COS 445 - PSet 2

Due online Monday, March 1st at 11:59 pm

Instructions:

- Some problems will be marked as no collaboration problems. This is to make sure you have experience solving a problem start-to-finish by yourself in preparation for the midterms/final. You cannot collaborate with other students or the Internet for these problems (you may still use the referenced sources and lecture notes). You may ask the course staff clarifying questions, but we will generally not give hints.
- Submit your solution to each problem as a **separate PDF** to codePost. Please make sure you're uploading the correct PDFs![1] If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well. Please anonymize your submission, although there are no repercussions if you forget.
- This cheatsheet gives problem solving tips, and guidelines for a "good proof" or "partial progress": `http://www.cs.princeton.edu/~smattw/Teaching/cheatsheet445.pdf`.
- Please reference the course collaboration policy here: `http://www.cs.princeton.edu/~smattw/Teaching/infosheet445sp21.pdf`.

[1] We will assign a minor deduction if we need to maneuver around the wrong PDFs. Please also note that depending on if/how you use Overleaf, you may need to recompile your solutions in between downloads to get the right files.

Problem 1: Two Candidates, Two Rules (20 points, no collaboration)

For this problem, there are n voters and $m = 2$ candidates, and $n \geq 3$ is odd. Recall also the following definitions:

Definition 1 (Unanimous) *A voting rule F is unanimous if for all candidates a, whenever all voters select a as their favorite candidate, F outputs a. Formally, F is unanimous if whenever there exists a candidate a such that $a \succ_i b$ for all i and $b \neq a$, then $F(\succ_1, \ldots, \succ_n) = a$.*

Definition 2 (Anonymous) *A voting rule F is anonymous if the identities of the voters do not matter. Put another way, F is anonymous if the output of F is invariant under relabeling the voters. Formally, F is anonymous if for all $\succ_1, \ldots, \succ_n$, and all permutations of voters σ, $F(\succ_1, \ldots, \succ_n) = F(\succ_{\sigma(1)}, \ldots, \succ_{\sigma(n)})$.*

Definition 3 (Neutral) *A voting rule F is neutral if the identities of the candidates do not matter. Put another way, F is neutral if relabeling the candidates causes the output of F to be similarly relabeled. Formally, F is neutral if for all $\succ_1, \ldots, \succ_n$, and all permutations of candidates, τ, $F(\tau(\succ_1), \ldots, \tau(\succ_n)) = \tau(F(\succ_1, \ldots, \succ_n))$. Here, we have abused notation and let $\tau(\succ_i)$ denote the ordering which places candidate $\tau(a)$ over candidate $\tau(b)$ if and only if $a \succ_i b$.*

Part a (10 points)

Design a voting rule which **is not** unanimous, **is** anonymous, and **is not** neutral (and briefly prove that it **is not** unanimous, **is** anonymous, and **is not** neutral).

Part b (10 points)

Design a voting rule which **is** unanimous, **is** anonymous, and **is not** neutral (and briefly prove that it **is** unanimous, **is** anonymous, and **is not** neutral).

Problem 2: Find the Bug! (30 points)

In a previous iteration of this class, I asked the class on a miderm to provide a voting rule that was Equivalent, unanimous, not a dictatorship, and not a Condorcet extension (all defined below) for $m \geq 3$ candidates. I also wrote an incorrect proof that a particular voting rule achieves all the intended properties. That proof was wrong (because the only rule that is Equivalent and unanimous is a dictatorship). This question asks you to establish that the proof is wrong.

Definition 4 (Equivalent) *Define two preferences $\succ, \succ'$ to be a-equivalent if for all b, $a \succ b \Leftrightarrow a \succ' b$ That is, $\succ'$ is the same as $\succ$ except that it may reorder candidates above a, and separately reorder candidates below a. A voting rule is Equivalent if whenever $F(\succ_1, \ldots, \succ_n) = a$, and $\succ_i, \succ_i'$ are a-equivalent for all i, then $F(\succ_1', \ldots, \succ_n') = a$ as well.*

Definition 5 (Unanimous) *A voting rule is unanimous if whenever a is everyone's favorite candidate, the rule selects a.*

Definition 6 (Dictatorship) *A voting rule is a dictatorship if there exists a voter i such that F always outputs i's favorite candidate.*

Definition 7 (Condorcet Extension) *Say that candidate a beats b if a strict majority of voters prefer a to b. A candidate is a Condorcet winner if it beats all other candidates. A rule is a Condorcet Extension if it always selects a Condorcet winner, when one exists.*

Here is the voting rule F: Say that candidate a dominates candidate b if every single voter prefers a to b. Index the candidates from a_1 to a_m. Let F output the minimum candidate a_i such that a_i is not dominated by any other candidate a_j.

Part a (10 points)

Prove that F is Equivalent, unanimous, not a dictatorship, and not a Condorcet extension when there are only $m = 2$ alternatives and any number $n \geq 3$ voters.

Part b (10 points)

Prove that F is not Equivalent when $m \geq 3$ (by counterexample).

You may not use code to find your example. When you find your example, your proof should be readable by a human.

Hint: An example exists with three candidates and two voters. But you are allowed to use different parameters.

Part c (10 points)

Find the specific line in my proof below that is incorrect. Quote (or paraphrase) the exact line below and prove that it's false (by counterexample). Your solution should actually identify a flaw in the logic below, and not simply declare that the last few lines are false because the rule is not Equivalent.

Label the candidates $a_1, \ldots, a_m$. Say that a candidate x dominates a candidate y if every single voter prefers x to y. Let i be the minimum index such that candidate a_i is not dominated by any other candidate (that is, we output a_1 if it is not dominated by any others, a_2 if any candidate dominates a_1, but nothing dominates a_2, etc.).

It is clear that this voting rule is unanimous: whenever every voter ranks a candiate first, that candidate dominates all others, so no other candidate can possibly be selected. Moreover, our rule is not a Condorcet extension. Consider preferences where one voter ranks a_1 first, and all other voters rank a_2 first. Then a_1 is not dominated by anything, so our rule will output a_1. However, because all other voters rank a_2 first, a_2 is certainly a Condorcet winner. So our rule is not a Condorcet extension.

Now we just need to prove it is Equivalent. Consider any preferences such that candidate a_i is selected. This means two things:

- Every voter prefers candidate a_i to candidate a_j, for all $j < i$. Therefore, all preferences that are a_i-equivalent also have every voter prefer a_i to a_j, so a_i will still dominate a_j and no candidate a_j will get selected for any $j < i$.
- For all $j > i$, some voter prefers candidate a_i to candidate a_j. Therefore, all preference that are a_i-equivalent also has, for all $j > i$, some voter who prefers a_i to a_j. So a_j will still not dominate a_i, and candidate a_i will get selected.

The above reasoning proves that our rule is Equivalent and unanimous, but not a Condorcet extension.

Problem 3: Representative Preferences (50 points)

This problem will involve the following three definitions (the first was studied in lecture, the second two are new to this problem). Throughout the problem, there are m candidates and n voters. A 'voter preference' is a strict ordering over the m candidates (from favorite to least favorite).

Definition 8 (Single-Peaked) *A (multi)-set*[2] $V \neq \emptyset$ *of voter preferences is* <u>*single-peaked*</u> *if there exists an ordering of the candidates* $c_1 \ldots, c_m$ *such that for all* $\succ \in V$*, the following holds: Let* c_i *denote the favorite candidate of* $\succ$*. Then for all* $j < k \leq i$, $c_k \succ c_j$*. Also, for all* $j > k \geq i$, $c_k \succ c_j$*.*

Definition 9 (Pairwise-Issue-Aligned) *A (multi)-set* $V \neq \emptyset$ *of* n *voter preferences is said to be* <u>*pairwise-issue-aligned*</u> *if there exists an ordering of the* <u>n *preferences in* V</u> $\succ_1, \ldots, \succ_n$ *such that for all candidates* a, b*, there exists a threshold index* $i_{a,b}$ *such that either:*

- *For all* $i \leq i_{a,b}$, $a \succ_i b$ *and for all* $i > i_{a,b}$, $b \succ_i a$*, or*
- *For all* $i \leq i_{a,b}$, $b \succ_i a$ *and for all* $i > i_{a,b}$, $a \succ_i b$*.*

That is, V *is pairwise-issue-aligned if there exists a way to order the preferences in* V *such that for all* a, b*, there exists a threshold index* $i_{a,b}$ *such that all preferences to the left of* $i_{a,b}$ *agree on* a *vs.* b*, and all preferences to the right of* $i_{a,b}$ *agree on* a *vs.* b*.*[3]

Definition 10 (Representative) *Let* V *be a (multi)-set of* n *voter preferences. We say that a preference ordering* $\succ$ *is* <u>*representative*</u> *of* V *if for all pairs of candidates* a, b, $a \succ b$ *if and only if a strict majority* ($> n/2$) *of voters in* V *prefer* a *to* b*. Note that a representative ordering does not necessarily exist.*

We say that $\succ$ *is a* <u>*strong representative*</u> *of* V *if* $\succ$ *is representative of* V*, and also* $\succ \in V$*.*

Part a (15 points)

Let V be pairwise-issue-aligned, and contain an odd number of voters. Prove that V has a strong representative.

Part b (20 points)

Let V be single-peaked, and contain an odd number of voters. Prove that V has a representative.

Part c (15 points)

Provide an example of a single-peaked V with an odd number of voters such that V does not have a strong representative. Prove that your example is correct.

You may not use code to find your example. When you find your example, your proof should be readable by a human.

Hint: An example exists with four candidates and three voters. But you are allowed to use different parameters.

[2] A multi-set is just a set that is allowed to contain repetitions. This just means that two voters in V might have the same preference.

[3] Note the order of quantifiers: there must exist a <u>single</u> ordering of preferences in V that does not change for each a, b. But the index $i_{a,b}$ depends on a, b.

Extra Credit: Why is nothing Strategyproof?

Recall that extra credit is not directly added to your PSet scores, but will contribute to your participation grade. Some extra credits are **quite** challenging and will contribute significantly.

For this problem, you may collaborate with any students and office hours. You may not consult course resources or external resources. In this problem we will guide you through the proof of a well-known result, so you should not copy the proof from one of the course texts (nor should you try to find a proof from external sources). You must follow the guide below (and not provide an alternative proof).

A Full-Ranking-Function (FRF) F is given a set of alternatives A and a profile of preferences over n voters, p, (just as with voting rules) except that now it must output a full ranking over A instead of a single winner.

Here are some desirable properties of FRFs:

- Unanimous: a FRF F is unanimous if whenever $a \succ_i b$ for all i, $\succ = F(\succ_1, \dots, \succ_n)$ has $a \succ b$ (whenever everyone likes a better than b, the final ranking has a above b).
- Independence of Irrelevant Alternatives: consider two profiles $p = (\succ_1, \succ_2, ..., \succ_n)$, $p' = (\succ'_1, \succ'_2, ..., \succ'_n)$ and let $\succ = F(p), \succ' = F(p')$.

  A FRF F satisfies Independence of Irrelevant Alternatives if for any two alternatives $a, b \in A$, if $a \succ_i b \iff a \succ'_i b, \forall i$ (every voter has the same preference between a and b) then $a \succ b \iff a \succ' b$ (the ordering output by F ranks a vs. b the same). Intuitively this property suggests that our preferences for c should not interfere with the ranking of a and b, and is related to strategyproof-ness.

Here is an undesirable property of FRFs:

- Dictatorship: voter i is a dictator in a in FRF F if for all $p = (\succ_1, \succ_2, ..., \succ_n)$, $\succ_i = F(p)$. That is to say, no matter what everyone else submits, the FRF chooses the ordering of the dictator. F is a dictatorship if some voter is a dictator.

It would be nice to produce FRFs that are unanimous with Independence of Irrelevant Alternatives, and are not dictatorships. Unfortunately, the following theorem says that for $|A| \geq 3$, this is not possible:

Theorem 11 *Every unanimous FRF F satisfying Independence of Irrelevant Alternatives over a set of more than 2 alternatives is a dictatorship.*

Part a

Prove the following lemma:

Lemma 12 *Let $p = (\succ_1, \succ_2, ..., \succ_n)$, $p' = (\succ'_1, \succ'_2, ..., \succ'_n)$ be two profiles such that for every player i, $a \succ_i b \iff c \succ'_i d$. Then if F is unanimous with Independence of Irrelevant Alternatives, $a \succ b \iff c \succ' d$, where $\succ = F(p), \succ' = F(p')$ (when there are > 2 alternatives).*

Note that a complete proof needs to consider all of the following cases:

1. $a = c, b = d$.

2. $a = c, b \neq d$.

3. $a \notin \{c, d\}, b = c$.

4. $a \notin \{c, d\}, b = d$.

5. $a \notin \{c, d\}, b \notin \{c, d\}$.

6. $a = d, b = c$.

7. $a = d, b \neq c$.

You do not need to provide a proof for all 7 cases, as many are similar. Provide a proof for cases One, Two, and Five.

Part b

Take any $a \neq b \in A$, and for every $0 \leq i \leq n$ define a preference profile π^i in which exactly the first i voters rank a above b, and the remaining voters rank b above a.

Prove that, if F is unanimous with Independence of Irrelevant Alternatives, there must be some $i^* \in [1, n]$ such that in $F(\pi^{i^*-1}), b \succ a$ but in $F(\pi^{i^*}), a \succ b$ (at this point, possibly not unique).

Part c

Use the lemma from part a to show the following:

Lemma 13 *If i^* is defined as in Part b, and F is unanimous with Independence of Irrelevant Alternatives, then for any $c \neq d \in A$: if $c \succ_{i^*} d$ then $c \succ d$.*

Conclude that i^* is a dictator for F.