COS 445 - PSet 4

Due online Monday, April 12th at 11:59 pm.

Instructions:

- Some problems will be marked as no collaboration problems. This is to make sure you have experience solving a problem start-to-finish by yourself in preparation for the midterms/final. You cannot collaborate with other students or the Internet for these problems (you may still use the referenced sources and lecture notes). You may ask the course staff clarifying questions, but we will generally not give hints.
- Submit your solution to each problem as a **separate PDF** to codePost. Please make sure you're uploading the correct PDFs![1] If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well. Please anonymize your submission, although there are no repercussions if you forget.
- This cheatsheet gives problem solving tips, and guidelines for a "good proof" or "partial progress": `http://www.cs.princeton.edu/~smattw/Teaching/cheatsheet445.pdf`.
- Please reference the course collaboration policy here: `http://www.cs.princeton.edu/~smattw/Teaching/infosheet445sp21.pdf`.

[1] We will assign a minor deduction if we need to maneuver around the wrong PDFs. Please also note that depending on if/how you use Overleaf, you may need to recompile your solutions in between downloads to get the right files.

Problem 1: Linear Programming (20 points, no collaboration)

Alice is trying to get enough oranges and bananas to host a fruit party. To successfully host a party she needs **at least 4 oranges** and **at least 3 bananas**. Unfortunately, her local grocery story only sells fruit in bundles. Bundle A costs 7 dollars and contains 2 oranges and 5 bananas. Bundle B costs 4 dollars and contains 3 oranges and 2 bananas. Fortunately, the grocery story will allow Alice to buy fractions of bundles (i.e. she can buy 2.5 bundle As). They will not allow Alice to buy negative bundles (i.e. she cannot buy -1 bundle As and 3 bundle Bs).

Alice would like to buy x_A bundle As and x_B bundle Bs to guarantee she has at least 4 oranges and at least 3 bananas. Moreover, she would like to find the solution that minimizes her dollars spent.

Part a (10 points)

Write a linear program whose solution is the optimal choice of x_A, x_B for Alice's problem.

Part b (10 points)

Take the dual of the linear program from part a.

Problem 2: Noisy Optimizers aren't Good Enough (40 points)

For this problem, you should assume that all bidders' values for the item are non-negative. This problem will try to address the "robustness" of the second-price auction (or more generally, ideas used for VCG) to underlying optimization algorithms which are imperfect. Consider the following error-prone algorithm A for computing the argmaximum of a set $\{b_1, ..., b_n\}$ of numbers:

- If the second-highest number is exactly one less than the largest number, then output the index of the second-highest number (break ties lexicographically).[2]
- Otherwise, output the index of the largest number (break ties lexicographically).

Consider the following error-prone version of the second-price auction with n buyers and a single item:

- Accept bids $b_1, \ldots, b_n$, all of which are ≥ 0.
- Award the item to the bidder $A(b_1, ..., b_n)$. Note that if A were not error-prone, this would be the highest bidder.
- Charge the winning bidder $b_{A(\vec{b}_{-i};-2)}$.[3] To clarify, $A(\vec{b}_{-i}; -2)$ means "replace b_i with -2 and keep all other bids the same. Then run A." Put another way,: find the bidder j which A selects as the winner on input $(\vec{b}_{-i}; -2)$, and charge bidder i b_j. Note that if A were not error prone, j would be the highest bidder among those $\neq i$.

Part a (10 points)

Prove that for any given $\vec{b}_{-i}$ (list of bids submitted by all bidders except for i), there exists a price p such that no matter what bid bidder i makes, bidder i will either win the item and pay p, or not get the item (and pay nothing).

Part b (10 points)

Say that $v_i > v_j$ for all $j \neq i$ (i is the highest bidder). Prove that if all other bidders tell the truth (that is, bid v_j), bidder i's best response is a bid which wins the item (you do not need to specify exactly what that bid is).

Part c (10 points)

Prove that the error-prone second-price auction is not incentive compatible by providing (and analyzing) a vector of values $v_1, \ldots, v_n$ such that if everyone tells the truth, the second-highest bidder wins and pays strictly more than their value (pick an n and provide a single example. It is OK to use non-integer values, if desired).

[2] To be extra clear: if the third-highest number is exactly one less than the largest number, but the second-highest number is not, then the index of the highest number is output. If there are two numbers with the same highest value, then the second-highest number is equal to the highest number.

[3] The choice of -2 is made just to guarantee that $A(\vec{b}_{-i}; -2) \neq i$ when all $b_j \geq 0$.

Part d (10 points)

Prove that the error-prone second-price auction is not incentive compatible by providing (and analyzing) a vector of values $v_1, \ldots, v_n$ such that if everyone tells the truth, the highest bidder wins, but the second-highest bidder would have been strictly happier by lying about their value (pick an n and provide a single example. It is OK to use non-integer values, if desired).

Problem 3: Revenue Equivalence (50 points)

This problem will recall the following definitions.

Definition 1 (Equal Revenue Curve) *The Equal Revenue Curve (denoted by ER) is a distribution with $F(x) = 1 - \frac{1}{x}$ for all $x \geq 1$, and $f(x) = \frac{1}{x^2}$ for all $x \geq 1$. For $x < 1$, $F(x) = 0$ and $f(x) = 0$.*

Definition 2 (All-Pay Auction) *In the All-Pay Auction, each bidder i submits a bid b_i. The item is awarded to the highest bidder (tie-breaking lexicographically), and all bidders pay their bids. So if bidder i wins the auction, their utility is $v_i - b_i$. If they lose, their utility is $-b_i$.*

Definition 3 (Bidding Strategy) *A bidding strategy is a function $b(\cdot)$ that takes as input a value v and proposes a bid $b(v)$ to make in the auction.*

Definition 4 (Bayes-Nash Equilibrium) *A bidding strategy $b(\cdot)$ is a Bayes-Nash equilibrium for the All-Pay Auction with two bidders drawn from ER if for all v_1, given that bidder 2 is going to draw a value $v_2 \leftarrow ER$ and bid $b(v_2)$, your expected utility is (weakly) maximized by bidding $b(v_1)$.*[4]

The following parts will guide you to find a Bayes-Nash Equilibrium using Revenue Equivalence. You should complete all parts and not provide an alternative proof.

Part a (10 points)

What is the expected revenue of the second-price auction when two bidders with values independently drawn from equal-revenue curves bid their true value?

Part b (10 points)

In the second-price auction, what is the expected payment made by bidder one, conditioned on bidding v_1, and that bidder two truthfully reports $v_2 \leftarrow ER$?

Note that we are **not** conditioning on bidder 1 winning. To be extra formal, let $P_1^{SPA}(v_1, v_2)$ denote the random variable that is equal to v_2 if $v_1 > v_2$, and 0 otherwise. For a fixed v_1, what is $\mathbb{E}_{v_2 \leftarrow ER}[P_1^{SPA}(v_1, v_2)]$?

Part c (10 points)

Consider the bidding strategy $b(\cdot)$ defined by $b(v_1) := \mathbb{E}_{v_2 \leftarrow ER}[P_1^{SPA}(v_1, v_2)]$. Prove that if both bidders use bidding strategy $b(\cdot)$ in the All-Pay auction, then the bidder with the highest value will always win the item.

Part d (10 points)

Assume that bidder two is using the bidding strategy $b(\cdot)$ from part c in the All-Pay auction. If bidder one's value is v_1, what is the expected utility that bidder one achieves by bidding b_1? Formally, if $U_1(v_1, b, b_1)$ is a random variable that is equal to $v_1 - b_1$ when $b(v_2) \leq b_1$, and $-b_1$ when $b(v_2) > b_1$, what is $\mathbb{E}_{v_2 \leftarrow ER}[U_1(v_1, b, b_1)]$?[5]

[4] You may want to see Lecture 16 for how we proved something is a Bayes-Nash equilbrium for two bidders drawn from Uniform($[0, 1]$). for the First-Price Auction.

[5] To be extra clear: for a given $v_1 \geq 1$, and $b_1 \geq 0$, your answer should say the expected utility that bidder one achieves by bidding b_1. So the variables b_1, v_1 should appear in your answer.

Part e (10 points)

Prove that the same $b(\cdot)$ from part c is a Bayes-Nash Equilibrium of the All-Pay auction for two bidders with values drawn independently from the equal-revenue curve. Your solution should include a (brief) justification of why the mathematical optimization you formulate correctly solves the problem, and also a (brief) justification of why you solved the mathematical optimization correctly.

Extra Credit: Walrasian Equilibria

Recall that extra credit is not directly added to your PSet scores, but will contribute to your participation grade. Some extra credits are **quite** challenging and will contribute significantly.

For this problem, you may collaborate with any students and office hours. You may not consult course resources or external resources, as this is a proof of a well-known result.[6]

In a combinatorial auction there are m items for sale to n buyers. Each buyer i has some valuation function $v_i(\cdot)$ which takes as input a set S of items and outputs that bidder's value for that set (so $v_i(S) = 5$ means that bidder i gets value 5 for receiving set S). These functions will always be monotone ($v_i(S \cup T) \geq v_i(S)$ for all S, T), and satisfy $v_i(\emptyset) = 0$. A Walrasian Equilibrium is a non-negative price for each item $\vec{p}$ such that:

- Each buyer i selects to purchase a set $B_i \in \arg\max_S\{v_i(S) - \sum_{j \in S} p_j\}$.
- The sets B_i are disjoint, and $\cup_i B_i = [m]$.

Prove that a Walrasian equilibrium exists for $v_1, \ldots, v_n$ if and only if the optimum of the LP relaxation below (called the configuration LP) is achieved at an integral point (i.e. where each $x_{i,S} \in \{0, 1\}$).

$$\max \sum_i \sum_S v_i(S) \cdot x_{i,S}$$
$$\forall i,\ \sum_S x_{i,S} = 1$$
$$\forall j,\ \sum_{S \ni j} \sum_i x_{i,S} \leq 1$$
$$\forall i, S,\ x_{i,S} \geq 0.$$

Finally, provide an example of two valuation functions v_1, v_2 over two items where a Walrasian equilibrium doesn't exist.

[6] You may consult course resources for general refreshers on Linear Programming, but not for anything specific to this problem.