Randomized Rounding

Fact 1. For any nonnegative $a_1, \ldots a_k$, the geometric mean is $\leq$ the arithmetic mean.

$$\sqrt[k]{a_1 \, a_2 \cdots a_k} \leq \tfrac{1}{k}(a_1 + a_2 + \cdots + a_k)$$

Theorem (Goemans-Williamson, 1994). The algorithm is an $e / (e-1) \approx 1.582$-approximation algorithm for MAX-SAT.

Proof. Consider an arbitrary clause C_j.

$$\begin{aligned} \Pr[\text{clause } C_j \text{ is not satisfied}] &= \prod_{i \in P_j} (1 - y_i^*) \prod_{i \in N_j} y_i^* \\ &\leq \left[\tfrac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{\ell_j} && \leftarrow \text{geometric-arithmetic mean} \\ &= \left[1 - \tfrac{1}{\ell_j} \left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right) \right]^{\ell_j} \\ &\leq \left(1 - \frac{z_j^*}{\ell_j} \right)^{\ell_j} && \leftarrow \text{LP constraint} \end{aligned}$$

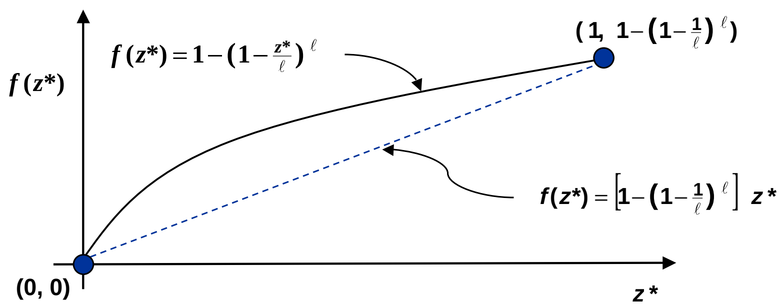
Maximum Satisfiability: Randomized Rounding

Proof (continued).

$$\Pr[\text{clause } C_j \text{ is satisfied}] \;\geq\; 1 - \left(1 - \tfrac{z_j^*}{\ell_j}\right)^{\ell_j}$$

$1 - (1 - z^* / \ell)^{\ell}$ is concave ⇒
$$\geq\; \left[1 - \left(1 - \tfrac{1}{\ell_j}\right)^{\ell_j}\right] z_j^*$$

$(1-1/x)^x$ converges to e^{-1} ⇒
$$\geq\; \left(1 - \tfrac{1}{e}\right) z_j^*$$



Maximum Satisfiability: Randomized Rounding

Proof (continued).

- From previous slide: $\Pr[\text{clause } C_j \text{ is satisfied}] \;\geq\; \left(1 - \tfrac{1}{e}\right) z_j^*$
- Let W = weight of clauses that are satisfied.

$$\begin{aligned} E[W] &= \sum_{j=1}^{m} w_j \Pr[\text{clause } C_j \text{ is satisfied}] \\ &\geq \left(1 - \tfrac{1}{e}\right) \sum_{j=1}^{m} w_j\, z_j^* \\ &= \left(1 - \tfrac{1}{e}\right) OPT_{LP} \\ &\geq \left(1 - \tfrac{1}{e}\right) OPT \end{aligned}$$

Corollary. If all clauses have length at most k

$$E[W] \;\geq\; \left[1 - \left(1 - \tfrac{1}{k}\right)^k\right] OPT.$$

Maximum Satisfiability: Best of Two

Observation. Two approximation algorithms are complementary.

- Johnson's algorithm works best when clauses are long.
- LP rounding algorithm works best when clauses are short.

How can we exploit this?

- Run both algorithms and output better of two.
- Re-analyze to get 4/3-approximation algorithm.
- Better performance than either algorithm individually!

Best-of-Two $(C_1, C_2, \ldots, C_m)$

```
(x¹, W¹) ← Johnson(C₁,…,Cₘ)
(x², W²) ← LPround(C₁,…,Cₘ)

IF (W¹ > W²)
   RETURN x¹
ELSE
   RETURN x²
```


Maximum Satisfiability: Best of Two

Theorem (Goemans-Williamson, 1994). The Best-of-Two algorithm is a 4/3-approximation algorithm for MAX-SAT.

Proof.

$$\begin{aligned} E\left[\max(W^1, W^2)\right] &\geq E\left[\tfrac{1}{2}W^1 + \tfrac{1}{2}W^2\right] \\ &= \tfrac{1}{2}\sum_j w_j \Pr[\text{clause } C_j \text{ is satisfied by Alg 1}] + \\ &\quad\; \tfrac{1}{2}\sum_j w_j \Pr[\text{clause } C_j \text{ is satisfied by Alg 2}] \\ &\geq \tfrac{1}{2}\sum_j w_j \left[\left(1 - \left(\tfrac{1}{2}\right)^{\ell_j}\right) + \left(1 - \left(1 - \tfrac{1}{\ell_j}\right)^{\ell_j}\right) z_j^*\right] \\ &\geq \tfrac{1}{2}\sum_j w_j \left(\tfrac{3}{2} z_j^*\right) \quad \text{(next slide)} \\ &\geq \tfrac{3}{4} OPT_{LP} \\ &\geq \tfrac{3}{4} OPT. \end{aligned}$$

next slide ⇒

Maximum Satisfiability: Best of Two

Lemma. For any integer $\ell \ge 1$, $\left(1-\left(\tfrac{1}{2}\right)^{\ell}\right) + \left\lfloor 1-\left(1-\tfrac{1}{\ell}\right)^{\ell} \right\rfloor z_j^* \;\ge\; \tfrac{3}{2} z_j^*$.

Proof.

- Case 1 ($\ell = 1$): $\tfrac{1}{2} + 1\, z_j^* \;\ge\; \tfrac{3}{2} z_j^*$.
- Case 2 ($\ell = 2$): $\tfrac{3}{4} + \tfrac{3}{4} z_j^* \;\ge\; \tfrac{3}{2} z_j^*$.
- Case 3 ($\ell \ge 3$):

$$\begin{aligned}\left(1-\left(\tfrac{1}{2}\right)^{\ell}\right) + \left\lfloor 1-\left(1-\tfrac{1}{\ell}\right)^{\ell} \right\rfloor z_j^* &\ge \left(1-\left(\tfrac{1}{2}\right)^{3}\right) + \left(1-\tfrac{1}{e}\right) z_j^* \\ &\ge \tfrac{7}{8} + \tfrac{5}{8} z_j^* \\ &= \tfrac{3}{2} z_j^*.\end{aligned}$$

Maximum Satisfiability: State of the Art

Observation. Can't get better than 4/3-approximation using our LP.

- If all weights = 1, $OPT_{LP} = 4$ but OPT = 3.

$$\begin{aligned} C_1 &= x_1 \vee x_2 \\ C_2 &= x_1 \vee \overline{x_2} \\ C_3 &= \overline{x_1} \vee x_2 \\ C_4 &= \overline{x_1} \vee \overline{x_2} \end{aligned}$$

Lower bound.

- Unless P = NP, can't do better than $8 / 7 \approx 1.142$.

Semi-definite programming.

- 1.275-approximation algorithm.
- 1.2-approximation algorithm if certain conjecture is true.

$$\text{(SDP)} \quad \begin{aligned} \max \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \\ & X \succeq 0 \end{aligned}$$

$C, X, B, A_i \in SR(n \times n)$

$$X \bullet Y = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}\, y_{ij}$$

$\succeq$ positive semi-definite

Open research problem.

- 4 / 3 - approximation algorithm without solving LP or SDP.

Appendix: Proof of LP Duality Theorem

LP Duality Theorem. For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty then max = min.

$$\text{(P)} \quad \begin{aligned} \max \; & c^T x \\ \text{s.t.} \; & Ax \le b \\ & x \ge 0 \end{aligned}$$

$$\text{(D)} \quad \begin{aligned} \min \; & y^T b \\ \text{s.t.} \; & y^T A \ge c \\ & y \ge 0 \end{aligned}$$

LP Weak Duality

LP Weak Duality. For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty, then max $\le$ min.

$$\text{(P)} \quad \begin{aligned} \max \; & c^T x \\ \text{s.t.} \; & Ax \le b \\ & x \ge 0 \end{aligned}$$

$$\text{(D)} \quad \begin{aligned} \min \; & y^T b \\ \text{s.t.} \; & y^T A \ge c \\ & y \ge 0 \end{aligned}$$

Proof (easy).

- Suppose $x \in \Re^m$ is feasible for (P) and $y \in \Re^n$ is feasible for (D).
  - $x \ge 0$, $y^T A \ge c$ $\Rightarrow$ $y^T A x \ge c^T x$.
  - $y \ge 0$, $Ax \le b$ $\Rightarrow$ $y^T A x \le y^T b$
  - combining two inequalities: $c^T x \le y^T A x \le y^T b$

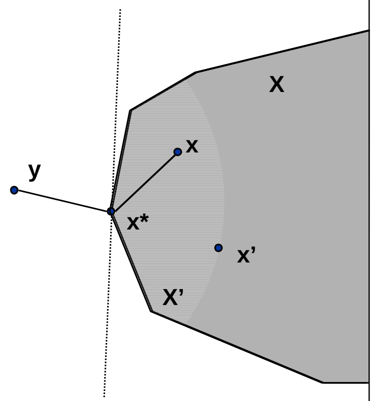
Closest Point

Weierstrass' Theorem. Let X be a compact set, and let f(x) be a continuous function on X. Then min {f(x) : $x \in X$} exists.

Lemma 1. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y. Moreover, for all $x \in X$ we have $(y - x^*)^T (x - x^*) \leq 0$.

Proof. (existence)

- Define f(x) = ||y - x||.
- Want to apply Weierstrass:
  - f is continuous
  - X closed, but maybe not bounded
- $X \neq \phi \Rightarrow$ there exists $x' \in X$.
- $X' = \{x \in X : ||y - x|| \leq ||y - x'||\}$ is closed and bounded.
- min {f(x) : $x \in X$} = min {f(x) : $x \in X'$}



Closest Point

Lemma 1. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y. Moreover, for all $x \in X$ we have $(y - x^*)^T (x - x^*) \leq 0$.

Proof. (moreover)

- x* min distance $\Rightarrow ||y - x^*||^2 \leq ||y - x||^2$ for all $x \in X$.
- By convexity: if $x \in X$, then $x^* + \varepsilon (x - x^*) \in X$ for all $0 \leq \varepsilon \leq 1$.
- $||y - x^*||^2 \leq ||y - x^* - \varepsilon(x - x^*)||^2$
  $= ||y - x^*||^2 + \varepsilon^2 ||(x - x^*)||^2 - 2\varepsilon (y - x^*)^T(x - x^*)$
- Thus, $(y - x^*)^T(x - x^*) \leq \frac{1}{2} \varepsilon ||(x - x^*)||^2$.
- Letting $\varepsilon \to 0^+$, we obtain the desired result.

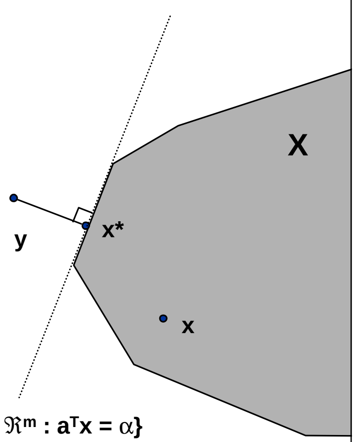
Separating Hyperplane Theorem

Separating Hyperplane Theorem. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists a hyperplane $H = \{x \in \Re^m : a^T x = \alpha\}$ where $a \in \Re^m$, $\alpha \in \Re$ that separates y from X.

- $a^T x \leq \alpha$ for all $x \in X$.
- $a^T y > \alpha$.

Proof.

- Let x* be closest point in X to y.
  - L1 $\Rightarrow (y - x^*)^T (x - x^*) \leq 0$ for all $x \in X$
- Choose $a = y - x^* \neq 0$ and $\alpha = a^T x^*$.
  - $a^T y = a^T (a + x^*) = ||a||^2 + \alpha > \alpha$
  - if $x \in X$, then $a^T(x - x^*) \leq 0$
    $\Rightarrow a^T x \leq a^T x^* = \alpha$



Fundamental Theorem of Linear Inequalities

Farkas' Theorem (Farkas 1894, Minkowski 1896). For $A \in \Re^{m \times n}$, $b \in \Re^m$ exactly one of the following two systems holds.

(I) $\exists\, x \in \Re^n$ s.t. $Ax = b$, $x \geq 0$

(II) $\exists\, y \in \Re^m$ s.t. $y^T A \leq 0$, $y^T b > 0$

Proof (not both). Suppose x satisfies (I) and y satisfies (II).

- Then $0 < y^T b = y^T A x \leq 0$, a contradiction.

Proof (at least one). Suppose (I) infeasible. We will show (II) feasible.

- Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.
  - there exists hyperplane $y \in \Re^m$, $\alpha \in \Re$ separating b from S: $y^T b > \alpha$, $y^T s \leq \alpha$ for all $s \in S$.
- $0 \in S \Rightarrow \alpha \geq 0 \Rightarrow y^T b > 0$
- $y^T A x \leq \alpha$ for all $x \geq 0 \Rightarrow y^T A \leq 0$ since x can be arbitrarily large.

Another Theorem of the Alternative

Corollary. For $A \in \Re^{m\times n}$, $b \in \Re^m$ exactly one of the following two systems holds.

$$\text{(I)}\quad \exists\, x \in \Re^n \quad \text{s.t.}\quad Ax \le b,\quad x \ge 0$$

$$\text{(II)}\quad \exists\, y \in \Re^m \quad \text{s.t.}\quad y^T A \ge 0,\quad y^T b < 0,\quad y \ge 0$$

Proof.

- Define $A' \in \Re^{m\times(m+n)} = [A \mid I]$, $x' = [x \mid s]$, where $s \in \Re^m$.
- Farkas' Theorem to A', b': exactly one of (I') and (II') is feasible.

$$\text{(I')}\quad \exists\, x \in \Re^n, s \in \Re^m \quad \text{s.t.}\quad Ax + Is = b,\quad x, s \ge 0$$

$$\text{(II')}\quad \exists\, y \in \Re^m \quad \text{s.t.}\quad y^T A \le 0,\quad Iy \le 0,\quad y^T b > 0$$

- (I') equivalent to (I), (II') equivalent to (II).

LP Strong Duality

LP Duality Theorem. For $A \in \Re^{m\times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty then max = min.

$$\text{(P)}\quad \max\ c^T x \quad \text{s.t.}\quad Ax \le b,\quad x \ge 0$$

$$\text{(D)}\quad \min\ y^T b \quad \text{s.t.}\quad y^T A \ge c,\quad y \ge 0$$

Proof (max ≤ min). Weak LP duality.

Proof (min ≤ max). Suppose max < α. We show min < α.

$$\text{(I)}\quad \exists\, x \in \Re^n \quad \text{s.t.}\quad Ax \le b,\quad -c^T x \le -\alpha,\quad x \ge 0$$

$$\text{(II)}\quad \exists\, y \in \Re^m, z \in \Re \quad \text{s.t.}\quad y^T A - cz \ge 0,\quad y^T b - \alpha z < 0,\quad y, z \ge 0$$

- By definition of α, (I) infeasible ⇒ (II) feasible by Farkas Corollary.

LP Strong Duality

$$\text{(II)}\quad \exists\, y \in \Re^m, z \in \Re \quad \text{s.t.}\quad y^T A - cz \ge 0,\quad y^T b - \alpha z < 0,\quad y, z \ge 0$$

Let ŷ, z be a solution to (II).

Case 1: z = 0.

- Then, $\{ y \in \Re^m : y^T A \ge 0,\ y^T b < 0,\ y \ge 0 \}$ is feasible.
- Farkas Corollary ⇒ $\{ x \in \Re^n : Ax \le b,\ x \ge 0 \}$ is infeasible.
- Contradiction since by assumption (P) is nonempty.

Case 2: z > 0.

- Scale ŷ, z so that ŷ satisfies (II) and z = 1.
- Resulting ŷ feasible to (D) and $\hat{y}^T b < \alpha$.