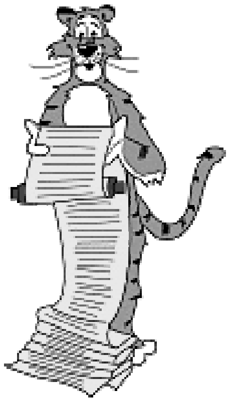
Competitive Analysis



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Beyond Worst Case Analysis

Worst-case analysis.

- Analyze running time as function of worst input of a given size.

Average case analysis.

- Analyze average running time over some distribution of inputs.
- Ex: quicksort.

Amortized analysis.

- Worst-case bound on sequence of operations.
- Ex: splay trees, union-find.

Competitive analysis.

- Make quantitative statements about online algorithms.
- Ex: paging, load balancing.

Online Algorithm and Competitive Analysis

Paging problem: Given two-level store consisting of fast memory (cache) that can hold k pages, and slow memory that can store infinitely many pages.

- Sequence of page requests p:
  - if page p already in cache, no cost incurred
  - otherwise, eject some other page q from cache and replace with p, and pay unit cost for page fault.
- If p not in cache, which page q should you evict?
- Most fundamental and practically important online problem in CS.

Online Algorithm and Competitive Analysis

Competitive analysis. (Sleator-Tarjan)

- Algorithm A is ρ-competitive if there exists some constant b such that for every sequence of inputs σ :

$$\text{cost}_A(\sigma) \;\leq\; \rho\, \text{cost}_{OPT}(\sigma) + b.$$

  where OPT is optimal offline algorithm.

- OPT = MIN: evict page whose next access is furthest away.
- A = LRU: evict page whose most recent access was earliest
  - Traditional analysis completely uninformative.
  - We show LRU is k-competitive.
- A = LIFO: evict page brought in most recently.
  - LIFO can have arbitrarily bad competitive ratio.
- Fact: no online paging algorithm is better than k-competitive.

Online Algorithm and Competitive Analysis

Theorem. LRU is k-competitive.

Proof: Let τ be a subsequence of σ on which LRU faults exactly k times, and τ does not contain fist access in σ. Let p denote page requested just before τ.

- Case 1: LRU faults in sequence τ on p.
  - τ requests at least k+1 different pages $\Rightarrow$ MIN faults at least once
- Case 2: LRU faults on some page, say q, at least twice in τ.
  - τ requests at least k+1 different pages $\Rightarrow$ MIN faults at least once

Online Algorithm and Competitive Analysis

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- Case 3: LRU does not fault on p, nor on any page more than once.
  - k different pages are accessed and faulted on, none of which is p
  - p is in MIN's cache at start of τ $\Rightarrow$ MIN faults at least once