• All problems on this exam are **no collaboration** problems.

• You **may not** discuss any aspect of any problems with anyone except for the course staff.

• You **may not** consult any external resources, the Internet, etc.

• You **may** consult the course lecture notes on Piazza, any of the five course readings, past Piazza discussion, or any notes directly linked on the course webpage (e.g. the cheatsheet, or notes on linear programming).

• You may discuss the test with the course staff, but we will only answer questions on clarification and will not give any guidance or hints. You should feel free to ask any questions and let us judge whether or not to answer, but just know that we may choose to politely decline to answer.

• If you choose to ask a question on Piazza, ask it **privately**. We will maintain a pinned FAQ for questions that are asked multiple times (please also reference this FAQ).

• Please upload each problem as a separate file via MTA.

• You **may not** use late days on the exam. You must upload your solution by May 20th at 11:59pm. If you are working down to the wire, upload your partial progress by in advance. There is no grace period for the exam. In case of a true emergency where you cannot upload, email me (smweinberg@princeton.edu) your solutions asap.

• If you miss the MTA deadline, we will not completely ignore your submission, but we will apply a hefty deduction before grading it. Please make sure you have something submitted by the deadline, and take into account that the server may be overloaded or sluggish near the end.

• There are **no exceptions, extensions, etc.** to the exam policy.
Problem 1: COS 445 Speedrun (120 points)

For each of the 12 problems below: unless otherwise specified you do not need to show any work and can just state the answer. However, if you simply state an incorrect answer with no justification, you will get no credit. You are encouraged to provide a very brief justification in order to receive partial credit in the event of a tiny mistake.  

Part a: Stable Matching (10 points)

Find the stable matching output by student-proposing deferred acceptance in the following example.

Alice: Princeton $\succ$ Yale $\succ$ Harvard  
Bob: Princeton $\succ$ Yale $\succ$ Harvard  
Charlie: Princeton $\succ$ Harvard $\succ$ Yale  
Princeton: Alice $\succ$ Bob $\succ$ Charlie  
Harvard: Alice $\succ$ Charlie $\succ$ Bob  
Yale: Bob $\succ$ Charlie $\succ$ Alice

Part b: Voting (10 points)

Definition 1 (Copeland Rule) For every pair of candidates $a, b$, give one point to whichever candidate a majority of voters prefer, tie-breaking in favor of the lexicographically-first candidate. Output the candidate with the most points, again tie-breaking in favor of the lexicographically-first candidate.

Find the candidate (the candidates are the foods — Alice, Bob, and Charlie are voting to pick a restaurant) output by the Copeland rule in the following example.

Alice: Tacos $\succ$ Seafood $\succ$ Pasta  
Bob: Tacos $\succ$ Seafood $\succ$ Pasta  
Charlie: Pasta $\succ$ Seafood $\succ$ Tacos

Part c: Game Theory (10 points)

Find a Nash equilibrium of the following game and state the expected payoff for both players. The first entry in each square denotes the payoff to the row player, and the second entry denotes the payoff to the column player. Recall that strategies as part of a Nash equilibrium may be mixed.

<table>
<thead>
<tr>
<th></th>
<th>Play</th>
<th>Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Play</td>
<td>(5,2)</td>
<td>(0,5)</td>
</tr>
<tr>
<td>Work</td>
<td>(3,3)</td>
<td>(2,0)</td>
</tr>
</tbody>
</table>

If otherwise specified, you should follow the otherwise specifications.
Part d: Extended Form Games (10 points)

There are two players (named 1 and 2) and three rounds. First player 1 plays, then player 2, then player 1 again. The numbers on the leaves denote the payoffs to the first and second players, respectively (as labeled at the internal nodes of the tree). The labels on the edges denote the names of the actions they can play at that turn. Find a subgame-perfect Nash equilibrium for this game.

![Extended Form Game Diagram](image)

Figure 1: An extended form game.

Recall that you must list a pure strategy for every internal node (even if it is never actually “played”). For example, \{L, X, A\} is not a complete answer, but \{L, X, Z, A, C, E, G\} is (i.e. you must still specify what actions “would have” been selected if those nodes were reached).

Part e: Linear Programming (10 points)

Write the dual of the following LP.
Maximize $3x + 2y$, such that:
- $4x + 4y \leq 1$.
- $3x + 7y \leq 3$.
- $x, y \geq 0$.

Part f: Scoring Rules (10 points)

Suppose you are asked to predict tomorrow’s weather. There’s four possible outcomes: it will be sunny, rainy, cloudy or snowy. You will be paid according to Brier’s scoring rule ($S(\vec{x}, i) = x_i - \sum_j x_j^2 / 2$). What is your expected payoff if you report the uniform distribution over these four outcomes?
Part g: Welfare-maximizing Auctions (10 points)

There are three bidders, and two ad slots. The first ad slot has a click-through rate of 1, and the second has a click-through rate of 1/2. The three bidders submit bids of $b_1 = 10, b_2 = 6, b_3 = 4$. The auctioneer is running a VCG auction.

For each of the three bidders, state the slot they win and their payment.

Part h: Revenue-maximizing Auctions

Suppose you are selling a pen to a single buyer. The buyer’s value is drawn uniformly from $[10, 14]$. What is the revenue-optimal auction/menu for you to sell the pen? What is the expected revenue you achieve?

Part i: Price of Anarchy (10 points)

Consider the following network. There are two nodes, $s$ and $t$, and one unit of flow traveling from $s$ to $t$. There are two directed edges from $s$ to $t$, one with cost $c(x) = 2$ and the other with cost $c(x) = 1 + x^2$. Compute the Price of Anarchy of this graph.

Part j: Cake cutting (10 points)

There is a single cake, the unit-interval $[0, 1]$. Alice, Bob, and Charlie all have normalized, additive valuations (that is, $v([0, 1]) = 1$, $v(\emptyset) = 0$, and $v(X \cup Y) = v(X) + v(Y)$ whenever $X \cap Y = \emptyset$, and $v(X) \geq 0$ for all $X$). Alice’s valuation satisfies $v_A([0, 1/4]) = 1$, distributed uniformly. Bob’s satisfies $v_B([0, 2/3]) = 1$, distributed uniformly. Charlie’s satisfies $v_C([1/2, 5/6]) = 1$, distributed uniformly.

Consider the allocation which awards Alice the interval $[0, 1/3]$, Bob the interval $[1/3, 2/3]$, and Charlie the interval $[2/3, 1]$. Is the allocation proportional? Is it envy-free? Is it equitable?

Part k: Behavioral Economics (10 points)

Recall that a utility function $f(\cdot)$ maps deterministic outcomes to a utility. Say that there are three possible outcomes, $A, B, C$. Define a utility function such that an expected utility maximizer with utility function $f(\cdot)$ prefers the randomized outcome which is $A$ with probability $1/2$, $B$ with
probability $1/4$, and $C$ with probability $1/4$ to the randomized outcome which is $A$ with probability $1/3$, $B$ with probability $1/3$, and $C$ with probability $1/3$.

**Part ℓ: Time-Inconsistent Planning (10 points)**

In the planning graph below, what path would be taken by a naive planner with present bias $b = 2$? What about a sophisticated planner with present bias $b = 2$? What is the shortest path from $s$ to $t$? **Note that this problem asks for you to provide three paths from $s$ to $t$.**
Problem 2: Switching Stable Matchings (40 points)

Consider a stable matching instance with \( n \) students and \( n \) universities (each with one slot). Let each student \( s \) have a complete, strict preference ordering \( \succ_s \) such that \( u \succ_s u' \) means that \( s \) prefers \( u \) to \( u' \) (and each university \( u \) has complete, strict preference ordering \( \succ_u \)). Let \( M \) and \( M' \) be any two stable matchings for the given preferences. Let \( A \) denote the set of students who strictly prefer their match in \( M \) to their match in \( M' \) (that is, \( M(s) \succ_s M'(s) \) for all \( s \in A \)), and \( B \) denote the set of universities who strictly prefer their match in \( M' \) to \( M \) (that is, \( M'(u) \succ_u M(u) \) for all \( u \in B \)). Prove that in both \( M \) and \( M' \), every student in \( A \) is matched to a university in \( B \).
Problem 3: Revenue vs. Welfare (60 points)

For a distribution with CDF $F$, consider the following three definitions:

- **Revenue** ($\text{REVENUE}(F)$) := $\max_p \{p \cdot (1 - F(p))\}$. That is, $\text{REVENUE}(F)$ denotes the maximum expected revenue that a seller with a single item for sale could achieve by selling to a single buyer whose value is drawn from $F$. Put another way, for a price $p$, $p \cdot (1 - F(p))$ is the expected revenue the seller achieves by setting price $p$ (price times probability the buyer purchases). $\text{REVENUE}(F)$ takes the maximum over all $p$.

- **Value** ($\text{VALUE}(F)$) := $\mathbb{E}_{v \sim F}\[v\] = \int_0^\infty 1 - F(x)dx$. That is, $\text{VALUE}(F)$ denotes the expected value of a single draw from $F$.

- For a number $H > 1$, a distribution with CDF $F$ is supported on $[1, H]$ if the following properties hold. For simplicity of notation, we’ll define $\mathcal{F}_H$ to be the set of all CDF’s which are supported on $[1, H]$.
  
  - $F(x) = 0$ for all $x < 1$.
  - $F(x) = 1$ for all $x > H$.
  - $F(x)$ is monotone non-decreasing on $(0, \infty)$.

  For a number, $H > 1$, define $\text{RATIO}(H)$, to be the maximum over all distributions $F$ supported on $[1, H]$ of $\text{VALUE}(F)/\text{REVENUE}(F)$. Put in pure math, $\text{RATIO}(H) := \max_{F \in \mathcal{F}_H} \{\text{VALUE}(F)/\text{REVENUE}(F)\}$.

Part a (10 points)

Prove that $\text{RATIO}(H) \leq H$. That is, prove that for all $F$ supported on $[1, H]$, $\text{VALUE}(F) \leq H \cdot \text{REVENUE}(F)$.

Part b (15 points)

When $H \geq 2$, prove that $\text{RATIO}(H) \geq 3/2$. That is, find an $F$ supported on $[1, H]$ so that $\text{VALUE}(F) \geq 1.5 \cdot \text{REVENUE}(F)$.

Hint: Try the uniform distribution on $[1, H]$, which has CDF $F(x) = (x - 1)/(H - 1)$ on $[1, H]$ and PDF $f(x) = 1/(H - 1)$ on $[1, H]$.

Part c (35 points)

Find $\text{RATIO}(H)$ for all $H > 1$, and prove that your answer is correct (that is, your answer should be given as a function of $H$). Note that a complete proof requires both an example distribution $F$ achieving $\text{VALUE}(F)/\text{REVENUE}(F) = \text{RATIO}(H)$, and a proof that no distribution $\overline{F}$ supported on $[1, H]$ can have $\text{VALUE}(\overline{F})/\text{REVENUE}(\overline{F}) > \text{RATIO}(H)$.

You will receive partial credit for providing suboptimal upper bounds on $\text{RATIO}(H)$ (i.e. improving part a), or suboptimal lower bounds on $\text{RATIO}(H)$ (i.e. improving part b).

Note: If you believe that your solution to part c resolves parts a and b, you may write “see part c” for parts a and b. But if you do this and your solution to part c is incorrect, you will also lose points on part a and b.
Problem 4: Almost-Envy-Free (80 points)

There are $n$ players and $m$ discrete items (i.e. you are dividing appliances, not cakes. You cannot cut a toaster to share it). Each player $i$ has value $v_{ij}$ for item $j$, and value $v_i(S) = \sum_{j \in S} v_{ij}$ for a set $S$ of items (that is, the valuations are additive). Your goal is to fairly allocate the items to the players. Like in Lecture 20, you should assume the players participate honestly. An allocation is a partition of the items into $S_1, \ldots, S_n$ ($S_i$ is the set of items allocated to player $i$) such that $S_i \cap S_j = \emptyset$ for all $i, j$ and $\cup_i S_i = [m]$ (that is, every item is allocated exactly once).

An allocation is envy-free if for all pairs of players $i, j$, player $i$ (weakly) prefers $S_i$ to $S_j$ (that is, $v_i(S_i) \geq v_i(S_j)$ for all $i, j$).

An allocation is almost-envy-free if for all pairs of players, $i, j$, there exists an item $k \in S_j$ such that $v_i(S_i) \geq v_i(S_j \setminus \{k\})$. That is, for all $i, j$, there exists an item $k$ such that player $i$ (weakly) prefers $S_i$ to $S_j$ after removing item $k$.

Part a (10 points)

Let there be two players and five items, and $v_{1j} = 1$ for all $j$, and $v_{2j} = j$, for all $j$. For each of the following allocations, state whether they are (i) envy free (or not), and (ii) whether they are almost-envy-free or not, and provide a very brief justification.

- $S_1 = \{1, 3, 5\}, S_2 = \{2, 4\}$.
- $S_1 = \{3, 4, 5\}, S_2 = \{1, 2\}$.
- $S_1 = \{1, 2, 3\}, S_2 = \{4, 5\}$.

Part b (20 points)

Prove that, for all $m$, and all $n$, and any $\vec{v}$, the following protocol produces an almost-envy-free allocation.

1. Input: $v_{ij}$, for all $i \in [n], j \in [m]$.
2. Initialize $i := 1$, $R = [m]$ ($R$ denotes the set of remaining items), $S_i = \emptyset$ for all $i$.
3. While $R \neq \emptyset$:
   (a) Ask player $i$ what is their favorite item $\ell$ in $R$ (i.e., $\arg \max_{j \in R} v_{ij}$, tie-breaking arbitrarily).
   (b) Remove item $\ell$ from $R$, and add it to $S_i$ (i.e. give item $\ell$ to player $i$, and item $\ell$ is no longer remaining).
   (c) Increase $i$ by one (unless $i = n$, then update $i := 1$).
4. Output: The partition $S_1, \ldots, S_n$. 

8
Part c (20 points)

Now, it is no longer the case that the valuations are additive. The only assumptions you should make on the valuations for this part are that the valuations are monotone (that is, \( v_i(S \cup T) \geq v_i(S) \) for all \( S \)) and normalized (that is, \( v_i(\emptyset) = 0 \)).

Prove that, for all \( m \), and \( n = 2 \), and any \( v_1(\cdot), v_2(\cdot) \), an almost-envy-free allocation always exists. That is, provide a protocol that takes as input \( v_1(\cdot), v_2(\cdot) \) and outputs \( S_1, S_2 \) such that \( S_1 \cap S_2 = \emptyset \) and \( S_1 \cup S_2 = [m] \), and is almost-envy-free.

Hint: Try a protocol that is similar to cut-and-choose from Lecture 20. It may be tricky to get the details correct.

Part d (30 points)

Again, it is no longer the case that the valuations are additive. The only assumptions you should make on the valuations for this part are that the valuations are monotone (that is, \( v_i(S \cup T) \geq v_i(S) \) for all \( S \)) and normalized (that is, \( v_i(\emptyset) = 0 \)).

Prove that, for all \( m \) and all \( n \), and any \( v_1(\cdot), \ldots, v_n(\cdot) \), the following protocol produces an almost-envy-free allocation.

1. **Input:** \( v_i(\cdot) \), for all \( i \in [n] \).

2. \( S_i = \emptyset \) for all \( i \).

3. At all times, let \( G \) denote the following graph: there is a node for each player \( i \in [n] \), put a directed edge from player \( i \) to player \( j \) if \( v_i(S_i) < v_i(S_j) \) (that is, put a directed edge from \( i \) to \( j \) if player \( i \) strictly prefers player \( j \)’s current set to their own).

4. For \( j := 1 \) to \( m \):
   
   a. **While** \( G \) contains a directed cycle:
      
      i. Pick such a cycle arbitrarily, and trade along that cycle. That is, if the cycle is players \( \langle i_1, i_2, \ldots, i_\ell \rangle \), simultaneously update \( S_{i_\ell} := S_{i_{\ell+1}} \) for all \( j \) (and \( S_{i_\ell} := S_{i_1} \)).

   ii. **After** trading, immediately update \( G \).

   b. Pick any player \( i \) that does not have an incoming edge in \( G \). Add item \( j \) to \( S_i \).

5. **Output:** \( S_1, \ldots, S_n \).

**Note:** If you believe that your solution to part d resolves part c, you may write “see part d” for part c. But if you do this and your solution to part d is incorrect, you will also lose points on part c. Note that your solution to part d cannot possibly solve part b, because you are analyzing a different algorithm.

**Note:** Note that for full credit, you should establish that the algorithm terminates in finite time (e.g. that the while loop will eventually break) and that the algorithm is well-defined (e.g. that the player \( i \) required in step 4b exists).