COS 445 - PSet 1

Due online Monday, February 15th at 11:59 pm

Instructions:

- Some problems will be marked as no collaboration problems. This is to make sure you have experience solving a problem start-to-finish by yourself in preparation for the midterms/final. You cannot collaborate with other students or the Internet for these problems (you may still use the referenced sources and lecture notes). You may ask the course staff clarifying questions, but we will generally not give hints.
- Submit your solution to each problem as a **separate PDF** to codePost. Please make sure you're uploading the correct PDFs![1] If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well. Please anonymize your submission, although there are no repercussions if you forget.
- This cheatsheet gives problem solving tips, and guidelines for a "good proof" or "partial progress": `http://www.cs.princeton.edu/~smattw/Teaching/cheatsheet445.pdf`.
- Please reference the course collaboration policy here: `http://www.cs.princeton.edu/~smattw/Teaching/infosheet445sp21.pdf`.

Problem 1: Instances with many stable matchings (20 points, no collaboration)

Prove that, for all even $n \geq 2$, there exists a stable matching instance with n students and n universities with one slot each, such that there are at least $2^{n/2}$ distinct stable matchings.

Hint: First try to prove the claim for $n = 2$. See if you can use this solution as a gadget to prove the claim for $n = 4$ and notice a pattern.

Problem 2: Both Sides Propose (30 points)

Consider the following algorithm, "Both-Proposing Deferred Acceptance:"

- Maintain a tentative matching M, initially empty.
- While there exists an unmatched student:

[1] We will assign a minor deduction if we need to maneuver around the wrong PDFs. Please also note that depending on if/how you use Overleaf, you may need to recompile your solutions in between downloads to get the right files.

– Pick an arbitrary unmatched student, s. s proposes to her favorite university who hasn't yet rejected her. If u prefers s to $t = M(u)$, update $M(s) = u$, $M(u) = s$, and $M(t) = \perp$.

– Pick an arbitrary unmatched university (if one still exists), u. u proposes to their favorite student who hasn't yet rejected them. If s prefers u to $v = M(s)$, update $M(u) = s$, $M(s) = u$, and $M(v) = \perp$.

Either prove that Both-Proposing Deferred Acceptance always terminates in a stable matching, or provide an example of preferences and order of proposals such that BPDA does not output a stable matching.

Problem 3: Random Preferences (40 points)

Consider an instance with n students, n universities, where each university has capacity one. Each student's preferences are drawn uniformly at random (over all possible orderings), and the universities' preferences are arbitrary.

Consider an execution of the student-proposing deferred acceptance algorithm, and let M denote the matching output. Let X_i denote the rank of student i's match under M (that is, one plus the number of universities student i prefers to their match). Prove that $\mathbb{E}[\sum_i X_i] = O(n \log n)$.

Hint 1: Prove that deferred acceptance terminates once every university has received a proposal.
Hint 2: You may want to make use of the Coupon Collector Problem from PS0!
Hint 3: Once you feel like you have the right intuition, you may want to consult the cheatsheet sections on the "principle of deferred decisions" and "coupling arguments" to make your intuition into a formal proof.

Extra Credit: Almost Unique Stable Matchings

Recall that extra credit is not directly added to your PSet scores, but will contribute to your participation. Some extra credits are **quite** challenging and will contribute significantly.

Consider an instance with n students and n universities where student preferences are uniformly random, and university preferences are arbitrary. However, instead of a full preference ordering over all n universities, each student <u>truncates</u> their preferences at the top $c = O(1)$ universities (that is, they prefer to be unmatched rather than partner with a school outside their top c). Prove that the expected number of universities with the same partner in <u>all</u> stable matchings is $n - o(n)$ (where "unmatched" counts as a partner).[2]

This is a long problem, and the following hints break down the key steps. You will still get some extra credit if you successfully complete some, but not all, of the key steps.

Hint 1: You may want to prove the following fact first. Let M, M' be any two stable matchings. Then every student who is matched in M is also matched in M' (and vice versa). Every university that is matched in M is also matched in M' (and vice versa).

Hint 2: You may also want to prove the following: Let M be output by student-proposing deferred acceptance (i.e. each student stops applying if they are rejected by all of their top c schools), and let $M(u) = s$. Now consider modifying u's preferences by "blacklisting" s and all s' that u likes less than s. That is, u declares that they would rather be unmatched that matched

[2] Observe that this means it barely matters which side proposes in this model because almost everyone has the same partner regardless.

to s or anyone below s. Then any matching M' where $M'(u) = s' \neq M(u)$ is stable for the new preferences if and only if it is stable for the original preferences.

Hint 3: You may next want to prove the following fact using Hints 1 and 2. Let M be output by student-proposing deferred acceptance (where each student only proposes to a university to which they apply), and let $M(u) = s$. Now consider modifying u's preferences by "blacklisting" s and all s' that u likes less than s. That is, u declares that they would rather be unmatched that matched to s or anyone below s. Let M' denote the matching output by student-proposing deferred acceptance with this modified preference (and all others the same). If u is unmatched, then u is matched to s in every stable matching.

Hint 4: To start wrapping up, you may want to use the fact that Student-Proposing Deferred Acceptance outputs the same matching, independent of the order in which students propose (this is a corollary of a theorem from lecture). In particular, you may want to choose the order in which students propose to make use of the earlier hints.

Hint 5: Finally, you may use the following fact without proof: imagine throwing kn balls into n bins uniformly at random. Then with probability $1 - e^{-\Omega(n)}$, at least $n \cdot e^{-k}/2$ bins are empty.