COS 445 - Midterm 2

Due online Wednesday, April 26th at 11:59 pm

- All problems on this midterm are no collaboration problems.
- You may not discuss any aspect of any problems with anyone except for the course staff.
- You may not consult any external resources, the Internet, etc.
- You may consult the course lecture notes on Piazza, any of the five course readings, and past Piazza discussion.
- You may discuss the test with the course staff. In general, we will only answer questions on clarification. You should feel free to ask any questions and let us judge whether or not to answer, but just know that we may choose to politely decline to answer.
- Please upload each problem as a separate file via CS Dropbox.
- You may not use late days on the midterm. You must upload your solution by April 26th at 11:59pm. If you are working down to the wire, upload your partial progress by 11:00pm. We will be reasonable, but please don’t rely on leniency.

**Problem 1: VCG = Vastly Changing Gains (15 points)**

**Part a: Alice and Bob Go Shopping (5 points)**

Consider the following setting. There are two items, say one apple and one banana (labeled $a$ and $b$), and three bidders Alice, Bob, and Charlie. Their valuation functions are such that:

- Alice only gets value when she gets both items: $v_A(S) = 1 + \epsilon$ iff $S = \{a, b\}$, and 0 otherwise (for some $\epsilon \in (0, 1)$).
- Bob only cares about getting an apple, getting the banana doesn’t affect him at all: $v_B(S) = 1$ iff $a \in S$, and 0 otherwise.
- Charlie only cares about getting the banana: $v_C(S) = 1$ iff $b \in S$, and 0 otherwise.

What outcome is selected by the VCG auction when only Alice and Bob are present, and how much revenue is generated? What outcome is selected by the VCG auction when Alice, Bob, and Charlie are all present, and how much revenue is generated?
Part b: Noisy Optimizers aren’t Good Enough (10 points)

Consider the following error-prone algorithm $A$ for computing the maximum of a set $\{b_1, ..., b_n\}$ of numbers:

- If the second-highest number is exactly one less than the largest number, then output the index of the second-highest number (break ties lexicographically).
- Otherwise, output the index of the largest number (break ties lexicographically).

Consider the following error-prone version of the VCG auction with $n$ buyers and a single item:

- Accept bids $b_1, \ldots, b_n$.
- Award the item to the bidder $A(b_1, ..., b_n)$.
- Charge the winning bidder $b_{A(\vec{b} - i; 0)}$ ($A(\vec{b} - i; 0)$ means “replace $b_i$ with 0 and run $A$”).

Prove that at least the error-prone Vickrey auction has the following “nice” properties:

1. Assuming that all other bidders tell the truth, it is always in the highest-value bidder’s interest to submit a bid that wins the item.
2. For any given $\vec{b} - i$ (bids except for bidder $i$), there exists a $p$ such that no matter what bid bidder $i$ makes, bidder $i$ will either win the item and pay $p$, or not get the item.

Prove that, despite the above properties, the error-prone Vickrey auction is not incentive compatible by providing the following examples:

1. A vector of values $v_1, \ldots, v_n$ such that the second-highest bidder wins and pays strictly more than their value (pick an $n$, non-integer values are OK).
2. A vector of values $v_1, \ldots, v_n$ such that the highest-bidder wins, but the second-highest bidder has a strict incentive to lie about their value (pick an $n$, non-integer values are OK).

Problem 2: Tit-for-Tat (15 points)

Recall the following payoff matrix that was used in repeated prisoner’s dilemma on PS3 for 1000 rounds:

<table>
<thead>
<tr>
<th>(Alice, Bob)</th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>(2,2)</td>
<td>(0,5)</td>
</tr>
<tr>
<td>Defect</td>
<td>(5,0)</td>
<td>(1,1)</td>
</tr>
</tbody>
</table>

Prove that if Alice plays Tit-for-Tat (cooperate on round one, then copy Bob’s strategy from the previous round), then no matter what Bob does, Bob’s total payoff after 1000 rounds is at least as large as Alice’s.
Problem 3: Rules and conditions may apply (15 points)

Part a: One to rule them all, one to bind them (5 points)

Recall that we use the notation $S(\vec{x}, i)$ to denote the payoff that a scoring rule $S$ awards to the predictor when she reports $\vec{x}$ and event $i$ occurs. Show that the scoring rule $S(\vec{x}, i) = x_i^{\alpha - 1} - \frac{\alpha - 1}{\alpha} \cdot (\sum_j x_j^\alpha)$ is strictly proper for any $\alpha > 1$. For what value of $\alpha$ is this rule equivalent to the quadratic/Brier’s scoring rule?

Part b: Scoring rules do not work for variance (10 points)

In this problem you’ll show that it’s important that the predictor be allowed to report their entire belief and not just properties of their belief. Say that all you care about is learning the variance of the predictor’s belief, and not the entire distribution. Prove that no scoring rule that only allows the predictor to report a variance strictly incentivizes the predictor to report the true variance of their beliefs. In other words, there is no function $S$ such that for all distributions $\vec{x}$:

$$\text{VARIANCE}(\vec{x}) = \arg \max_r \mathbb{E}_{\vec{r} \leftarrow \vec{x}} S(r, i).$$

Hint: Assume for contradiction that such an $S$ existed. Consider two distributions $\vec{x}_1$ and $\vec{x}_2$ with the same variance, and a third distribution $\vec{y}$ that is a convex combination of $\vec{x}_1$ and $\vec{x}_2$ (i.e. $\vec{y} = \alpha \vec{x}_1 + (1 - \alpha) \vec{x}_2$ for some $\alpha \in [0, 1]$). What can you say about $\arg \max_r \mathbb{E}_{\vec{r} \leftarrow \vec{y}} S(r, i)$?

Problem 4: Nakamoto’s Intuition? (20 points)

Recall our (very slightly modified) definition of the block-reward model Bitcoin mining game:

- At all times, every miner $m$ is aware of a directed tree $G_m$.
- Every time step a miner is selected to mine a new block. They add a new node $v$ to $G_m$ pointing to any existing node in $G_m$ that they choose.
- At every time step, miners may decide to announce any nodes in $G_m$ that they wish, or not.
- The first time there is an announced node $v^*$ whose distance to the root is $T$, the game stops. For every node $v$ along the path from $v^*$ to the root, the miner who created that node receives reward $1/T$.
- There is no latency, or any other “real world” aspects to the model.

Part a (10 points)

Prove that if every other miner is choosing to point to the current longest chain, tie-breaking in favor of the earliest-announced-chain, and announce every block they find immediately, that it is a best response for you to do so as well if you know, with certainty, that you will only mine one block ever.
Part b (10 points)

Prove that for all $\epsilon, T$, there exists a sufficiently small $\delta$ such that if at every step you are selected independently with probability $\delta$ (and some other miner is selected with probability $(1 - \delta)$), and all other miners are pointing to the current longest chain and announcing every block they find, that it is an $\epsilon$-best response for you to do so as well (it is within expected payoff $\epsilon$ of being a best response).

Problem 5: Revenue Equivalence (30 points)

Consider a single-item auction with two bidders whose values are drawn from the equal-revenue curve $ER$, $(F(x) = 1 - 1/x$ for all $x \geq 1$, and $f(x) = 1/x^2$ for all $x \geq 1)$. Find a bidding strategy $b(\cdot)$ that is a Bayes-Nash equilibrium: for all $v_1$, given that bidder 2 is going to draw a value $v_2 \leftarrow ER$ and bid $b(v_2)$, your expected utility from participating in the First-Price Auction is (weakly) maximized by bidding $b(v_1)$. Recall that your utility is equal to $v_1 - b$ if you win and bid $b$, and zero otherwise.

You may use any method you like to find your equilibrium. The parts below guide through a search using Revenue Equivalence, and you may receive partial credit for each step you complete.

Part a (5 points)

What is the expected revenue of the second-price auction when two bidders with values independently drawn from equal-revenue curves bid their true value?

Part b (5 points)

In the second-price auction, what is the expected payment made by bidder one, conditioned on bidding $v_1$, and that bidder two truthfully reports $v_2 \leftarrow ER$?

Note that we are not conditioning on bidder 1 winning. To be extra formal, let $P^{SPA}_1(v_1)$ denote the random variable that is $v_2$ if $v_1 > v_2$, and 0 otherwise. What is $E_{v_2 \leftarrow ER}[P^{SPA}_1(v_1)]$?

Part c (10 points)

For a given bidding strategy $b(\cdot)$, define $P^{FPA}_1(v_1, b)$ to be the random variable that is $b(v_1)$ if $v_1 > v_2$, and 0 otherwise. Find a bidding strategy $b(\cdot)$ such that:

- $b(\cdot)$ is strictly monotone increasing on $[1, \infty)$ ($b(v) > b(v') \iff v > v'$). That is, bidder 1 will win exactly when $v_1 > v_2$ if both bidders use strategy $b(\cdot)$.

- For all $v_1 \in [1, \infty)$, $E_{v_2 \leftarrow ER}[P^{FPA}_1(v_1, b)] = E_{v_2 \leftarrow ER}[P^{SPA}_1(v_1)]$. That is, the expected payment made by bidder 1, conditioned on $v_1$ is the same in both auctions.

Part d (10 points)

Prove that the strategy you found in Part c is a Bayes-Nash Equilibrium of the first-price auction for two bidders with values drawn from the equal-revenue curve.