On Extreme Points of the Dual Ball of a Polyhedral Space

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Abstract: We prove that every separable polyhedral Banach space X is isomorphic to a polyhedral Banach space Y such that, the set ext $B_Y^*$ cannot be covered by a sequence of balls $B(y_i, \epsilon_i)$ with $0 < \epsilon_i < 1$ and $\epsilon_i \to 0$. In particular ext $B_Y^*$ cannot be covered by a sequence of norm compact sets. This generalizes a result from [7] where an equivalent polyhedral norm $|||\cdot|||$ on $c_0$ was constructed such that ext $B_{(c_0,|||\cdot|||)^*}$ is uncountable but can be covered by a sequence of norm compact sets.

Key words: Polyhedral Banach space, boundary, extreme points.


In [8] V. Klee introduced the following definition of a polyhedral Banach space.

DEFINITION 1. A Banach space X is called polyhedral if the unit ball of every finite dimensional subspace of X is a polytope.

Recall that a subset $B \subseteq S_X^*$ of the unit sphere of the dual Banach space $X^*$ is called a boundary of X if for any $x \in X$ there is $f \in B$ with $f(x) = ||x||$. In [3] (see also [5] and [10]), it was proved that any separable polyhedral space has a countable boundary. The converse is true under a suitable renorming (see [2]).

By the Krein-Milman Theorem, the set ext $B_X^*$ is a boundary for any Banach space X. In [7], a separable polyhedral Banach space X was constructed (actually X is isomorphic to $c_0$) such that ext $B_X^*$ is uncountable. Of course, being separable polyhedral, X admits a countable boundary. However, it is easily seen from the construction in [7] that the set ext $B_X^*$ can be covered by a sequence of norm compact sets, i.e. although ext $B_X^*$ is uncountable it is in a sense “close” to a countable set.

DEFINITION 2. Let L be a Banach space and $C \subset E$. We say that C has property (A) if for each sequence $\epsilon_i \to 0$, $0 < \epsilon_i < 1$ and any sequence of balls $B(z_i, \epsilon_i) = \{x \in L : ||x - z_i|| \leq \epsilon_i\}$, we have $C \not\subseteq \bigcup_{i=1}^\infty B(z_i, \epsilon_i)$. 

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Clearly, if $C$ has (A) then $C$ cannot be covered by a sequence of norm compact sets.

The main result of this paper is the following

**Theorem 1.** Let $Y$ be a separable polyhedral Banach space. Then $Y$ is isomorphic to a polyhedral Banach space $Z$ such that the set $\text{ext} B_{Z^*}$ has property (A).

**Remark.** It follows from Theorem 3 [4], that if a Banach space $Y$ is not isomorphic to a polyhedral space then $\text{ext} B_{Y^*}$ has property (A) in any equivalent norm on $Y$.

We prove Theorem 1 in two steps. First we prove Theorem 1 for $Y = c_0$. Here we use some ideas from [7]. Then, by using that any polyhedral space contains an isomorphic copy of $c_0$ (see [3]) we finish the proof.

**Theorem 2.** There exists a separable polyhedral Banach space $X$, isomorphic to $c_0$, such that the set $\text{ext} B_{X^*}$ has property (A).

**Proof.** Let $\{e_i\}_{i=1}^{\infty}$ be the natural basis of $c_0$ and $\{e_i^*\}_{i=1}^{\infty}$ be its biorthogonal sequence in $l_1 = c_0^*$. Fix $\varrho \in (0, \frac{1}{2})$ and denote

$$
\lambda_i = \frac{1}{2^i}, \quad i = 1, 2, \ldots, \quad a = \frac{1}{\lambda_1}, \quad a_n = \frac{a \sum_{i=1}^{n} \lambda_i}{1 - \varrho \sum_{i=n+1}^{\infty} \lambda_i}, \quad n = 1, 2, \ldots.
$$

Let $G_m$ be the family of all injective, non-decreasing mappings from $\{1, \ldots, m\}$ to $\mathbb{N}$ and $G_\infty$ be the family of all injective, non-decreasing mappings from $\mathbb{N}$ to $\mathbb{N}$. Next define:

$$
A_m = \left\{ a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=1}^{m} \epsilon_k \lambda_k e_{g(k)}^* : \epsilon_k = \pm 1, \ g \in G_m \right\}.
$$

Clearly, each $A_m$ is countable. Denote

$$
B = \bigcup_{m=1}^{\infty} A_m, \quad U^* = \overline{\text{conv}}^{w^*} B,
$$

and define a new norm on $c_0$ as follows

$$|||x||| = \sup \{ f(x) : f \in U^* \}, \quad x \in c_0.$$
It is easily seen that the norm $|||\cdot|||$ on $c_0$ is equivalent to the original one (note that $A_1 = \{\pm a_1 e_k^*: k = 1, 2, \ldots\}$). Put $X = (c_0, |||\cdot|||)$. Also a standard argument shows that $B_X^* = U^*$.

For every subset $A$ of $X^*$, denote $A'$ the set of all $w^*$-limit points of the set $A$.

**Claim 1.** Every $f \in B'$ with $|||f||| = 1$ (if any) does not attain its norm $|||f|||$ at an element of the unit ball of $X$.

**Proof.** Take $f \in B'$, $f \neq 0$. We first assume that $f \in A'_m$ for some $m \geq 2$. Since $e_n^* \to w^*$ 0 we get

$$f = a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^*, $$

for some $n < m$ and $g \in G_n$.

$$|||f||| = \left| a_m \left( \sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^* \right| < 1. $$

Next assume that $f \in B'$ and $f \notin A'_m$, $m = 1, 2, \ldots$. It is easy to see that either $f$ is of the form

$$f = a \sum_{k=1}^{\infty} \epsilon_k \lambda_k e_{g(k)}^*, \quad \epsilon_k = \pm 1, \ g \in G_{\infty}, \quad (1)$$

or

$$f = a \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^*, \quad \epsilon_k = \pm 1, \ g \in G_n \quad (2)$$

If $f$ satisfies (2) then $|||f||| < 1$. So we assume that $f$ satisfies (1). Assume to the contrary, that there is $x \in c_0$, $|||x||| = 1$, such that $f(x) = 1$. Choose $s$ so large that $a \cdot \max \{|x_{g(k)}|\}_{k=s+1}^{\infty} < \frac{\theta}{2}$. Then the definition of $|||\cdot|||$ implies
\[ 1 = f(x) = a \sum_{k=1}^{s} \epsilon_k \lambda_k x_g(k) + a \sum_{k=s+1}^{\infty} \epsilon_k \lambda_k x_g(k) \]
\[ \leq \frac{a}{a_s} \left[ a_s \left( \sum_{i=1}^{s} \lambda_i \right)^{-1} \sum_{k=1}^{s} \lambda_k |x_g(k)| \right] \sum_{i=1}^{s} \lambda_i + \left( a \cdot \max_{k>s} |x_g(k)| \right) \sum_{k=s+1}^{\infty} \lambda_k \]
\[ < \frac{a}{a_s} \cdot \sum_{i=1}^{s} \lambda_i + \frac{2}{\infty} \sum_{i=s+1}^{\infty} \lambda_i < 1. \]

The last inequality follows from the following equality:
\[ \frac{a}{a_s} \sum_{i=1}^{s} \lambda_i + \frac{2}{\infty} \sum_{i=s+1}^{\infty} \lambda_i = 1. \]

**Claim 2.** \( B \) is a countable boundary for \( X \) and \( X \) is polyhedral.

**Proof.** Since each \( A_m \) is countable and \( B = \bigcup_{m=1}^{\infty} A_m \), it follows that \( B \) is countable. The rest of the claim is a direct result of Claim 1 and Proposition 6.11 from [6]. We give a proof for the sake of completeness. Since \( U^* = \overline{conv} B^w \), \( \overline{B^w}^w = B \cup B' \) is a boundary for \( X \). As a result of Claim 1, none of the elements in \( B' \) attain their norm at \( B_X \) hence \( B \) is a boundary for \( X \). Now let \( F \) be a finite dimensional subspace of \( X \) and assume \( F^* \) has infinitely many extreme points, By Milman’s theorem, these would be restrictions to \( F \) of elements of \( \overline{B^w}^w \). Since \( F \) is finite-dimensional, any \( w^* \)-cluster point of the set of the extreme points of \( B_F^w \) attains its norm at an element of \( B_F^w \). But this contradicts Claim 1. Hence \( F^* \) has only finitely many extreme points, and \( F \) is polyhedral. \( \blacksquare \)

**Claim 3.** For any \( g \in G_\infty \) and \( \{ \epsilon_i \}_{i=1}^{\infty} \) a sequence of signs, we have \( f = a \sum_{k=1}^{\infty} \epsilon_k \lambda_k e_g(k) \in \text{ext } U^* \).

**Proof.** First note that from the definition of the norm \(||||\cdot|||\) (the supremum over the set \( B \)) follows that
\[
\left\| \sum_{i=1}^{n} \epsilon_i e_g(i) \right\| \leq 2
\]
Next the series \( \sum_{i=1}^{\infty} \epsilon_i e_{g(i)} \) converges in the \( w^* \)-topology of \( X^{**} \cong \ell_\infty \) and it follows that \( ||\sum_{i=1}^{\infty} \epsilon_i e_{g(i)}|| \leq 2 \). Moreover, setting \( z^{**} = \sum_{i=1}^{\infty} \epsilon_i e_{g(i)} \) and \( b^* = a \sum_{i=1}^{\infty} \epsilon_i \lambda_i e^{*}_{g(i)} \) we see that \( b^* \in B_{X^*} \) and \( z^{**}(b^*) = 2 \). Therefore \( z^{**} \) attains its norm at the element \( b^* \in B_{X^*} \) and \( ||z^{**}|| = 2 \). Moreover, setting \( z^{**}(b^*) = 2 \) then it follows that \( ||z^{**}|| = 2 \). Therefore \( z^{**} \) attains its norm at the element \( b^* \in B_{X^*} \) and \( ||z^{**}|| = 2 \). By a classical result [1], since \( X^* \) is separable, \( z^{**} \) attains its norm at an extreme point of \( B_{X^*} \) too.

The latter set of points, in view of Milman’s theorem, is contained in \( \overline{B_{X^*}} \). It is easy to check that \( z^{**} \) does not attain its norm at a finitely supported (with respect to \( (e^{*}_i) \)) element of \( \overline{B_{X^*}} \). Among the infinitely supported members of \( \overline{B_{X^*}} \), it is clear that only \( b^* \) satisfies \( z^{**}(b^*) = 2 \), hence \( b^* \) is an extreme point of \( B_{X^*} \).

**Claim 4.** The set \( \text{ext} U^* \) has property (A).

**Proof.** Denote \( E = \{ a \sum_{i=1}^{\infty} \lambda_i e^{*}_{g(i)} : g \in G_{\infty} \} \). By Claim 3, \( E \subseteq \text{ext} U^* \).

So it is enough to prove that \( E \) has property (A). Our proof relies on the following easily verified fact.

**Fact 1.** For each two elements \( u, v \in E \), if \( u = a \sum_{i=1}^{\infty} \lambda_i e^{*}_{g_u(i)} \) and \( v = a \sum_{i=1}^{\infty} \lambda_i e^{*}_{g_v(i)} \) and \( g_u(j) \neq g_v(j) \) then \( \|u - v\| > \frac{1}{2^j} \).

Assume to the contrary that

\[
E \subseteq \bigcup_{i=1}^{\infty} B_{X^*}(x_i, \epsilon_i), \quad \epsilon_i \to 0.
\]

Since \( B_{X^*} \subseteq 2B_{\ell_1} \) it follows that

\[
E \subseteq \bigcup_{i=1}^{\infty} B_{\ell_1}(x_i, 2\epsilon_i).
\]

Obviously, we can suppose that each \( B_{\ell_1}(x_i, 2\epsilon_i) \) intersects \( E \). For each \( i \) choose a representative \( y_i \in B_{\ell_1}(x_i, 2\epsilon_i) \cap E \).

Choose \( m_0 \) sufficiently large so that for \( m > m_0 \) it holds that \( 2\epsilon_m < \frac{1}{4} \).

Choose \( n_0 \) sufficiently large so that if \( y \in E \) and \( g_y(1) > n_0 \) then

\[
\max\{4\epsilon_1, \ldots, 4\epsilon_{m_0}\} < \|y - y_j\|
\]

for each \( j \leq m_0 \) (this is possible since \( 4\epsilon_i < 4 \) and \( E \subseteq 2S_{\ell_1} \)). Denote by \( G_0 \) the set \( \{1, 2, \ldots, n_0\} \). Choose \( m_1 > m_0 \) sufficiently large such that if \( m > m_1 \)
then $2\epsilon_m < \frac{1}{8}$. Denote by $G_1$ the set $\{g_{ym_0+1}(1), \ldots, g_{ym_1}(1)\}$. By Fact 1 if $x \in E$ and $g_x(1) \notin G_1$ then $\|x - y_j\| > \frac{1}{2}$ for $m_0 < j \leq m_1$. Hence, $x \notin \bigcup_{i=m_0+1}^{m_1} B_{\ell_1}(x_i, 2\epsilon_i)$. Next we define inductively $m_n$ and $G_n$ such that

1) For every $m > m_n$, $2\epsilon_m < \frac{1}{2^{n+1}}$.

2) $G_n$ is finite.

3) If $g_x(n) \notin G_n$ then $x \notin \bigcup_{i=m_{n-1}+1}^{m_n} B_{\ell_1}(x_i, 2\epsilon_i)$.

Choose $m_{n+1}$ so that for $m > m_{n+1}$ it holds that $2\epsilon_m < \frac{1}{2^{n+1}}$. Denote by $G_{n+1}$ the set $\{g_{ym_{n+1}}(n+1), \ldots, g_{ym_{n+1}}(n+1)\}$. For every $x \in E$ and $m_n < j \leq m_{n+1}$ if $g_x(n+1) \notin G_{n+1}$ then by Fact 1 $\|x - y_j\| > \frac{1}{2^{n+1}} > 4\epsilon_j$ and $x \notin \bigcup_{i=m_{n+1}}^{m_n} B_{\ell_1}(x_i, 2\epsilon_i)$. Define $b_1 = \max(G_0 \cup G_1) + 1$ and $b_n$ to be $\max(\bigcup_{i=0}^{n} G_n \cup \{b_1, \ldots, b_{n-1}\}) + 1$. Next define $g \in G_{\infty}$ to be $g(n) = b_n$, $n = 1, 2, \ldots$, and $x = \sum_{i=1}^{\infty} \lambda_i e^{*}_{g(i)}$. From our construction follows that $x \notin \bigcup_{i=1}^{\infty} B_{\ell_1}(x_i, 2\epsilon_i)$, a contradiction.

The proof of Theorem 2 is complete.

**Proof of Theorem 1.** By [3] $Y$ contains $c_0$ (actually $Y$ is $c_0$-saturated). Since $Y$ is separable it follows [9] that $c_0$ is complemented in $Y$. Hence $Y$ is isomorphic to the direct sum of $Y_1$ and $c_0$, where $Y_1$ is isometric to some subspace of $Y$ and hence polyhedral. By Theorem 2, $c_0$ is isomorphic to a polyhedral Banach space $X$ with the set $\text{ext} B_{X^*}$ having property (A). Put $Z = (Y_1 \oplus_{\infty} X)$. Clearly, $Z$ is polyhedral and $Y \cong Z$. Since $\text{ext} B_{Z^*} = \text{ext} B_{Y^*} \cup \text{ext} B_{X^*}$ it follows that the set $\text{ext} B_{Z^*}$ has property (A). The proof is complete.

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**References**


