Reductions

- designing algorithms
- proving limits
- classifying problems
- NP-completeness
Desiderata.
Classify problems according to their computational requirements.

Frustrating news.
Huge number of fundamental problems have defied classification.

Desiderata'.
Suppose we could (couldn't) solve problem X efficiently.
What else could (couldn't) we solve efficiently?

Give me a lever long enough and a fulcrum on which to place it, and I shall move the world. - Archimedes
Def. Problem $X$ reduces to problem $Y$ if you can use an algorithm that solves $Y$ to help solve $X$.

Ex. Euclidean MST reduces to Voronoi.
To solve Euclidean MST on $N$ points
  • solve Voronoi for those points
  • construct graph with linear number of edges
  • use Prim/Kruskal to find MST in time proportional to $N \log N$
Reduction

**Def.** Problem $X$ reduces to problem $Y$ if you can use an algorithm that solves $Y$ to help solve $X$

\[
\text{Cost of solving } X = M \times (\text{cost of solving } Y) + \text{cost of reduction.}
\]

number of times $Y$ is used

**Applications**
- designing algorithms: given algorithm for $Y$, can also solve $X$.
- proving limits: if $X$ is hard, then so is $Y$.
- classifying problems: establish relative difficulty of problems.
designing algorithms
proving limits
classifying problems
NP-completeness
Reductions for algorithm design

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X

Cost of solving X = M*(cost of solving Y) + cost of reduction.

Applications.
- designing algorithms: given algorithm for Y, can also solve X.
- proving limits: if X is hard, then so is Y.
- classifying problems: establish relative difficulty of problems.

Mentality: Since I know how to solve Y, can I use that algorithm to solve X?

Programmer’s version: I have code for Y. Can I use it for X?
**Reductions for algorithm design: convex hull**

**Sorting.** Given N distinct integers, rearrange them in ascending order.

**Convex hull.** Given N points in the plane, identify the extreme points of the convex hull (in counter-clockwise order).

**Claim.** Convex hull reduces to sorting.

**Pf.** Graham scan algorithm.

Cost of convex hull = cost of sort + cost of reduction

- linearithmic
- linear
Claim. Shortest paths reduces to path search in graphs (PFS)

Pf. Dijkstra's algorithm

Cost of shortest paths = cost of search + cost of reduction
Reductions for algorithm design: maxflow

Claim: Maxflow reduces to PFS (!)

A forward edge is an edge in the same direction of the flow

An backward edge is an edge in the opposite direction of the flow

An augmenting path is along which we can increase flow by adding flow on a forward edge or decreasing flow on a backward edge

Theorem [Ford-Fulkerson] To find maxflow:
• increase flow along any augmenting path
• continue until no augmenting path can be found

Reduction is not linear because it requires multiple calls to PFS
Cost of maxflow $= M \times $ (cost of PFS) + cost of reduction

depends on path choice!
Reductions for algorithm design: bipartite matching

Bipartite matching reduces to maxflow

Proof:
- construct new vertices $s$ and $t$
- add edges from $s$ to each vertex in one set
- add edges from each vertex in other set to $t$
- set all edge weights to 1
- find maxflow in resulting network
- matching is edges between two sets

Note: Need to establish that maxflow solution has all integer (0-1) values.
Bipartite matching reduces to maxflow

Proof:
- construct new vertices $s$ and $t$
- add edges from $s$ to each vertex in one set
- add edges from each vertex in other set to $t$
- set all edge weights to 1
- find maxflow in resulting network
- matching is edges between two sets

Note: Need to establish that maxflow solution has all integer (0-1) values.

Cost of matching = cost of maxflow + cost of reduction

\[
\text{linear}
\]
Some reductions we have seen so far:

- LP (standard form)
- convex hull
- median finding
- sorting
- element distinctness
- arbitrage
- shortest paths (neg weights)
- shortest paths
- PFS
- maxflow
- bipartite matching
- Voronoi
- Euclidean MST
- closest pair
- LP (standard form)
Reductions for algorithm design: a caveat

**PRIME.** Given an integer \( x \) (represented in binary), is \( x \) prime?

**COMPOSITE.** Given an integer \( x \), does \( x \) have a nontrivial factor?

**PRIME reduces to COMPOSITE**

```java
public static boolean isPrime(BigInteger x) {
    if (isComposite(x)) return false;
    else return true;
}
```

**COMPOSITE reduces to PRIME**

```java
public static boolean isComposite(BigInteger x) {
    if (isPrime(x)) return false;
    else return true;
}
```

A possible real-world scenario:

- System designer specs the interfaces for project.
- Programmer A implements `isComposite()` using `isPrime()`.
- Programmer B implements `isPrime()` using `isComposite()`.
- **Infinite reduction loop!**

whose fault?
designing algorithms
proving limits
classifying problems
polynomial-time reductions
NP-completeness
Linear-time reductions to prove limits

Def. Problem X linear reduces to problem Y if X can be solved with:
- linear number of standard computational steps for reduction
- one call to subroutine for Y.

Applications.
- designing algorithms: given algorithm for Y, can also solve X.
- proving limits: if X is hard, then so is Y.
- classifying problems: establish relative difficulty of problems.

Mentality:
If I could easily solve Y, then I could easily solve X.
I can’t easily solve X.
Therefore, I can’t easily solve Y.

Purpose of reduction is to establish that Y is hard.
Proving limits on convex-hull algorithms

**Lower bound on sorting:** Sorting $N$ integers requires $\Omega(N \log N)$ steps.

Claim. SORTING reduces to CONVEX HULL [see next slide].

Consequence.
Any ccw-based convex hull algorithm requires $\Omega(N \log N)$ steps.
Sorting linear-reduces to convex hull

**Sorting instance.** \( X = \{ x_1, x_2, \ldots, x_N \} \)

**Convex hull instance.** \( P = \{ (x_1, x_1^2), (x_2, x_2^2), \ldots, (x_N, x_N^2) \} \)

**Observation.** Region \( \{ x : x^2 \geq x \} \) is convex \( \Rightarrow \) all points are on hull.

**Consequence.** Starting at point with most negative \( x \), counter-clockwise order of hull points yields items in ascending order.

To sort \( X \), find the convex hull of \( P \).
**Claim.** 3-SUM reduces to 3-COLLINEAR.

**Conjecture.** Any algorithm for 3-SUM requires $\Omega(N^2)$ time.

**Consequence.** Sub-quadratic algorithm for 3-COLLINEAR unlikely.
3-SUM reduces to 3-COLLINEAR (continued)

**Claim.** $3$-$SUM \leq L 3$-COLLINEAR.

- **3-SUM instance:** $x_1, x_2, \ldots, x_N$
- **3-COLLINEAR instance:** $(x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)$

**Lemma.** If $a$, $b$, and $c$ are distinct, then $a + b + c = 0$ if and only if $(a, a^3), (b, b^3), (c, c^3)$ are collinear.

**Pf.** [see next slide]
Lemma. If \( a, b, \) and \( c \) are distinct, then \( a + b + c = 0 \) if and only if \( (a, a^3), (b, b^3), (c, c^3) \) are collinear.

Pf. Three points \( (a, a^3), (b, b^3), (c, c^3) \) are collinear iff:

\[
\frac{a^3 - b^3}{a - b} = \frac{b^3 - c^3}{b - c} \quad \text{slopes are equal}
\]

\[
\frac{(a - b)(a^2 + ab + b^2)}{a - b} = \frac{(b - c)(b^2 + bc + c^2)}{b - c} \quad \text{factor numerators}
\]

\[
(a^2 + ab + b^2) = (b^2 + bc + c^2) \quad \text{a-b and b-c are nonzero}
\]

\[
a^2 + ab - bc - c^2 = 0 \quad \text{collect terms}
\]

\[
(a - c)(a + b + c) = 0 \quad \text{factor}
\]

\[
a + b + c = 0 \quad \text{a-c is nonzero}
\]
Reductions for proving limits: summary

Establishing limits through reduction is an important tool in guiding algorithm design efforts.

Want to be convinced that no linear-time convex hull alg exists?
Hard way: long futile search for a linear-time algorithm
Easy way: reduction from sorting

Want to be convinced that no subquadratic 3-COLLINEAR alg exists?
Hard way: long futile search for a subquadratic algorithm
Easy way: reduction from 3-SUM
designing algorithms
proving limits
classifying problems
NP-completeness
Reductions to classify problems

**Def.** Problem $X$ linear reduces to problem $Y$ if $X$ can be solved with:
- Linear number of standard computational steps.
- One call to subroutine for $Y$.

**Applications.**
- Design algorithms: given algorithm for $Y$, can also solve $X$.
- Establish intractability: if $X$ is hard, then so is $Y$.
- Classify problems: establish relative difficulty between two problems.

**Ex:** Sorting linear-reduces to convex hull.
Convex hull linear-reduces to sorting.
Thus, sorting and convex hull are equivalent

Most often used to classify problems as either
- **tractable** (solvable in polynomial time)
- **intractable** (exponential time seems to be required)
Polynomial-time reductions

**Def.** Problem $X$ polynomial reduces to problem $Y$ if arbitrary instances of problem $X$ can be solved using:
- Polynomial number of standard computational steps for reduction
- One call to subroutine for $Y$.

**Notation.** $X \leq_p Y$.

**Ex.** Any linear reduction is a polynomial reduction.

**Ex.** All algorithms for which we know poly-time algorithms poly-time reduce to one another.

Poly-time reduction of $X$ to $Y$ makes sense only when $X$ or $Y$ is not known to have a poly-time algorithm.
Polynomial-time reductions for classifying problems

**Goal.** Classify and separate problems according to relative difficulty.
- **tractable** problems: can be solved in polynomial time.
- **intractable** problems: seem to require exponential time.

**Establish tractability.** If $X \leq_p Y$ and $Y$ is tractable then so is $X$.
- Solve $Y$ in polynomial time.
- Use reduction to solve $X$.

**Establish intractability.** If $Y \leq_p X$ and $Y$ is intractable, then so is $X$.
- Suppose $X$ can be solved in polynomial time.
- Then so could $Y$ (through reduction).
- Contradiction. Therefore $X$ is intractable.

**Transitivity.** If $X \leq_p Y$ and $Y \leq_p Z$ then $X \leq_p Z$.

**Ex:** all problems that reduce to LP are tractable
3-satisfiability

**Literal**: A Boolean variable or its negation. \( x_i \) or \( \neg x_i \)

**Clause**: A disjunction of 3 distinct literals. \( C_j = (x_1 \lor \neg x_2 \lor x_3) \)

**Conjunctive normal form**: A propositional formula \( \Phi \) that is the conjunction of clauses. \( \text{CNF} = (C_1 \land C_2 \land C_3 \land C_4) \)

**3-SAT**: Given a CNF formula \( \Phi \) consisting of \( k \) clauses over \( n \) literals, does it have a satisfying truth assignment?

**yes instance**

\[
(\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_4) \land (\neg x_2 \lor x_3 \lor x_4)
\]

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**no instance**

\[
(\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_4) \land (\neg x_2 \lor x_3 \lor x_4)
\]

**Applications**: Circuit design, program correctness, [many others]
3-satisfiability is intractable

**Good news:** easy algorithm to solve 3-SAT
   [ check all possible solutions ]
**Bad news:** running time is exponential in input size.
   [ there are $2^n$ possible solutions ]
**Worse news:**
   no algorithm that guarantees subexponential running time is known

**Implication:**
- suppose 3-SAT poly-reduces to a problem $A$
- poly-time algorithm for $A$ would imply poly-time 3-SAT algorithm
- we suspect that no poly-time algorithm exists for $A$!

Want to be convinced that a new problem is intractable?
**Hard way:** long futile search for an efficient algorithm (as for 3-SAT)
**Easy way:** reduction from a known intractable problem (such as 3-SAT)

hence, intricate reductions are common
**Graph 3-colorability**

**3-COLOR.** Given a graph, is there a way to color the vertices red, green, and blue so that no adjacent vertices have the same color?
Graph 3-colorability

3-COLOR. Given a graph, is there a way to color the vertices red, green, and blue so that no adjacent vertices have the same color?

yes instance
**Graph 3-colorability**

**3-COLOR.** Given a graph, is there a way to color the vertices red, green, and blue so that no adjacent vertices have the same color?

*no instance*
Claim. $3$-SAT $\leq_p 3$-COLOR.

Pf. Given 3-SAT instance $\Phi$, we construct an instance of 3-COLOR that is 3-colorable if and only if $\Phi$ is satisfiable.

Construction.
(i) Create one vertex for each literal and 3 vertices $\overline{F}$, $T$, $B$.
(ii) Connect $F$, $T$, $B$ in a triangle and connect each literal to $B$.
(iii) Connect each literal to its negation.
(iv) For each clause, attach a 6-vertex gadget [details to follow].

![Diagram](attachment:diagram.png)
3-satisfiability reduces to graph 3-colorability

Claim. If graph is 3-colorable then $\Phi$ is satisfiable.

Pf.
- Consider assignment where $\mathbf{F}$ corresponds to false and $\mathbf{T}$ to true.
- (ii) [triangle] ensures each literal is true or false.
Claim. If graph is 3-colorable then \( \Phi \) is satisfiable.

Pf. \( \Rightarrow \) Suppose graph is 3-colorable.

- Consider assignment where \( \mathbf{F} \) corresponds to false and \( \mathbf{T} \) to true.
- (ii) [triangle] ensures each literal is true or false.
- (iii) ensures a literal and its negation are opposites.
Claim. If graph is 3-colorable then $\Phi$ is satisfiable.

Pf.

- Consider assignment where $\text{F}$ corresponds to false and $\text{T}$ to true.
- (ii) [triangle] ensures each literal is true or false.
- (iii) ensures a literal and its negation are opposites.
- (iv) [gadget] ensures at least one literal in each clause is true.

$$(x_1 \lor \neg x_2 \lor x_3)$$
3-satisfiability reduces to graph 3-colorability

**Claim.** If graph is 3-colorable then $\Phi$ is satisfiable.

**Pf.**

- Consider assignment where $\text{F}$ corresponds to false and $\text{F}$ to true.
- (ii) [triangle] ensures each literal is true or false.
- (iii) ensures a literal and its negation are opposites.
- (iv) [gadget] ensures at least one literal in each clause is true.

Therefore, $\Phi$ is satisfiable.

$$ (x_1 \lor \neg x_2 \lor x_3) $$
3-satisfiability reduces to graph 3-colorability

**Claim.** If $\Phi$ is satisfiable then graph is 3-colorable.

**Pf.**
- Color nodes corresponding to false literals $\square$ and to true literals $\bigcirc$.

$(x_1 \lor \neg x_2 \lor x_3)$

at least one in each clause
Claim. If $\Phi$ is satisfiable then graph is 3-colorable.

Pf.
- Color nodes corresponding to false literals and to true literals.
- Color vertex below one vertex, and vertex below that.
3-satisfiability reduces to graph 3-colorability

Claim. If $\Phi$ is satisfiable then graph is 3-colorable.

Pf.

• Color nodes corresponding to false literals and to true literals .
• Color vertex below one vertex , and vertex below that .
• Color remaining middle row vertices .

$(x_1 \lor \neg x_2 \lor x_3)$
3-satisfiability reduces to graph 3-colorability

Claim. If \( \Phi \) is satisfiable then graph is 3-colorable.

Pf.
- Color nodes corresponding to false literals \( \text{\ding{55}} \) and to true literals \( \text{\ding{58}} \).
- Color vertex below one \( \text{\ding{58}} \) vertex \( \text{\ding{55}} \), and vertex below that \( \text{\ding{59}} \).
- Color remaining middle row vertices \( \text{\ding{59}} \).
- Color remaining bottom vertices \( \text{\ding{55}} \) or \( \text{\ding{58}} \) as forced.

Works for all gadgets, so graph is 3-colorable. □

\[(x_1 \lor \neg x_2 \lor x_3)\]
**Claim.** 3-SAT $\leq_p$ 3-COLOR.

**Pf.** Given 3-SAT instance $\Phi$, we construct an instance of 3-COLOR that is 3-colorable if and only if $\Phi$ is satisfiable.

**Construction.**

(i) Create one vertex for each literal.

(ii) Create 3 new vertices T, F, and B; connect them in a triangle, and connect each literal to B.

(iii) Connect each literal to its negation.

(iv) For each clause, attach a gadget of 6 vertices and 13 edges

**Conjecture:** No polynomial-time algorithm for 3-SAT

Implication: No polynomial-time algorithm for 3-COLOR.

**Reminder**

Construction is not intended for use, just to prove 3-COLOR difficult
designing algorithms
proving limits
classifying problems
polynomial-time reductions
NP-completeness
More Poly-Time Reductions

Conjecture: no poly-time algorithm for 3-SAT. (and hence none of these problems)
Cook’s Theorem

$\text{NP}$: set of problems solvable in polynomial time by a nondeterministic Turing machine

**THM.** Any problem in $\text{NP} \leq_p \text{3-SAT}$. 

**Pf sketch.**

Each problem $P$ in $\text{NP}$ corresponds to a TM $M$ that accepts or rejects any input in time polynomial in its size.

*Given* $M$ and a problem instance $I$, construct an instance of $\text{3-SAT}$ that is satisfiable iff the machine accepts $I$.

**Construction.**

- Variables for every tape cell, head position, and state at every step.
- Clauses corresponding to each transition.
- [many details omitted]
Implications of Cook's theorem

All of these problems (any many more) polynomial reduce to 3-SAT.

Stephen Cook '82 Turing award
Implications of Karp + Cook

All of these problems poly-reduce to one another!

Conjecture: no poly-time algorithm for 3-SAT.
(and hence none of these problems)
Poly-Time Reductions: Implications

“I can’t find an efficient algorithm, I guess I’m just too dumb.”
Poly-Time Reductions: Implications

“I can’t find an efficient algorithm, because no such algorithm is possible!”
Poly-Time Reductions: Implications

“I can’t find an efficient algorithm, but neither can all these famous people.”
Summary

Reductions are important in **theory** to:
- Establish tractability.
- Establish intractability.
- Classify problems according to their computational requirements.

Reductions are important in **practice** to:
- Design algorithms.
- Design reusable software modules.
  - stack, queue, sorting, priority queue, symbol table, set, graph
  - shortest path, regular expressions, linear programming
- Determine difficulty of your problem and choose the right tool.
  - use exact algorithm for tractable problems
  - use heuristics for intractable problems