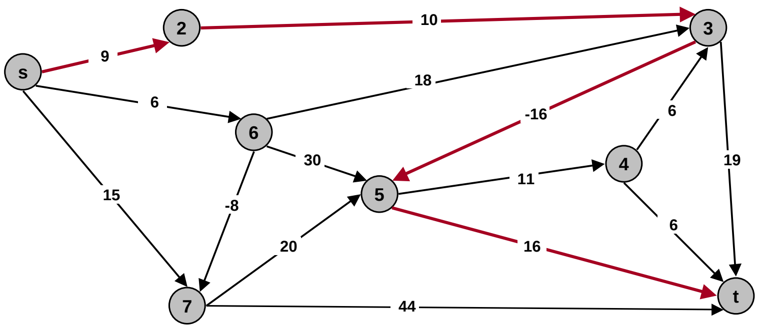
Shortest Path With Negative Weights



Contents

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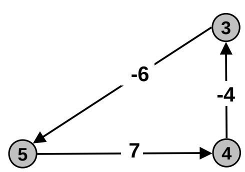
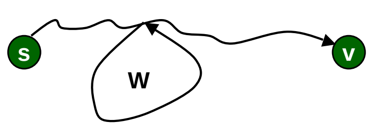
- Directed shortest path with negative weights.
- Negative cycle detection.
  - application: currency exchange arbitrage
- Tramp steamer problem.
  - application: optimal pipelining of VLSI chips

Shortest Paths with Negative Weights

Negative cost cycle.



If some path from s to v contains a negative cost cycle, there does not exist a shortest s-v path; otherwise, there exists one that is simple.



c(W) < 0

Shortest Paths with Negative Weights

OPT(i, v) = length of shortest s-v path using at most i arcs.

- Let P be such a path.
- Case 1: P uses at most i-1 arcs.
- Case 2: P uses exactly i arcs.
  - if (u, v) is last arc, then OPT selects best s-u path using at most i-1 arcs, and then uses (u, v)

$$OPT(i,v) = \begin{cases} 0 & \text{if } i = 0 \\ \min\left\{ OPT(i-1, v),\ \min\limits_{(u,v)\in E} \left\{ OPT(i-1, u) + c(u,v) \right\} \right\} & \text{otherwise} \end{cases}$$

Goal: compute OPT(n-1, t) and find a corresponding s-t path.

Shortest Paths with Negative Weights: Algorithm

Dynamic Programming Shortest Path

```
INPUT: G = (V, E), s, t
n = |V|

ARRAY: OPT[0..n, V]
FOREACH v ∈ V
   OPT[0, v] = ∞

OPT[0,s] = 0
FOR i = 1 to n
   FOREACH v ∈ V
      m  = OPT[i-1, v]
      m' = ∞
      FOREACH (u, v) ∈ E
         m' = min (m', OPT[i-1, u] + c[u,v])
      OPT[i, v] = min(m, m')

RETURN OPT[n-1, t]
```

$$\min_{(u,v)\in E} \{ OPT(i-1, u) + c(u,v) \}$$

Shortest Paths: Running Time

Dynamic programming algorithm requires $\Theta(mn)$ time and space.

- Outer loop repeats n times.
- Inner loop for vertex v considers indegree(v) arcs.

$$\sum_{v \in V} \text{indegree}(v) = m$$

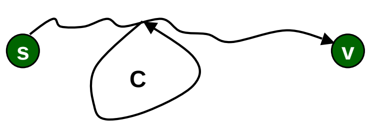
Finding the shortest paths.

- Could maintain predecessor variables.
- Alternative: compute optimal distances, consider only zero reduced cost arcs.

Shortest Paths: Detecting Negative Cycles

L1: if OPT(n,v) < OPT(n-1,v) for some node v, then (any) shortest path from s to v using at most n arcs contains a cycle; moreover any such cycle has negative cost.

- Proof (by contradiction).
- Since OPT(n,v) < OPT(n-1,v), P has n arcs.
- Let C be any directed cycle in P.
- Deleting C gives us a path from s to v of fewer than n arcs $\Rightarrow$ C has negative cost.



c(C) < 0

Shortest Paths: Detecting Negative Cycles

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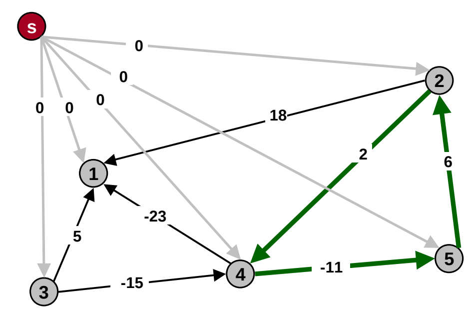
Corollary: can detect negative cost cycle in O(mn) time.

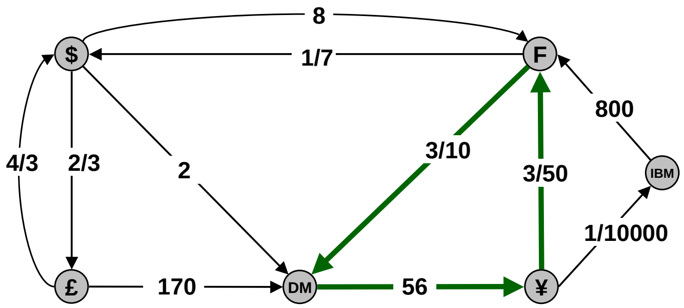
- Need to trace back through sub-problems.



Detecting Negative Cycles: Application

Currency conversion.

- Given n currencies (financial instruments) and exchange rates between pairs of currencies, is there an arbitrage opportunity?
- Fastest algorithm very valuable!



Shortest Paths: Practical Improvements

Practical improvements.

- If OPT(i, v) = OPT(i-1, v) for all nodes v, then OPT(i, v) are the shortest path distances.
  - ✎ Consequence: can stop algorithm as soon as this happens.
- Maintain only one array OPT(v).
  - ✎ Use O(m+n) space; otherwise Θ(mn) best case.
- No need to check arcs of the form (u, v) unless OPT(u) changed in previous iteration.
  - ✎ Avoid unnecessary work.

Overall effect.

- Still O(mn) worst case, but O(m) behavior in practice.

Shortest Paths: Practical Improvements

Negative cycle tweak: stop if any node enqueued n times.

Bellman-Ford FIFO Shortest Path

```
INPUT: G = (V, E), s, t
n = |V|

ARRAY: OPT[V], pred[V]
FOREACH v ∈ V
   OPT[v]  = ∞, pred[v] = ϕ

OPT[s] = 0, Q = QUEUEinit(s)
WHILE (Q ≠ ϕ)
   u = QUEUEget()
   FOREACH (u, v) ∈ E
      IF (OPT[u] + c[u,v] < OPT[v])
         OPT[v] = OPT[u] + c[u,v]
         pred[v] = u
         IF (v ∉ Q)
            QUEUEput(v)

RETURN OPT[n-1]
```


Shortest Paths: State of the Art

All times below are for single source shortest path in directed graphs with no negative cycle.

O(mn) time, O(m + n) space.

- Shortest path: straightforward.
- Negative cycle: Bellman-Ford predecessor variables contain shortest path or negative cycle (not proved here).

$O(mn^{1/2} \log C)$ time if all arc costs are integers between –C and C.

- Reduce to weighted bipartite matching (assignment problem).
- "Cost-scaling."
- Gabow-Tarjan (1989), Orlin-Ahuja (1992).

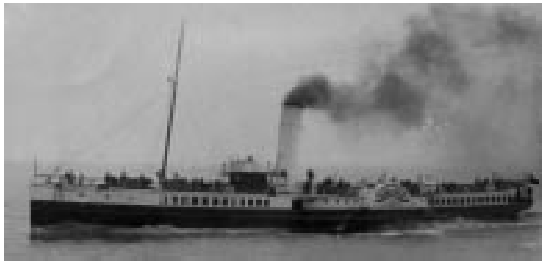
$O(mn + n^2 \log n)$ undirected shortest path, no negative cycles.

- Reduce to weighted non-bipartite matching.
- Beyond the scope of this course.

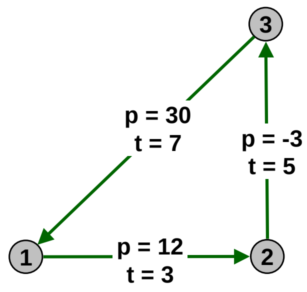
Tramp-Steamer Problem

Tramp-steamer (min cost to time ratio) problem.

- A tramp steamer travels from port to port carrying cargo. A voyage from port v to port w earn p(v,w) dollars, and requires t(v,w) days.
- Captain wants a tour that achieves largest mean daily profit.



Westward Ho (1894 – 1946)



mean daily profit = $\dfrac{30+12-3}{7+3+5} = \dfrac{39}{15}$

Tramp-Steamer Problem

Tramp-steamer (min cost to time ratio) problem.

- Input: digraph G = (V, E), arc costs c, and arc traversal times t > 0.
- Goal: find a directed cycle W that minimizes ratio $\mu(W) = \dfrac{\sum_{e \in W} c_e}{\sum_{e \in W} t_e}$.

Novel application.

- Minimize cycle time (maximize frequency) of logic chip on IBM processor chips by adjusting clocking schedule.

Special case.

- Find a negative cost cycle.

Tramp-Steamer Problem

Linearize objective function.

- Let μ^* be value of minimum ratio cycle.
- Let μ be a constant.
- Define $\ell_e = c_e - \mu\, t_e$.

Case 1: there exists negative cost cycle W using lengths ℓ_e .

$$\sum_{e \in W} (c_e - \mu\, t_e) < 0 \quad \Leftrightarrow \quad \mu > \frac{\sum_{e \in W} c_e}{\sum_{e \in W} t_e} \geq \mu^*.$$

Case 2: every directed cycle has positive cost using lengths ℓ_e.

$$\sum_{e \in W} (c_e - \mu\, t_e) > 0 \quad \text{for every cycle } W \quad \Leftrightarrow$$

$$\mu < \frac{\sum_{e \in W} c_e}{\sum_{e \in W} t_e} \quad \text{for every cycle } W \quad \Leftrightarrow \quad \mu < \mu^*.$$

Tramp-Steamer Problem

Linearize objective function.

- Let μ^* be value of minimum ratio cycle.
- Let μ be a constant.
- Define $\ell_e = c_e - \mu\, t_e$.

Case 3: every directed cycle has nonnegative cost using lengths ℓ_e , and there exists a zero cost cycle W*.

$$\sum_{e \in W} (c_e - \mu\, t_e) \geq 0 \quad \text{for every cycle W} \quad \Leftrightarrow$$

$$\mu \leq \frac{\sum_{e \in W} c_e}{\sum_{e \in W} t_e} \quad \text{for every cycle } W \quad \Leftrightarrow \quad \mu \leq \mu^*.$$

$$\frac{\sum_{e \in W^*} c_e}{\sum_{e \in W^*} t_e} = \mu \quad \Rightarrow \quad \mu = \mu^*.$$

Tramp-Steamer Problem

Linearize objective function.

- Let μ^* be value of minimum ratio cycle.
- Let μ be a constant.
- Define $\ell_e = c_e - \mu\, t_e$.

Case 1: there exists negative cost cycle W using lengths ℓ_e .

- $\mu^* < \mu$

Case 2: every directed cycle has positive cost using lengths ℓ_e.

- $\mu^* > \mu$

Case 3: every directed cycle has nonnegative cost using lengths ℓ_e , and there exists a zero cost cycle W*.

- $\mu^* = \mu$

Tramp-Steamer: Sequential Search Procedure

Sequential Tramp Steamer

```
Let μ be a known upper bound on μ*.

REPEAT (forever)
   ℓe ← ce – μ
   Solve shortest path problem with lengths ℓe
   IF (negative cost cycle W w.r.t. ℓe)
      μ ← μ(W)
   ELSE
      Find a zero cost cycle W* w.r.t. ℓe.
      RETURN W*.
```

Theorem: sequential algorithm terminates.

- Case 1 $\Rightarrow$ μ strictly decreases from one iteration to the next.
- μ is the ratio of some cycle, and only finitely many cycles.

Tramp-Steamer: Binary Search Procedure

Binary Search Tramp Steamer

```
W ← cycle
left  ← -C, right ← C          ⇐ left ≤ μ* ≤ right

REPEAT (forever)
   IF (μ(W) = μ*)
      RETURN W
   μ ← (left + right) / 2
   ℓe ← ce – μ
   Solve shortest path problem with lengths ℓe
   IF (negative cost cycle w.r.t. ℓe)
      right ← μ
      W ← negative cost cycle w.r.t. ℓe
   ELSE IF (zero cost cycle W*)
      RETURN W*.
   ELSE
      left ← μ
```


Tramp-Steamer: Binary Search Procedure

Invariant: interval `[left, right]` and cycle W satisfy:
`left ≤ μ* ≤ μ(W) < right`.

- Proof by induction follows from cases 1-2.

Lemma. Upon termination, the algorithm returns a min ratio cycle.

- Immediate from case 3.

Assumption.

- All arc costs are integers between –C and C.
- All arc traversal times are integers between –T and T.

Lemma. The algorithm terminates after O(log(nCT)) iterations.

- Proof on next slide.

Theorem. The algorithm finds min ratio cycle in O(mn log (nCT)) time.

Tramp-Steamer: Binary Search Procedure

Lemma. The algorithm terminates after O(log(nCT)) iterations.

- Initially, `left = -C, right = C`.
- Each iteration halves the size of the interval.
- Let c(W) and t(W) denote cost and traversal time of cycle W.
- We show any interval of size less than $1 / (n^2T^2)$ contains at most one value from the set { c(W) / t(W) : W is a cycle }.
  - let W_1 and W_2 cycles with $\mu(W_1) > \mu(W_2)$

$$\frac{c(W_1)}{t(W_1)} - \frac{c(W_2)}{t(W_2)} > 0 \iff \frac{c(W_1)\,t(W_2) - c(W_2)\,t(W_1)}{t(W_1)\,t(W_2)} > 0.$$

  - numerator of RHS is at least 1, denominator is at most n^2T^2
- After $1 + \log_2 ((2C)(n^2T^2)) = O(\log(nCT))$ iterations, at most one ratio in the interval.
- Algorithm maintains cycle W and interval `[left, right]` s.t. `left` $\le \mu^* \le \mu$`(W) < right`.

Tramp Steamer: State of the Art

Min ratio cycle.

- O(mn log (nCT)).
- $O(n^3 \log^2 n)$ dense. (Megiddo, 1979)
- $O(n^3 \log n)$ sparse. (Megiddo, 1983)

Minimum mean cycle.

- Special case when all traversal times = 1.
- $\Theta(mn)$. (Karp, 1978)
- $O(mn^{1/2} \log C)$. (Orlin-Ahuja, 1992)
- O(mn log n). (Karp-Orlin, 1981)
  - parametric simplex - best in practice

Optimal Pipelining of VLSI Chip

Novel application.

- Minimize cycle time (maximize frequency) of logic chip on IBM processor chips by adjusting clocking schedule.

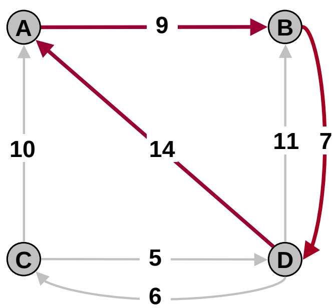
If clock signal arrive at latches simultaneously, min cycle time = 14.

Allow individual clock arrival times at latches.

Clock signal at latch:

- A: 0, 10, 20, 30, . . .
- B: -1, 9, 19, 29, . . .
- C: 0, 10, 20, 30, . . .
- D: -4, 6, 16, 26, . . .

Optimal cycle time = 10.
Max mean weight cycle = 10.



Latch Graph