Problem 1: Voting Examples (10 points, NO COLLABORATION)

In class, we saw the following two voting rules:

- Instant Runoff Voting: Iteratively remove the candidate with the fewest first-place votes until only one remains.

- Copeland Rule: Say that candidate \( a \) beats candidate \( b \) if \( \geq \frac{n}{2} \) voters prefer \( a \) to \( b \). The winner is the candidate who beats the most other candidates.

We also saw the following property:

**Definition 1** A voting rule \( F \) is consistent if whenever \( F(\succ_1, \ldots, \succ_n) = F(\succ_{n+1}, \ldots, \succ_{n+k}) = a \), \( F(\succ_1, \ldots, \succ_{n+k}) = a \) as well. In other words, if \( F \) applied to two sub-populations outputs \( a \), then \( F \) applied on the entire population outputs \( a \) as well.

**Part a: IRV (10 points)**

Prove that IRV is not consistent. Your example must not have a need for tie-breaking. In other words, your example must have a distinct candidate with the fewest first place votes in every round.

**Hint:** An example exists with three candidates, but you are free to use more.
Part b: Copeland.

Bonus. Not mandatory. Solving/skipping this problem will not affect your score on this assignment. Prove that Copeland is not consistent. Your example must not have a need for tie-breaking. In other words, your example for must have a distinct candidate who wins the most “matches.”

Problem 2: Tribal Council (20 points)

There’s an upcoming election between candidates A and B in the town of Econometrica, which has $5^k$ voters. The electoral system works as follows: they use a voting tree of height $k + 1$ where every node except the leaves has exactly 5 children. Each non-leaf node is initially empty. Each voter labels a leaf of the tree according to their vote. Then each unlabelled node looks at its 5 children and takes the label of the majority of their labels. The winner of the election is the candidate whose label appears in the root of the tree.

Part a: Gerrymandering Lower Bounds (10 points)

Suppose candidate A gets to choose where in the tree each voter deposits their vote. What is the smallest number $n$ of votes that A needs in order to secure the election? In order to show this is the smallest $n$ you need to show both how to arrange the $n$ votes and that if $A$ had only $n - 1$ votes there is no arrangement that declares $A$ as the winner.

Part b: Random Populations (10 points)

Suppose that each voter in Econometrica independently chooses candidate A over B with probability $p$ (and B over A with probability $1 - p$). As $k \to \infty$, what is the asymptotic probability that A is declared the winner? How does the answer change depending on $p$?

Hint: Let $p_k$ denote the probability that $A$ wins in a tree of height $k$ (so $p_1 = p$). Write a recurrence relation to write $p_k$ as a function of $p_{k-1}$. You may use the following fact about fixed points without proof. Let $f$ be a continuous function from $[0, x_0]$ to $[0, x_0]$ with a fixed point at 0 ($f(0) = 0$) and $f(y) < y$, $\forall y \in (0, x_0]$. Then for the sequence of values $x_0, x_1, ...$ such that $x_{i+1} = f(x_i)$, $\lim_{i \to \infty} x_i = 0$.

Problem 3: Proving the Impossible (20 points, see below for collaboration policy)

For this problem, you may collaborate with any students. You may not consult course resources or external resources.

In this problem we will guide you through the proof of a well-known result, so you should not copy the proof from one of the course texts (nor should you try to find a proof from external sources). You must follow the guide below (and not provide an alternative proof), and you may receive partial credit for solving later parts without earlier parts.

1But we will grade it and give you feedback, and it will make a minor contribution towards the “participation” portion of your final grade.
A Full-Ranking-Function (FRF) \( F \) is given a set of alternatives \( A \) and a profile of preferences over \( n \) voters, \( p \), (just as with voting rules) except that now it must output a full ranking over \( A \) instead of a single winner.

Here are some desirable properties of FRFs:

- **Sensible**: a FRF \( F \) is sensible if whenever \( a \succ_i b \) for all \( i \), \( \succ = F(\succ_1, \ldots, \succ_n) \) has \( a \succ b \) (whenever everyone likes \( a \) better than \( b \), the final ranking has \( a \) above \( b \)).

- **No Externalities**: consider two profiles \( p = (\succ_1, \succ_2, \ldots, \succ_n) \), \( p' = (\succ'_1, \succ'_2, \ldots, \succ'_n) \) and let \( \succ = F(p), \succ' = F(p') \).

A FRF \( F \) has no externalities if for any two alternatives \( a, b \in A \), if \( a \succ_i b \iff a \succ'_i b, \forall i \) (every voter has the same preference between \( a \) and \( b \)) then \( a \succ b \iff a \succ' b \) (the ordering output by \( F \) ranks \( a \) vs. \( b \) the same). Intuitively this property suggests that our preferences for \( c \) should not interfere with the ranking of \( a \) and \( b \), and is related to strategyproof-ness.

Here is an undesirable property of FRFs:

- **Dictatorship**: voter \( i \) is a dictator in \( a \) in FRF \( F \) if for all \( p = (\succ_1, \succ_2, \ldots, \succ_n) \), \( \succ_i = F(p) \).

That is to say, no matter what everyone else submits, the FRF chooses the ordering of the dictator. \( F \) is a dictatorship if some voter is a dictator.

It would be nice to produce FRFs that are sensible with no externalities, and are not dictatorships. Unfortunately, the following theorem says that for \( |A| \geq 3 \), this is not possible:

**Theorem 2** Every sensible FRF \( F \) with no externalities over a set of more than 2 alternatives is a dictatorship.

**Part a (8 points)**

Prove the following lemma:

**Lemma 3** Let \( p = (\succ_1, \succ_2, \ldots, \succ_n) \), \( p' = (\succ'_1, \succ'_2, \ldots, \succ'_n) \) be two profiles such that for every player \( i \), \( a \succ_i b \iff c \succ'_i d \). Then if \( F \) is sensible with no externalities, \( a \succ b \iff c \succ' d \), where \( \succ = F(p), \succ' = F(p') \).

**Hint 1**: Observe that no externalities tells you that whether or not \( a \succ b \) in the output of \( F \) only depends on how the voters rank \( a \) vs. \( b \). Can you use this to build a single voter profile \( p'' \) where you know how \( F(p'') \) ranks \( a \) vs. \( b \) and \( c \) vs. \( d \)?

**Hint 2**: Once you have such a \( p'' \), can you move voter preferences around without swapping their preference for \( a \) vs. \( b \) and \( c \) vs. \( d \) to create \( p''' \) so that sensibility tells you something more about \( F(p''') \). What does no externalities then tell you about \( F(p''') \)?

**Part b (4 points)**

Take any \( a \neq b \in A \), and for every \( 0 \leq i \leq n \) define a preference profile \( \pi^i \) in which exactly the first \( i \) voters rank \( a \) above \( b \), and the remaining voters rank \( b \) above \( a \).

Prove that, if \( F \) is sensible with no externalities, there must be some \( i^* \in [1, n] \) such that in \( F(\pi_{i^*-1}), b \succ a \) but in \( F(\pi_{i^*}), a \succ b \) (at this point, possibly not unique).
Part c: A dictator arises (8 points)

Use the lemma from part a to show the following:

**Lemma 4** If $i^*$ is defined as in Part b, and $F$ is sensible with no externalities, then for any $c \neq d \in A$: if $c \succ_{i^*} d$ then $c \succ d$.

Conclude that $i^*$ is a dictator for $F$.

**Hint:** Place $a \notin \{c, d\}$ in each voter’s ranking so that you can invoke Part a) with our “special profile” from Part b. Can you place $a$ cleverly so that sensibility guarantees you something about $c$ vs. $a$ in the output of $F$?