Category Theory PSet 6

November 7, 2018

1. A *complete semi-lattice* is a partial order $\mathcal{Q} = (Q \leq)$ in which every subset $S \subseteq Q$ has a supremum. Let $\mathcal{P}$ be the covariant power set functor on **Set**, so that for any function $f\colon S \to T$, $\mathcal{P}f(X) = f(X) \in \mathcal{P}T$ where $X \in \mathcal{P}S$. For each S and each $s \in S$ let

$$\begin{aligned} \eta_S\colon S &\to \mathcal{P}S \\ s &\mapsto \{s\} \end{aligned}$$

and

$$\begin{aligned} \mu_S\colon \mathcal{P}\mathcal{P}S &\to \mathcal{P}S \\ \{X_i \subseteq S\}_{i\in I} &\mapsto \bigcup_{i\in I} X_i \end{aligned}$$

   (a) Prove that $\langle \mathcal{P}, \eta, \mu \rangle$ defines a monad on **Set**.

   (b) Prove that each $\mathcal{P}$-algebra (S, h) is a complete semi-lattice with $s \leq_h s'$ defined by $h(\{s, s'\}) = s'$.

   (c) Prove conversely that every small complete semi-lattice is a $\mathcal{P}$-algebra in this way.

   (d) Let **CompSLat** be the category of all (small) complete semi-lattices with morphisms the order and sup-preserving functions. Conclude from the above that the forgetful functor $U\colon \mathbf{CompSLat} \to \mathbf{Set}$ is monadic (you can find the definition of a monadic functor on the nLab: monadic functor).

2. An *ultrafilter* F on a set X is a family of subsets of X such that

   (i) $\varnothing \notin F$

   (ii) If $B \in F$ and $B \subseteq A$ then $A \in F$

   (iii) If $A, B \in F$ then $A \cap B \in F$

   (iv) For every subset A of X, either $A \in F$ or $X \setminus A \in F$

We call an ultrafilter F *principal* if there is an element $x \in A$ such that $x \in A$ for all $A \in F$. For any subset of $A \subseteq X$ let $[A]$ denote the set of all ultrafilters on X that contain A. Show the following:

(a) For any set X let $\mathcal{U}X$ denote the set of ultrafilters on X. Show that $\mathcal{U}$ defines a functor $\mathbf{Set} \to \mathbf{Set}$

(b) For any set X, take any $\mathcal{F} \in \mathcal{U}\mathcal{U}X$ (an ultrafilter of ultrafilters.) Show that the set $\mu(\mathcal{F}) = \{A | [A] \in \mathcal{F}\}$ is an ultrafilter.

(c) Show that $\mathcal{U}$ defines a monad on $\mathbf{Set}$ as follows: its unit $\eta_X \colon X \to \mathcal{U}X$ sends an element of X to the principal ultrafilter generated by that element and its multiplication $\mu_X \colon \mathcal{U}\mathcal{U}X \to X$ is defined as above.

(d) A lattice homomorphism between (Boolean) algebras P, Q is a map $h \colon P \to Q$ that preserves meets ($\wedge$) and joins ($\vee$). Show that ultrafilters on X correspond to lattice homomorphisms $\mathcal{P}X \to \mathbf{2}$ (with $\mathbf{2}$ regarded as the two element ordered Boolean algebra $0 \to 1$.) Briefly explain how this fact can be used to characterize the adjunction that gives rise to the ultrafilter monad. (If not familiar, you can find the definition of a boolean algebra on the nLab: boolean algebra. While they are many ways to define them, I personally find it most useful to think of them as a full subcategory of $\mathbf{PoSet}$ as a full subcategory of $\mathbf{Cat}$).

3. Work on your final projects!